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Generalized crossed modules and group-groupoids

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Abstract: In this present work, we present the concept of a crossed module over generalized groups and we call it a "generalized crossed module". We also define a generalized group-groupoid. Furthermore, we show that the category of generalized crossed modules is equivalent to that of generalized group-groupoids whose object sets are abelian generalized group.

Key words: Groupoid, crossed module, generalized group

1. Introduction

The generalized group, first defined by Molaei [13] in 1999, is an interesting generalization of groups. While there is only one identity element in a group, each element in a generalized group has a unique identity element. With this property, every group is a generalized group. After Molaei gave the definition of a generalized group, this concept was studied in terms of algebraic, topological, and differentiable in large various areas of mathematics [1, 2, 8, 12–15].

Another algebraic concept covered in the present study is the crossed module. The concept of crossed module was defined over groups by Whitehead [19]. Afterwards, crossed modules were studied extensively in many areas of mathematics by defining them also over other algebraic structures [3, 6, 16, 17].

We also define the concept of crossed module over generalized groups (called the generalized crossed module). A generalized crossed module is a generalization of the crossed module over groups. We construct the category of generalized crossed modules and their homomorphisms by giving some concrete examples about generalized crossed modules.

The concept of groupoid was first introduced by Brandt [4] in 1926 as an algebraic notion. However, in the category of theoretical approach, a groupoid is a small category whose every morphism is an isomorphism. After the introduction of topological and differentiable groupoids by Ehresmann [7] in the 1950s, they have been studied by many mathematicians with different approaches [5, 9, 10]. One of these different approaches is the structured groupoid, which is obtained by adding another algebraic structure such that the composition of the groupoid is compatible with the operation of the added algebraic structure [6, 8, 11, 16]. The best known of the structured groupoids is the concept of group-groupoid. The group-groupoid, which is a group object in the category of groupoids, was defined by Brown and Spencer [6]. They showed that the category of group-groupoids is equivalent to that of crossed modules over groups. Then in [18] this result was generalized to the group with

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operations and internal groupoids. We note that the object groups in group-groupoids and the target groups in crossed modules are the only groups. However, in the present study, different from the main result in [6], the stated groups in the generalized group-groupoids and generalized crossed modules, respectively, are abelian.

In this study, we extend the concept of group-groupoid to the concept of generalized group-groupoid by adding the structure of generalized group to a groupoid such that the composition of the groupoid and the multiplication of the generalized group are compatible. In other words, a generalized group-groupoid is a generalized group object in the category of groupoids. Thus, we construct the category of generalized group-groupoids. Then, as a main result of the present study, we prove the more general case of the equivalences given in [6] and [18].

2. Preliminaries

This section of the paper is devoted to giving fundamental definitions and concepts related to generalized groups and groupoids. We will consider these fundamental concepts under two headings: generalized groups and groupoids.

2.1. Generalized groups

In this subsection, some basic recalls of the concept of generalized group first defined by Molaei are given.

Definition 2.1 *[13] A generalized group G is a nonempty set admitting an operation called multiplication subject to the set of rules given below:*

i) $(ab)c = a(bc)$, *for all* $a, b, c \in G$

ii) For each $a \in G$, *there exists a unique* $e(a) \in G$ *such that* $ae(a) = e(a)a = a$

iii) For each $a \in G$, *there exists* $a^{-1} \in G$ *such that* $aa^{-1} = a^{-1}a = e(a)$.

Let us give some results related to the structure of generalized groups via the following lemma.

Lemma 2.1 *[13] Let G be a generalized group. Then*

i) For each $a \in G$, *there is a unique element* $a^{-1} \in G$.

ii) For each $a \in G$, *we have* $e(a) = e(a^{-1})$ *and* $e(e(a)) = e(a)$.

iii) For each $a \in G$, *we have* $(a^{-1})^{-1} = a$.

It is easily seen from Definition 2.1 that every group is a generalized group, but it is not true in general that every generalized group is a group.

Let us state the relation between group and generalized group by the following lemma.

Lemma 2.2 *[15] Let G be a generalized group and $ab = ba$ for all $a, b \in G$. Then G is a group.*

In other words, every abelian generalized group is a group.

Example 2.3 *[15] Let $G = IR \times (IR \setminus \{0\})$. Then G with the multiplication $(a, b) \cdot (c, d) = (bc, bd)$ is a generalized group in which for all $(a, b) \in G$, $e(a, b) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$.*

Example 2.4 *[8] Let G with the multiplication m be a generalized group. Then $G \times G$ with the multiplication*

$$m_1((a,b),(c,d)) = (m(a,c), m(b,d))$$

is a generalized group. For any element $(a,b) \in G \times G$, the identity element is $e_1(a,b) = (e(a), e(b))$ and the inverse element is $(a,b)^{-1} = (a^{-1}, b^{-1})$.

Definition 2.2 *[12] If $e(ab) = e(a)e(b)$ for all $a, b \in G$, then G is called a normal generalized group.*

Definition 2.3 *[12] Let G and H be two generalized groups. A generalized group homomorphism from G to H is a map $f : G \to H$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.*

Definition 2.4 *[12] A nonempty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.*

Definition 2.5 *A generalized subgroup N of the generalized group G is called a generalized normal subgroup if there exist a generalized group H and a homomorphism $f : G \to H$ such that for all $a \in G$, $N_a = \emptyset$ or $N_a = ker f_a$, where $N_a = N \cap G_a$, $G_a = \{g \in G \mid e(g) = e(a)\}$, and $f_a = f\,|_{G_a}$.*

Example 2.5 *[15] Let G be a generalized group of Example 2.3. Then $N = \{(a,b) : a = b \text{ or } a = 3b\}$ is a generalized normal subgroup of G.*

Theorem 2.6 *[12] Let $f : G \to H$ be a homomorphism of the distinct generalized groups G and H. Then*

i) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.

ii) $f(a^{-1}) = (f(a))^{-1}$

iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.

