Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2017) 41: 1552 – 1569
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doi:10.3906/mat-1604-34

Universal central extensions of $\mathfrak{sl}(m,n,A)$ over associative superalgebras

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Received: 08.04.2016 • **Accepted/Published Online:** 02.02.2017 • **Final Version:** 23.11.2017

Abstract: We find the universal central extension of the matrix superalgebras $\mathfrak{sl}(m,n,A)$, where A is an associative superalgebra and $m+n=3,4$, and its relation with the Steinberg superalgebra $\mathfrak{st}(m,n,A)$. We calculate $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ and $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$. Finally, we introduce a new method using the nonabelian tensor product of Lie superalgebras to find the connection between $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ and the cyclic homology of associative superalgebras for $m+n \geq 3$.

Key words: Lie superalgebras, Steinberg superalgebras, universal central extensions

1. Introduction

The study of central extensions plays an important role in the theory of groups or Lie algebras and has numerous applications going through physics, representation theory, or homological algebra. They have been studied by many people in the context of Lie algebras as [8, 15], etc. The universal central extension is a key object in this study, since it simplifies the task of finding all central extensions and, moreover, its kernel is the second homology group. In [4] the universal central extension of Lie algebras is constructed as a nonabelian tensor product, extended to Lie superalgebras in [7]; and in [9, 13] some of the results of [8] are extended to Lie superalgebras and the universal central extension is constructed. The main problem of these constructions is that they are usually hard to compute.

The concrete problem of finding the universal central extension of $\mathfrak{sl}_n(A)$ for $n \geq 5$ was solved in [11]. It is a very important result that involves Steinberg Lie algebras (see [2, 5]) and allowed the development of the additive K-theory. If $n \geq 5$, $\mathfrak{st}_n(A)$ is the universal central extension of $\mathfrak{sl}_n(A)$ and if A is K-free, the kernel is isomorphic to the first cyclic homology $\mathrm{HC}_1(A)$. The problem of finding the universal central extension of $\mathfrak{sl}_n(A)$ and $\mathfrak{st}_n(A)$ for $n = 3,4$ was solved years later in [6]. In [12] the universal central extension of the Lie superalgebras $\mathfrak{sl}(m,n,A)$ and $\mathfrak{st}(m,n,A)$ is computed with $m+n \geq 5$, where A is an associative algebra, and the remaining cases where $m+n=3,4$ are solved in [14].

If A is an associative superalgebra, the universal central extension of $\mathfrak{sl}_n(A)$ is computed in [3] for all $n \geq 3$. The case $\mathfrak{sl}(m,n,A)$ is studied in [7] for $m+n \geq 5$, leaving as an open problem the cases $m+n=3,4$. In this paper, we will solve these specific cases in order to complete the computation of the universal central extension of $\mathfrak{sl}(m,n,A)$ where A is an associative superalgebra and $m+n \geq 3$, and therefore giving a complete characterization of the second homology $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ for $m+n \geq 3$ (Theorem 8.1). Moreover, we introduce

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2010 *AMS Mathematics Subject Classification:* 17B60, 17B55, 17B05

a new technique using the nonabelian tensor product of Lie superalgebras defined in [7] to relate $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ and the cyclic homology of associative superalgebras for $m+n\geq 3$ (Theorem 8.2).

The organization of this paper is as follows. In Section 2 we give some preliminary well-known results and some technical lemmas about $\mathfrak{sl}(m,n,A)$ and $\mathfrak{st}(m,n,A)$. In Section 3 we adapt the classical construction of a central extension from a super 2-cocycle in Lie superalgebras. In Section 4 we start with the case of $\mathfrak{sl}(2,1,A)$ and we show that its universal central extension is $\mathfrak{st}(2,1,A)$, constructing a (unique) homomorphism to any central extension. In Section 5 we find the universal central extension of $\mathfrak{st}(3,1,A)$ (which consequently will be the universal central extension of $\mathfrak{sl}(3,1,A)$) via the construction of a super 2-cocycle, repeating the procedure for $\mathfrak{st}(2,2,A)$ in Section 6. In Section 7 we relate the second homology of $\mathfrak{sl}(m,n,A)$ with cyclic homology. Finally, in Section 8 we give concluding remarks establishing a combination of the results presented here with results of [3, 7] to give the full computation of $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ and $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ for $m+n\geq 3$.