Now we state a theorem without proof from [2] that will be used later in the proof of Theorem 5.2.

Theorem 2.7 *Let G be a normal generalized group in which $e(a)b^{-1} = b^{-1}e(a)$, $\forall a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1}$, $\forall a, b \in G$.*

Let us now recall the generalized action of a generalized group on a set, which is defined by Molaei.

Definition 2.6 *[15] We say that a generalized group G acts on a set S if there exists a function*

$$\dot{} : G \times S \to S$$

$$(g, x) \mapsto g\,\dot{}\,x,$$

which is called a generalized action such that

i) $(g_1 g_2)\dot{}\,x = g_1\,\dot{}\,(g_2\,\dot{}\,x)$, for all $g_1, g_2 \in G$ and $x \in S$.

ii) For all $x \in S$, there exists $e(g) \in G$ such that $e(g)\,\dot{}\,x = x$.

Example 2.8 *[15] Let $G = IR \times (IR \setminus \{0\})$ with the multiplication $(a, b) \cdot (c, d) = (bc, bd)$. Since*

$$f : G \to IR, (a, b) \mapsto \frac{a}{b}$$

is a generalized group homomorphism, when the multiplication of IR is $ab = b$, the function

$$: G \times IR \to IR$$

$$((a, b), c)) \mapsto \left(\frac{ac}{b}\right)$$

is a generalized action.

Corollary 2.9 *If we take G as an abelian generalized group in Definition 2.6, then we obtain the known action of the group G on the set S.*

If we take set S in the definition of Molaei as a generalized group, we can define the generalized action of a generalized group G on a generalized group S as follows. This definition will be used for defining the concept of a generalized crossed module in section 4.

Definition 2.7 *Let G and S be two generalized groups. Then a generalized action of the generalized group G on the generalized group S is a function*

$$\dot{} : G \times S \to S, \ (g, x) \mapsto g \,\dot{}\, x$$

such that the following conditions are satisfied.

i) $(g_1 g_2)\dot{}\,x = g_1 \,\dot{}\,(g_2 \,\dot{}\,x)$, $\forall g_1, g_2 \in G$ *and* $x \in S$

ii) $g \,\dot{}\,(x_1 x_2) = (g \,\dot{}\,x_1)(g \,\dot{}\,x_2)$, $\forall g \in G$ *and* $x_1, x_2 \in S$

iii) For $\forall x \in S$, *there exists an element* $e(g) \in G$ *such that* $e(g) \,\dot{}\,x = x$.

We can also add the following condition to these conditions. However, this condition can be easily obtained from the other conditions.

iv) $g \,\dot{}\,e(x) = e(x)$ *for* $\forall g \in G$ *and* $x \in S$.

Example 2.10 *A generalized group G acts on itself with the product $g \cdot h = h$.*

Let us show that the conditions in Definition 2.7 hold.

i) We have $(gh) \cdot k = k$ and $g \cdot (h \cdot k) = g \cdot k = k$ for $\forall g, h, k \in G$. Hence $(gh) \cdot k = g \cdot (h \cdot k)$. That is, the first condition is verified.

ii) We have $g \cdot (hk) = hk$ and $(g \cdot h)(g \cdot k) = hk$ for $\forall g, h, k \in G$. Hence, we obtain the equality $g \cdot (hk) = (g \cdot h)(g \cdot k)$.

iii) Since $e(g) \cdot h = h$ for $\forall h \in G$, there exists element $e(g) \in G$. That is, the existence of $e(g)$ follows directly from the definition of the action.

iv) Since $g \cdot (e(h)) = e(h)$ for $\forall g, h \in G$, the fourth condition is verified.

Proposition 2.11 *The semidirect product of two generalized groups is also a generalized group.*

Proof Let A and B be two generalized groups. We consider the generalized action of B on A. Namely, we have the map

$$\cdot : B \times A \to A,\ (b, a) \mapsto b \cdot a$$

satisfying the conditions in Definition 2.7. Let us now define the multiplication on the set $B \ltimes A$ as follows:

$$(b, a)(b_1, a_1) = (bb_1, a(b \cdot a_1))$$

According to this multiplication, the identity of $(b, a) \in B \ltimes A$ is $e(b, a) = (e(b), e(a))$, and the inverse of $(b, a) \in B \ltimes A$ is $(b, a)^{-1} = \left(b^{-1}, (b^{-1} \cdot a^{-1})\right)$.

Indeed,

$$\begin{aligned}
(b, a)\,(e(b, a)) &= (b, a)\,(e(b), e(a)) \\
&= (be(b), a\,(b \cdot e(a))) \\
&= (b, ae(a)) \\
&= (b, a)
\end{aligned}$$

and

$$\begin{aligned}
(e(b, a))\,(b, a) &= (e(b), e(a))\,(b, a) \\
&= (e(b)b, e(a)\,(e(b) \cdot a)) \\
&= (b, e(a)a) \\
&= (b, a).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(b, a)\,(b, a)^{-1} &= (b, a)\left(b^{-1}, b^{-1} \cdot a^{-1}\right) \\
&= \left(bb^{-1}, a\left(b \cdot \left(b^{-1} \cdot a^{-1}\right)\right)\right) \\
&= \left(e(b), a\left(\left(bb^{-1}\right) \cdot a^{-1}\right)\right) \\
&= \left(e(b), a\left(e(b) \cdot a^{-1}\right)\right) \\
&= \left(e(b), aa^{-1}\right) \\
&= (e(b), e(a)) \\
&= e(b, a)
\end{aligned}$$

and

$$\begin{aligned}(b,a)^{-1}(b,a) &= \left(b^{-1}, b^{-1}\cdot a^{-1}\right)(b,a)\\ &= \left(b^{-1}b, \left(b^{-1}\cdot a^{-1}\right)\left(b^{-1}\cdot a\right)\right)\\ &= \left(e(b), b^{-1}\cdot (a^{-1}a)\right)\\ &= \left(e(b), b^{-1}\cdot e(a)\right)\\ &= (e(b), e(a))\\ &= e(b,a).\end{aligned}$$

If we show that the associativity law holds, then the proof is completed.