2. The Lie superalgebras $\mathfrak{sl}(m,n,A)$ and $\mathfrak{st}(m,n,A)$

Throughout this paper we consider K as a unital commutative ring and $A=A_{\bar{0}}\oplus A_{\bar{1}}$ an associative unital K-superalgebra. For any $m,n\in\mathbb{Z}_+$, let $\{1,\dots,m\}\cup\{m+1,\dots,m+n\}$ be the $\mathbb{Z}_2$-graded set, where the first set is the even part and the second one the odd part. We now consider $\mathrm{Mat}(m,n,A)$ the $(m+n)\times(m+n)$ matrices with coefficients in A. It is defined a $\mathbb{Z}_2$-graduation where homogeneous elements are matrices, denoted by $E_{ij}(a)$, having $a\in A_{\bar{0}},A_{\bar{1}}$ at position (i,j) and zero elsewhere and $|E_{ij}(a)|=|i|+|j|+|a|$. With this graduation we define the associative superalgebra $\mathfrak{gl}(m,n,A)$ whose underlying set is $\mathrm{Mat}(m,n,A)$ with the usual matrix product and it is endowed by a Lie superalgebra structure with the usual bracket $[x,y]=xy-(-1)^{|x||y|}yx$.

Assuming that $m+n\geq 3$, we define the *special Lie superalgebra*

$$\mathfrak{sl}(m,n,A)=[\mathfrak{gl}(m,n,A),\mathfrak{gl}(m,n,A)].$$

It is generated by the elements $E_{ij}, 1\leq i\neq j\leq m+n,\ a\in A$, with bracket

$$[E_{ij}(a),E_{kl}(b)]=\delta_{jk}E_{il}(ab)-(-1)^{|E_{ij}(a)||E_{kl}(b)|}\delta_{li}E_{kj}(ba).$$

In [1] is introduced a generalization of the supertrace for $x\in\mathfrak{gl}(m,n,A)$, defined as follows:

$$\mathrm{Str}_1(x)=\sum_{i=1}^{m+n}(-1)^{|i|(|i|+|x_{ii}|)}x_{ii},$$

where x_{ii} represents the element of x in the position (i,i). It is straightforward that $\mathfrak{sl}(m,n,A)=\{x\in\mathfrak{gl}(m,n,A):\mathrm{Str}_1(x)\in[A,A]\}$ and that $\mathfrak{sl}(m,n,A)$ is perfect.

For $m+n\geq 3$, the *Steinberg Lie superalgebra* $\mathfrak{st}(m,n,A)$ is defined as the Lie superalgebra over K generated by homogeneous $F_{ij}(a)$, $1\leq i\neq j\leq m+n$, and $a\in A$ homogeneous, with grading $|F_{ij}(a)|=|i|+|j|+|a|$, satisfying the following relations:

$$a\mapsto F_{ij}(a)\ \text{ is a } K\text{-linear map,}\tag{1}$$

$$[F_{ij}(a),F_{jk}(b)]=F_{ik}(ab),\quad\text{for distinct } i,j,k,\tag{2}$$

$$[F_{ij}(a),F_{kl}(b)]=0,\quad\text{for } j\neq k, i\neq l,\tag{3}$$

where $a, b \in A$, $1 \leq i, j, k, l \leq m+n$. Note that $\mathfrak{st}(m,n,A)$ is a perfect Lie algebra and there is a canonical central extension

$$\varphi\colon \mathfrak{st}(m,n,A) \to \mathfrak{sl}(m,n,A), \qquad \varphi\big(F_{ij}(a)\big) \mapsto E_{ij}(a).$$

Using a completely new technique, in [7] it is shown that if $m+n \geq 5$, this epimorphism is the universal central extension of $\mathfrak{sl}(m,n,A)$. The remaining cases, when $m+n=3$ or 4, are left as an open problem and they are the object of study of this paper. Our procedure to solve the problem is to find the universal central extension of $\mathfrak{st}(m,n,A)$ and by [13, Corollary 1.9] it will be the universal central extension of $\mathfrak{sl}(m,n,A)$.