For all $(b,a),(b_1,a_1),(b_2,a_2)\in B\ltimes A$,

$$\begin{aligned}((b,a)(b_1,a_1))(b_2,a_2) &= (bb_1, a(b\cdot a_1))(b_2,a_2)\\ &= ((bb_1)b_2, a(b\cdot a_1)((bb_1)\cdot a_2))\\ &= (bb_1b_2, a(b\cdot(a_1(b_1\cdot a_2))))\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}(b,a)((b_1,a_1)(b_2,a_2)) &= (b,a)(b_1b_2, a_1(b_1\cdot a_2))\\ &= (b(b_1b_2), a(b\cdot(a_1(b_1\cdot a_2))))\end{aligned}\tag{2.2}$$

Hence from (2.1) and (2.2), we get $((b,a)(b_1,a_1))(b_2,a_2)=(b,a)((b_1,a_1)(b_2,a_2))$.

Thus, the semidirect product $B\ltimes A$ is a generalized group. $\square$

2.2. Groupoids

In this section, we introduce the elementary concepts of the groupoid theory. Then some recalls about the concept of group-groupoid, which is a group object in the category of groupoids, are given.

Definition 2.8 *[5, 9] A groupoid consists of two sets G and G_0, called the groupoid and the base, respectively, together with two maps α and β from G to G_0, called respectively the source and the target maps, a map $\epsilon: G_0\to G$, $x\mapsto \epsilon(x)=\widetilde{x}=1_x$, called the object inclusion map, a map $i: G\to G$, $x\mapsto i(x)=x^{-1}$, called the inversion, and a partial multiplication $(x,y)\mapsto m(x,y)=xy$ in G defined on the set $G_2=G*G=\{(x,y)\mid \beta(x)=\alpha(y)\}$. These maps verify the following conditions:*

G1) (associativity): $x(yz)=(xy)z$ for all $x,y,z\in G$ such that $\beta(x)=\alpha(y)$ and $\beta(y)=\alpha(z)$.

G2) (units): For each $x\in G$, we have $(\epsilon(\alpha(x)),x)\in G_2$, $(x,\epsilon(\beta(x)))\in G_2$ and $\epsilon(\alpha(x))x=x\epsilon(\beta(x))=x$.

G3) (inverses): For each $x\in G$, we have $(x,i(x))\in G_2$, $(i(x),x)\in G_2$ and $xi(x)=\epsilon(\alpha(x))$, $i(x)x=\epsilon(\beta(x))$.

The maps $\alpha,\beta,m,\epsilon,i$ are called structure maps of groupoids. For a groupoid G on G_0 and $x,y\in G_0$, we will write St_Gx for $\alpha^{-1}(x)$, $CoSt_Gy$ for $\beta^{-1}(y)$, and $G(x,y)$ for $St_Gx\cap CoSt_Gy$. The set St_Gx is the star of G at x and $CoSt_G y$ is the co-star of G at y. The set $G(x,x)$, which is obviously a group under the restriction of the partial multiplication in G, is called the vertex group at x.

Although the following examples of groupoids are well known, they are given here since they are going to be essential later.

Example 2.12 *[5, 9] A group can be regarded as a groupoid with only one object.*

Example 2.13 *[5, 9] Any set G can be regarded as a groupoid on itself with $\alpha = \beta = id_G$ and every element a unity.*

Example 2.14 *[5] For a set X, the cartesian product $X \times X$ is a groupoid over X, called the Banal groupoid. The maps α and β are the natural projections onto the second and first factors, respectively. The object inclusion map is $x \mapsto (x,x)$ and the partial multiplication is given by $(x,y)(y,z) = (x,z)$. The inverse of (x,y) is simply (y,x).*

Definition 2.9 *[5, 9] Let G and $G^{'}$ be groupoids on B and $B^{'}$, respectively. A homomorphism $G \to G^{'}$ is a pair of (f,f_0) of maps $f : G \to G^{'}$, $f_0 : B \to B^{'}$ such that $\alpha^{'} \circ f = f_0 \circ \alpha$, $\beta^{'} \circ f = f_0 \circ \beta$, and $f(ab) = f(a)f(b)$ $\forall (a,b) \in G_2$.*

We denote the groupoid homomorphism (f,f_0) by f for brevity.

Thus, we can construct the category *Gpd* of the groupoids and their homomorphisms.

Now let us recall the concept of group-groupoid, which is a group object in the category of groupoids.

Let $(G,\alpha,\beta,m,\epsilon,i,G_0)$ be a groupoid. We suppose that on G is defined a group structure $w : G \times G \to G$, $(x,y) \mapsto w(x,y) = x+y$. The identity element of the group is denoted by e, that is $v : \{\lambda\} \to G$, $\lambda \mapsto v(\lambda) = e$ (here $\{\lambda\}$ is a singleton). The inverse of $x \in G$ is denoted by $\bar{x}$, that is $\sigma : G \to G$, $x \mapsto \sigma(x) = \bar{x}$. Moreover, we suppose that on G_0 is defined as a group structure $w_0 : G_0 \times G_0 \to G_0$, $(x,y) \mapsto w(u,v) = u+v$. The identity element of the group G_0 is denoted by e_0, that is $v_0 : \{\lambda\} \to G_0$, $\lambda \mapsto v_0(\lambda) = e_0$. The inverse of $u \in G_0$ is denoted by $\bar{u}$, that is $\sigma_0 : G_0 \to G_0$, $u \mapsto \sigma_0(u) = \bar{u}$.

Definition 2.10 *A group-groupoid is a groupoid (G,G_0) such that the following conditions hold:*

i) (G,w,v,σ) *and* (G_0,w_0,v_0,σ_0) *are groups.*

ii) The maps $(w,w_0) : (G \times G, G_0 \times G_0) \to (G,G_0)$, $v : \{\lambda\} \to G$ *and* $(\sigma,\sigma_0) : (G,G_0) \to (G,G_0)$ *are groupoid homomorphisms.*

Moreover, there exists an interchange law between the groupoid composition and the group multiplication:

$$w\left(m\left(b,a\right),m\left(d,c\right)\right) = m\left(w\left(b,d\right),w\left(a,c\right)\right).$$

We shall denote a group-groupoid by $(G,\alpha,\beta,m,\epsilon,i,+,G_0)$ or $(G,\alpha,\beta,m,+,G_0)$ [6].

The above interchange law is generally denoted by $(b \circ a) + (d \circ c) = (b+d) \circ (a+c)$.

Example 2.15 *[16] Let G be a group. Then the cartesian product $G\times G$ is a group-groupoid with the object set G.*

Definition 2.11 *[6, 16] Let G and H be two group-groupoids. A homomorphism $f : G \to H$ of group-groupoids is a homomorphism of underlying groupoids preserving group structure, that is $f\left(m_1(a,b)\right) = m_2\left(f(a), f(b)\right)$, where m_1 and m_2 are compositions of G and H, respectively.*

Thus, the group-groupoids and their homomorphisms form a category denoted by GGd.

3. Generalized group-groupoids

In this section we present the concept of generalized group-groupoid, which is a generalized group object in the category of groupoids. Additionally, we construct the category of generalized group-groupoids.

Definition 3.1 *A generalized group-groupoid is a groupoid (G, G_0) such that the following conditions hold:*

i) (G, w, v, σ) and $(G_0, w_0, v_0, \sigma_0)$ are generalized groups.

ii) The maps $(w, w_0) : (G\times G, G_0\times G_0) \to (G, G_0)$, $v : \{\lambda\} \to G$ and $(\sigma, \sigma_0) : (G, G_0) \to (G, G_0)$ are groupoid homomorphisms.

Furthermore, there exists an interchange law between the groupoid composition and the generalized group operation:

$$w\left(m\left(b,a\right), m\left(d,c\right)\right) = m\left(w\left(b,d\right), w\left(a,c\right)\right).$$

We shall denote a generalized group-groupoid by $(G, \alpha, \beta, m, \epsilon, i, +, G_0)$.

We use the following equality for the interchange law:

$$(b\circ a) + (d\circ c) = (b+d)\circ(a+c).$$

Example 3.1 *Let G be a generalized group. Then we constitute a generalized group-groupoid $G\times G$ with object set G. Indeed, it is obvious that $G\times G$ is a groupoid over G from Example 2.14. On the other hand, since G is a generalized group, $G\times G$ is also a generalized group with the operation $(x,y)+(z,t) = (x+z, y+t)$ defined by the operation of G. The identity element of $(x,y)\in G\times G$ is $(e(x), e(y))$, and the inverse of (y,x) is $(-y,-x)$.*

Now let us show that the generalized group structure maps of $G\times G$ are groupoid homomorphisms.

For $w : (G\times G)\times(G\times G) \to (G\times G)$,

$$\begin{aligned}\left((z,y)+\left(z',y'\right)\right)\circ\left((y,x)+\left(y',x'\right)\right) &= \left(z+z', y+y'\right)\circ\left(y+y', x+x'\right)\\ &= \left(z+z', x+x'\right)\end{aligned}$$

and

$$\begin{aligned}((z,y)\circ(y,x))+\left(\left(z',y'\right)\circ\left(y',x'\right)\right) &= (z+x)+\left(z',x'\right)\\ &= \left(z+z', x+x'\right).\end{aligned}$$

Hence, the generalized group operation is a groupoid homomorphism.

Similarly, it can be shown that the unit map and the inverse map of the generalized group are groupoid homomorphisms.

Consequently, $G \times G$ is a generalized group-groupoid.

Definition 3.2 *Let G and H be two generalized group-groupoids. A homomorphism $f : G \to H$ of generalized group-groupoids is a homomorphism of underlying groupoids preserving the generalized group structure.*

Thus, the generalized group-groupoids and their homomorphisms form a category denoted by $GG - Gd$.

Proposition 3.2 *There is a functor from the category GG of generalized groups to the category $GG - Gd$ of the generalized group-groupoids.*

Proof Let G be a generalized group. Then, from Example 3.1, the cartesian product $G \times G$ is a generalized group-groupoid. If $f : G_1 \to G_2$ is a homomorphism of the generalized groups, then

$$\Gamma(f) : G_1 \times G_1 \longrightarrow G_2 \times G_2$$
$$(a, b) \longmapsto (f(a), f(b))$$

is a homomorphism of the generalized group-groupoids. Thus, Γ is a functor from the category GG to the category $GG - Gd$. □

Remark 3.3 *In the last section, we prove the category of generalized group-groupoids whose object sets are abelian generalized groups and the category of generalized crossed modules. Thus, we have to emphasize the following at this point: the category of generalized group-groupoids whose object sets are abelian generalized groups is a full subcategory of the category of generalized group-groupoids and we denote it by $GG - Gd_{/Ab}$.*

4. Generalized crossed modules

In this section, we define the concept of crossed module over the generalized group,s that is a generalized crossed module. A generalized crossed module is a generalization of the crossed module over groups.

Definition 4.1 *A generalized crossed module (M, P, δ) consists of two generalized groups M and P together with a generalized group homomorphism $\delta : M \to P$ and a generalized action of P on M, written $(m, p) \mapsto m{\cdot}p$, such that the following conditions are satisfied.*

GCM1) $\delta(m \cdot p) = p\delta(m)p^{-1}$, $\forall m \in M$ *and* $\forall p \in P$

GCM2) $\delta(m) \cdot n = mnm^{-1}$, $\forall m, n \in M$.