We begin giving some relations in $\mathfrak{st}(m,n,A)$ that will be useful. Let

$$H_{ij}(a,b) = [F_{ij}(a), F_{ji}(b)],$$
$$h(a,b) = H_{1j}(a,b) - (-1)^{|a||b|}H_{1j}(1,ba),$$

for $1 \leq i \neq j \leq m+n, a \in A$. It is well defined since $h(a,b)$ does not depend on j, for $j \neq 1$. We recall that $|H_{ij}(a,b)| = |a|+|b|$ for homogeneous $a, b \in A$.

Lemma 2.1 *We have the following identities in* $\mathfrak{st}(m,n,A)$*,*

$$H_{ij}(a,b) = -(-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)}H_{ji}(b,a), \tag{4}$$
$$[H_{ij}(a,b), F_{ik}(c)] = F_{ik}(abc), \tag{5}$$
$$[H_{ij}(a,b), F_{ki}(c)] = -(-1)^{(|a|+|b|)(|i|+|k|+|c|)}F_{ki}(cab), \tag{6}$$
$$[H_{ij}(a,b), F_{kj}(c)] = (-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)+(|a|+|b|)(|j|+|k|+|c|)}F_{kj}(cba) \tag{7}$$
$$[H_{ij}(a,b), F_{ij}(c)] = F_{ij}\big(abc + (-1)^{(|i|+|j|+|a||b|+|b||c|+|c||a|)}cba\big), \tag{8}$$
$$[H_{ij}(a,b), F_{kl}(c)] = 0, \tag{9}$$
$$[h(a,b), F_{1i}(c)] = F_{1i}\big((ab-(-1)^{|a||b|}ba)c\big), \tag{10}$$
$$[h(a,b), F_{jk}(c)] = 0 \textit{ for } j,k \geq 2. \tag{11}$$

for homogeneous $a, b, c \in A$ *and* i, j, k, l *distinct.*

Proof Relations (4)–(9) are just consequences of antisymmetry and Jacobi identities. To check (10) and (11) we need to apply (5) and (9) to the definition of $h(a,b)$. □

The following lemma gives a better understanding of the structure of $\mathfrak{st}(m,n,A)$.

Lemma 2.2 *Let* $F_{ij}(A)$ *be the subalgebra generated by* $F_{ij}(a)$*,* $\mathcal{N}^+$ *the subalgebra generated by* $F_{ij}(a)$ *for* $1 \leq i < j \leq m+n$*,* $\mathcal{N}^-$ *the subalgebra generated by* $F_{ij}(a)$ *for* $1 \leq j < i \leq m+n$*, and* $\mathcal{H}$ *the subalgebra generated by* $H_{ij}(a,b)$*, for all* $a, b \in A$*. Then*

$$\mathcal{N}^+ = \bigoplus_{1\leq i<j\leq m+n} F_{ij}(A),$$
$$\mathcal{N}^- = \bigoplus_{1\leq j<i\leq m+n} F_{ij}(A),$$
$$\mathcal{H} = h(A,A) \oplus \Bigg(\bigoplus_{j=2}^{m+n} H_{1j}(1,A)\Bigg),$$

and we have the decomposition

$$\mathfrak{st}(m,n,A) = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^- = h(A,A) \oplus \left(\bigoplus_{j=2}^{m+n} H_{1j}(1,A) \right) \bigoplus_{1 \le i \ne j \le m+n} F_{ij}(A).$$