Example 4.1 *Let K be a generalized group and the set*

$$I(K) = \Big\{ f_k \mid f_k : K \to K,\ f_k(k^{'}) = kk^{'}k^{-1},\ k, k^{'} \in K \Big\}$$

be generalized group of the inner automorphisms of K. Then we obtain a generalized crossed module with the generalized group homomorphism

$$\delta : K \longrightarrow I(K),\ k \longmapsto \delta(k) = f_k$$

and the generalized action

$$I(K) \times K \longrightarrow K$$

$$\left(f_k, k^{'}\right) \longmapsto f_k \cdot k = f_k(k^{'}) = kk^{'}k^{-1}$$

Indeed,

GCM1) $\delta\left(f_k \cdot k^{'}\right) = \delta\left(f_k(k^{'})\right) = \delta\left(kk^{'}k^{-1}\right) = \delta\left(k\right)\delta\left(k^{'}\right)\delta\left(k^{-1}\right) = f_k\delta\left(k^{'}\right)f_k^{-1}$,

GCM2) $\delta\left(k\right) \cdot k^{'} = f_k \cdot k^{'} = f_k(k^{'}) = kk^{'}k^{-1}$.

Example 4.2 *Let G be a generalized group and N be generalized normal subgroup of G. Then we constitute a generalized crossed module with the inclusion homomorphism $\delta = i : N \to G,\ n \mapsto n$, and the generalized action*

$$G \times N \to N,\ (g, n) \mapsto g \cdot n = gng^{-1}.$$

Indeed,

GCM1) $\delta\left(g \cdot n\right) = \delta\left(gng^{-1}\right) = gng^{-1} = g\delta(n)g^{-1}$,

GCM2) $\delta(n) \cdot n^{'} = n \cdot n^{'} = nn^{'}n^{-1}$.

Now, for this example to be more elementary, let us apply this to the generalized group G and the generalized normal subgroup N in Example 2.3 and Example 2.5, respectively.

Example 4.3 *Consider the inclusion homomorphism $\delta = i : N \to G$, $(a, b) \mapsto (a, b)$ and the generalized action*

$$G \times N \to N$$

$$((a, b), (c, d)) \mapsto \left(\frac{da}{b}, d\right).$$

The generalized action is defined by gng^{-1} using the operations in G and N. Namely,

$$\begin{aligned}(a, b) \cdot (c, d) &= (a, b)(c, d)(a, b)^{-1} \\ &= (a, b)\left[(c, d)\left(\frac{a}{b^2}, \frac{1}{b}\right)\right] \\ &= (a, b)\left(\frac{da}{b^2}, \frac{d}{b}\right) \\ &= \left(\frac{bda}{b^2}, \frac{bd}{b}\right) \\ &= \left(\frac{da}{b}, d\right).\end{aligned}$$

Now let us control the conditions to be a generalized crossed module.

GCM1) For $(a,b) \in G$ *and* $(c,d) \in N$,

$$\delta\left((a,b)\cdot(c,d)\right) = \delta\left(\frac{da}{b}, d\right) = \left(\frac{da}{b}, d\right)$$

and

$$\begin{aligned}(a,b)\delta(c,d)(a,b)^{-1} &= (a,b)\delta(c,d)\left(\frac{a}{b^2}, \frac{1}{b}\right)\\ &= (a,b)(c,d)\left(\frac{a}{b^2}, \frac{1}{b}\right)\\ &= \left(\frac{da}{b}, d\right).\end{aligned}$$

Hence, it follows $\delta\left((a,b)\cdot(c,d)\right) = (a,b)\delta(c,d)(a,b)^{-1}$.

GCM2) For $(c,d),(m,n) \in N$,

$$\delta(c,d)\cdot(m,n) = (c,d)\cdot(m,n) = \left(\frac{nc}{d}, n\right)$$

and

$$(c,d)(m,n)(c,d)^{-1} = (c,d)(m,n)\left(\frac{c}{d^2}, \frac{1}{d}\right) = \left(\frac{nc}{d}, n\right).$$

Hence, it follows that $\delta(c,d)\cdot(m,n) = (c,d)(m,n)(c,d)^{-1}$

Now let us define the concept of homomorphism of generalized crossed modules.

Definition 4.2 *Let* (C,G,δ) *and* $(C^{'},G^{'},\delta^{'})$ *be two generalized crossed modules. A generalized crossed module homomorphism* $(\varphi,\psi):(C,G,\delta)\to(C^{'},G^{'},\delta^{'})$ *is a pair of generalized group homomorphisms* $\varphi: C\to C^{'}$ *and* $\psi: G\to G^{'}$ *satisfying the equalities* $\psi\delta(c) = \delta^{'}\varphi(c)$ *and* $\varphi(g\cdot c) = \psi(g)\cdot\varphi(c)$ *for all* $c\in C$ *and* $g\in G$.

Thus, we obtain the category GCM of the generalized crossed modules and their homomorphisms.

5. The equivalence of the categories

In this section, we show that the category of the generalized crossed modules is equivalent to that of generalized group-groupoids whose object sets are abelian generalized groups.

Theorem 5.1 *Let* G *be a generalized group-groupoid whose object set is an abelian generalized group. Then* G *induces a generalized crossed module* $\varphi(G)$.

Proof We proceed as follows to obtain a generalized crossed module $\varphi(G) = (A,B,\delta)$ within a generalized group-groupoid G.

i) $A = CoSt_G e(x)$, $\forall x\in G_0$, is a generalized group.

ii) The set $B = G_0$ is a generalized group.

iii) $\delta : A \to B$ is a generalized group homomorphism.

iv) $\cdot : B \times A \to A$ is a generalized action.

When we set up the above construction, we obtain a generalized crossed module within a generalized group-groupoid. Let us examine these steps in detail now.

i) Let us show that the set $A = CoSt_G e(x)$, $\forall x \in G_0$, has a structure of a generalized group. Since the object set G_0 is an abelian generalized group, it is actually a group. Thus it has a unique identity. Hence, the identity $e(x)$ is actually one element for all the object $x \in G_0$. Let us denote this unique identity by e. On the other hand, the elements of A are arrows ended at e. According to these,

- Let us take $\alpha(m_1) = x$, $\alpha(m_2) = y$ and $\alpha(m_3) = z$ for all $m_1, m_2, m_3 \in CoSt_G e$. Since G_0 is an abelian generalized group, we have $(x + y) + z = x + (y + z)$. Hence, the equality $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$ follows.
- For all $m \in A = CoSt_G e$ with $\alpha(m) = x$, there exists only one arrow $e(m) \in A$ with $\alpha(e(m)) = e(x)$ such that
$$m + e(m) = e(m) + m = m.$$
Moreover, since $e(x) = e$ for all x in G_0, we have $\alpha\left(e(m)\right) = e$.
- For all $m \in A = CoSt_G e$ with $\alpha(m) = x$, there exists only one arrow $-m \in A$ with $\alpha(-m) = -x$ such that
$$m + (-m) = (-m) + m = e(m).$$

Thus, $A = CoSt_G e$ is a generalized group.

ii) Because of the hypothesis, the set $B = G_0$ is a generalized group.

iii) $\delta : A = CoSt_G e \to B = G_0$ is a generalized group homomorphism. Indeed, since δ is a restriction to A of the source map β and β is a generalized group homomorphism, δ is also a generalized group homomorphism.

iv) Let us define a generalized action of the generalized group $B = G_0$ on the generalized group $A = CoSt_G e$ by
$$\cdot : B \times A \to A$$
$$(x, m) \mapsto x \cdot m = 1_x + m - 1_x.$$

Now let us show that the conditions of generalized action are satisfied.

- For all $x, x_1 \in B$ and $m \in A$,
$$\begin{aligned}(x + x_1) \cdot m &= 1_{x+x_1} + m - 1_{x+x_1} \\ &= 1_x + 1_{x_1} + m - 1_x - 1_{x_1} \\ &= 1_x + (x_1 \cdot m) - 1_x \\ &= x \cdot (x_1 \cdot m).\end{aligned}$$

- For all $x \in B$ and $m \in A$, we have $e(x) \cdot m = m$. Indeed, since $e(x) = e$, the source of the arrow $e \cdot m = 1_e + m - 1_e$ is x. That is, the arrows $e \cdot m$ and m are the same.
- For all $x \in B$ and $m_1, m_2 \in A$,

$$\begin{aligned} x \cdot (m_1 + m_2) &= 1_x + (m_1 + m_2) - 1_x \\ &= 1_x + m_1 - 1_x + 1_x + m_2 - 1_x \\ &= x \cdot m_1 + x \cdot m_2. \end{aligned}$$

- We have $x \cdot e = e$, because the action $x \cdot e = 1_x + e - 1_x$ is the arrow e such that $\alpha(e) = e = \beta(e)$.

Thus, all conditions of the generalized action hold.

Finally, let us show that the conditions of the generalized crossed module are satisfied.

GCM1) For all $x \in B$ and $m \in A$, we have

$$\begin{aligned} \delta(x \cdot m) &= \delta(1_x + m - 1_x) \\ &= \delta(1_x) + \delta(m) + \delta(-1_x) \\ &= \delta(1_x) + \delta(m) + \delta(1_{-x}) \\ &= x + \delta(m) - x. \end{aligned}$$

GCM2) For all $m, m_1 \in A$ with $\alpha(m) = x$ and $\alpha(m_1) = y$, we have

$$\begin{aligned} \delta(m) \cdot m_1 = 1_{\delta(m)} + m_1 - 1_{\delta(m)} &= 1_{\alpha(m)} + m_1 - 1_{\alpha(m)} \\ &= 1_x + m_1 - 1_x. \end{aligned}$$

On the other hand, the source of the arrow $m + m_1 - m$ is also $x + y - x$. Hence, it follows that $\delta(m) \cdot m_1 = m + m_1 - m$.

Consequently, $\varphi(G) = (CoSt_G e, G_0, \beta|_{CoSt_G e})$ is a generalized crossed module. □

From now on, we assume that B is an abelian generalized group and A is a normal generalized group satisfying the equality $e(a)b^{-1} = b^{-1}e(a)$ for any $a, b \in A$. Thus, we have a new category whose objects consist of the generalized crossed modules satisfying the above conditions. This category is a full subcategory of the category GCM, and we denote it by GCM^*.

Theorem 5.2 *Let (A, B, δ) be a generalized crossed module. Then (A, B, δ) induces a generalized group-groupoid.*

Proof Since (A, B, δ) is a generalized crossed module, there exists a generalized action

$$\cdot : B \times A \to A, \ (b, a) \mapsto b \cdot a$$

of B on A such that the following conditions hold:

GCM1) $\delta(b \cdot a) = b\delta(a)b^{-1}$

GCM2) $\delta(a) \cdot a_1 = aa_1a^{-1}$.

Let us now show how a generalized group-groupoid from the generalized crossed module is obtained.

Set of objects : Our object set is the generalized group B.

Set of morphisms : Our morphism set is the semidirect product $B \ltimes A$, which is a generalized group from Proposition 2.11.

The source map: The source map $\alpha : B \ltimes A \to B$ is defined by $\alpha(b, a) = b$. Moreover, α is a generalized group homomorphism, because

$$\begin{aligned}\alpha\left((b, a)(b_1, a_1)\right) &= \alpha\left(bb_1, a(b \cdot a_1)\right) \\ &= bb_1 \\ &= \alpha(b, a)\alpha(b_1, a_1).\end{aligned}$$

The target map : The target map $\beta : B \ltimes A \to B$ is defined by $\beta(b, a) = \delta(a)b$. We can prove that β is a generalized group homomorphism in the following way:

$$\begin{aligned}\beta\left((b, a)\left(b_1, a_1\right)\right) &= \beta\left(bb_1, a\left(b \cdot a_1\right)\right) \\ &= \delta\left(a\left(b \cdot a_1\right)\right)bb_1 \\ &= \delta(a)\delta(b \cdot a_1)bb_1 \\ &= \delta(a)b\delta(a_1)b^{-1}bb_1 \\ &= \delta(a)b\delta(a_1)e(b)b_1 \qquad (5.1)\end{aligned}$$

and

$$\beta\left(b, a\right)\beta\left(b_1, a_1\right) = \delta(a)b\delta(a_1)b_1. \qquad (5.2)$$

If we remember that B is an abelian generalized group, it follows that the equalities (5.1) and (5.2) are the same. Thus, β is also a generalized group homomorphism.

The object map : For all $a \in A$, the object map $\epsilon_a : B \to B \ltimes A$ is defined by $b \mapsto \epsilon_a(b) = (b, e(a))$. Now let us show that the object map is a generalized group homomorphism. For $\forall a \in A$ and $\forall b, b_1 \in B$,

$$\epsilon_a(bb_1) = (bb_1, e(a)) \qquad (5.3)$$

and

$$\begin{aligned}\epsilon_a(b)\epsilon_a(b_1) &= (b, e(a))(b_1, e(a)) \\ &= (bb_1, e(a)\left(b \cdot e(a)\right)) \\ &= (bb_1, e(a)e(a)) \\ &= (bb_1, e(a)). \qquad (5.4)\end{aligned}$$

Hence from (5.3) and (5.4), it follows that β is a generalized group homomorphism.

The inverse map : We define the inverse map $\eta : B \ltimes A \to B \ltimes A$ by $(b, a) \mapsto (b, a)^{-1} = \left(\delta(a)b, a^{-1}\right)$. It follows that η is a generalized group homomorphism after the following steps. For $\forall a, a_1 \in A$ and $\forall b, b_1 \in B$,

$$\begin{aligned}
\eta\left((b,a)(b_1,a_1)\right) &= \eta\left(bb_1, a\left(b\cdot a_1\right)\right)\\
&= \left(\delta\left(a\left(b\cdot a_1\right)\right)bb_1, \left(a\left(b\cdot a_1\right)\right)^{-1}\right)\\
&= \left(\delta(a)\delta(b\cdot a_1)bb_1, \left(a\left(b\cdot a_1\right)\right)^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b^{-1}bb_1, \left(a\left(b\cdot a_1\right)\right)^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, \left(a\left(b\cdot a_1\right)\right)^{-1}\right)
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
\eta\left(b,a\right)\eta\left(b_1,a_1\right) &= \left(\delta(a)b, a^{-1}\right)\left(\delta(a_1)b_1, a_1^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, a^{-1}\left((\delta(a)b)\cdot a_1^{-1}\right)\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, a^{-1}\left(\delta(a)\cdot\left(b\cdot a_1^{-1}\right)\right)\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, a^{-1}a\left(b\cdot a_1^{-1}\right)a^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, e(a)\left(b\cdot a_1^{-1}\right)a^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, \left(b\cdot a_1^{-1}\right)a^{-1}e(a)\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, \left(b\cdot a_1^{-1}\right)a^{-1}\right)\\
&= \left(\delta(a)b\delta(a_1)b_1, \left(a\left(b\cdot a_1\right)\right)^{-1}\right).
\end{aligned} \tag{5.6}$$

As can be seen from the above steps, in order to show that (5.5) and (5.6) are the same, we have used that A has the property $e(a_1)a_2^{-1} = a_2^{-1}e(a_1)$ for any $a_1, a_2 \in A$.

The composition of groupoid : We define the composition of the groupoid by

$$\circ : B \ltimes A \times B \ltimes A \to B \ltimes A$$

$$((b,a),(b_1,a_1)) \mapsto (b_1,a_1)\circ(b,a) = (b, a_1 a),$$

where $\beta(b,a) = \delta(a)b = b_1 = \alpha(b_1,a_1)$ for $b, b_1 \in B$ and $a, a_1 \in A$.

Let us show that the interchange law between the composition of the groupoid and the multiplication of the generalized group is verified. Namely, we must show that the equality

$$\left[(b_1,a_1)\circ(b,a)\right]\left[(b_3,a_3)\circ(b_2,a_2)\right] = \left[(b_1,a_1)\left(b_3,a_3\right)\right]\circ\left[(b,a)\left(b_2,a_2\right)\right] \tag{5.7}$$

holds.

Firstly, consider the following part, which is the left-hand side of the equality (5.7):

$$\left[(b_1,a_1)\circ(b,a)\right]\left[(b_3,a_3)\circ(b_2,a_2)\right] \tag{5.8}$$

The statement 5.8 is defined under the cases

$$\beta(b,a) = \delta(a)b = b_1 = \alpha(b_1,a_1) \tag{5.9}$$

and

$$\beta(b_2, a_2) = \delta(a_2)b_2 = b_3 = \alpha(b_3, a_3). \tag{5.10}$$

Then we have

$$\begin{aligned}
[(b_1, a_1) \circ (b, a)] \, [(b_3, a_3) \circ (b_2, a_2)] &= (b, a_1 a) \, (b_2, a_3 a_2) \\
&= (b b_2, a_1 a \, (b \cdot (a_3 a_2))) \\
&= (b b_2, a_1 a \, (b \cdot a_3) \, (b \cdot a_2)) \, ...(*)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[(b_1, a_1) \, (b_3, a_3)] \circ [(b, a) \, (b_2, a_2)] &= (b b_2, a \, (b \cdot a_2)) \circ (b_1 b_3, a_1 \, (b_1 \cdot a_3)) \\
&= (b b_2, a_1 \, (b_1 \cdot a_3) \, a \, (b \cdot a_2)) \, ...(**)
\end{aligned}$$

In order for $(*)$ and $(**)$ to be equal, we must satisfy the following equality

$$a_1 a \, (b \cdot a_3) \, (b \cdot a_2) = a_1 \, (b_1 \cdot a_3) \, a \, (b \cdot a_2) \, . \tag{5.11}$$

Equality (5.11) follows directly from $b_1 = \delta(a)b$ and the property $e(a)b^{-1} = b^{-1}e(a)$, $a, b \in A$ for the normal generalized group A. □

Finally, using the Theorems 5.1 and 5.2, let us state the following theorem, which is the main result of this paper.

Theorem 5.3 *The categories $GG - Gd_{/Ab}$ and GCM^* are equivalent.*

Proof Let $M = (A, B, \delta)$ and $M^{'} = (A^{'}, B^{'}, \delta^{'})$ be two generalized crossed modules, and $(f_1, f_2) : (A, B, \delta) \to (A^{'}, B^{'}, \delta^{'})$ be a generalized crossed module homomorphism. Then, according to Theorem 5.1, there exists a functor $\theta : GCM^* \to GG - Gd_{/Ab}$ defined by $\theta(f_1, f_2) = (f_2 \times f_1, f_2)$ on morphisms and by $\theta(M) = (B, B \ltimes A)$ on objects.

$$\begin{tikzcd}
B \ltimes A \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "f_2 \times f_1"'] & B \arrow[d, "f_2"] \\
B^{'} \ltimes A^{'} \arrow[r, shift left, "\alpha^{'}"] \arrow[r, shift right, "\beta^{'}"'] & B^{'}
\end{tikzcd}$$

Conversely, let $\mathcal{G} = (G_0, G_1)$ and $\mathcal{H} = (H_0, H_1)$ be two generalized group-groupoids, and $f = (f_0, f_1) : \mathcal{G} \to \mathcal{H}$ be a generalized group-groupoid homomorphism. Then, according to Theorem 5.2, there exists a functor

$$\begin{tikzcd}
\mathrm{Ker}(\alpha) \arrow[r, "\beta|_{\mathrm{Ker}(\alpha)}"] \arrow[d, "f_1|_{\mathrm{Ker}(\alpha)}"'] & G_0 \arrow[d, "f_0"] \\
\mathrm{Ker}(\alpha) \arrow[r, "\beta|_{\mathrm{Ker}(\alpha)}"'] & H_0
\end{tikzcd}$$

$\Gamma : GG{-}Gd_{/Ab} \to GCM^*$ defined by $\Gamma(f_0, f_1) = (f_1|_{Ker\alpha}, f_0)$ on morphisms and by $\Gamma(\mathcal{G}) = (Ker\alpha, G_0, \beta|_{Ker\alpha})$ on objects.

It is obvious that $\theta\Gamma \simeq 1_{GG-Gd_{/Ab}}$ and $\Gamma\theta \simeq 1_{GCM^*}$. □

References

[1] Ahmadi SA. Generalized topological groups and genetic recombination. J Dyn Syst Geom Theor 2013; 11: 51-58.

[2] Akinmoyewa JT. A study of some properties of generalized groups. Octogon Mathematical Magazine 2009; 17: 125-134.

[3] Alemdar N, Mucuk O. The liftings of R-modules to covering groupoids. Hacet J Math Stat 2012; 41: 813-822.

[4] Brandt H. Uber eine verallgemeinerung des gruppenbegriffes. Math Ann 1926; 96: 360-366.

[5] Brown R. Topology and groupoids. Deganwy, UK: BookSurge LLC, 2006.

[6] Brown R, Spencer CB. G-groupoids, crossed modules and the fundamental groupoid of a topological group. P K Ned Akad A Math 1976; 79: 196-302.

[7] Ehresmann C. Catégories topologiques et catégories differentiables. Colloque de géometrie différentielle globale, Cenrtre Belge de Recherches Mathematiques, Bruxelles 1958; 137-150.

[8] Farhangdoost MR, Nasirzade T. Geometrical categories of generalized Lie groups and Lie group-groupoids. Iran J Sci Technol A 2013; 69-73.

[9] Gürsoy MH, Icen I. The homomorphisms of topological groupoids. Novi Sad J Math 2014; 44: 129-141.

[10] Icen I, Gürsoy MH, Ozan AF. Coverings of Lie groupoids. Turk J Math 2011; 35: 207-218.

[11] Ivan M. Vector space-groupoids. Eur J Pure Appl Math 2013; 6: 469-484.

[12] Mehrabi M, Molaei MR, Olomi A. Generalized subgroups and homomorphisms. Arab J Math Sci 2000; 6: 1-7.

[13] Molaei MR. Generalized groups. Bul Inst Politeh Iasi Sect I Mat Mec Teor Fiz 1999; 45: 21-24.

[14] Molaei MR. Generalized structures based on completely simple semigroups. Palm Harbor, FL, USA: Hadronic Press, 2005.

[15] Molaei MR. Generalized actions. In: Mladenov IM, Naber GL, editors. Intenational Conference on Geometry, Integrability and Quantization, 1–10 September 1999; Varna, Bulgaria. Sofia: Coral Press, 1999, pp.175-179.

[16] Mucuk O, Sahan T, Alemdar N. Normality and quotients in crossed modules and group-groupoids. Appl Categor Struct 2015; 23: 415-428.

[17] Mutlu A, Mutlu B. Quasi 3-crossed modules. Theor Math Appl 2012; 2: 101-118.

[18] Porter T. Extensions, crossed modules and internal categories in categories of groups with operations. Proc Edinb Math Soc 1987; 30: 373-381.

[19] Whitehead JHC. Combinatorial homotopy II. B Am Math Soc 1949; 55: 453-456.