Definition 2.3 *Let $\mathcal{I}_m$ be the graded ideal of A generated by the elements ma (i.e. $a + \cdots + a$, m times) and $ab - (-1)^{|a||b|}ba$. Let $A_m = A/\mathcal{I}_m$ be the quotient algebra and denote by $\bar{a} = a + \mathcal{I}_m$ its elements.*

Lemma 2.4 ([3]) *$\mathcal{I}_m = mA + A[A,A]$ and $[A,A]A = A[A,A]$.*

3. Central extensions of $\mathfrak{st}(m,n,A)$ and cocycles

Definition 3.1 *Let L be a Lie superalgebra and $\mathcal{W}$ be a K-free supermodule. A* super 2-cocycle *is a K-bilinear map $\psi \colon L \times L \to \mathcal{W}$ such that*

$$\psi(x,y) = -(-1)^{|x||y|}\psi(y,x),$$
$$(-1)^{|x||z|}\psi([x,y],z) + (-1)^{|x||y|}\psi([y,z],x) + (-1)^{|y||z|}\psi([z,x],y) = 0,$$
$$\psi(x_{\bar{0}}, x_{\bar{0}}) = 0,$$

for all $x,y,z \in L$, $x_{\bar{0}} \in L_{\bar{0}}$.

Given an even super 2-cocycle ψ, we can construct a central extension ([13]) $L \oplus \mathcal{W} \to L$, $(x,w) \mapsto x$, where the bracket is given by $[(x,w_1),(y,w_2)] = \big([x,y], \psi(x,y)\big)$ (see [13]). In the particular case of $L = \mathfrak{st}(m,n,A)$ and the super 2-cocycle being surjective, this construction can be described in a different way using generators and relations.

Definition 3.2 *Let $\psi \colon \mathfrak{st}(m,n,A) \times \mathfrak{st}(m,n,A) \to \mathcal{W}$ be an even super 2-cocycle, i.e. a super 2-cocycle such that $|\psi(x,y)| = |x| + |y|$ for homogeneous $x, y \in \mathfrak{st}(m,n,A)$. Let $\mathfrak{st}(m,n,A)^\sharp$ be the Lie superalgebra generated by the elements $F_{ij}(a)^\sharp$ with homogeneous $a \in A$, $1 \le i \ne j \le m+n$, with degree $|F^\sharp_{ij}(a)| = |i| + |j| + |a|$ and by the elements of $\mathcal{W}$, with the relations*

$$a \mapsto F^\sharp_{ij}(a) \quad \text{is a } K\text{-linear map,}$$
$$[\mathcal{W},\mathcal{W}] = [F^\sharp_{ij}(a), \mathcal{W}] = 0,$$
$$[F^\sharp_{ij}(a), F^\sharp_{jk}(b)] = F^\sharp_{ik}(ab) + \psi\big(F_{ij}(a), F_{jk}(b)\big) \quad \text{for distinct } i,\ j,\ k,$$
$$[F^\sharp_{ij}(a), F^\sharp_{kl}(b)] = \psi\big(F_{ij}(a), F_{kl}(b)\big) \text{ for } i \ne j \ne k \ne l \ne i,$$

where $a, b \in A$.

Lemma 3.3 *If $\mathfrak{st}(m,n,A)' = \mathfrak{st}(m,n,A) \oplus \mathcal{W}$ is a central extension constructed from a surjective super 2-cocycle $\psi \colon \mathfrak{st}(m,n,A) \times \mathfrak{st}(m,n,A) \to \mathcal{W}$ then there is an isomorphism $\rho \colon \mathfrak{st}(m,n,A)^\sharp \to \mathfrak{st}(m,n,A)'$ where $\rho\big(F^\sharp_{ij}(a)\big) = F_{ij}$ and $\rho(w) = w$.*

Proof The proof of [3, Lemma 1] can be easily adapted. □

As before, we denote $H^\sharp_{ij}(a,b) = [F^\sharp_{ij}(a), F^\sharp_{ji}(b)]$ and $h^\sharp(a,b) = H^\sharp_{1j}(a,b) - (-1)^{|a||b|} H^\sharp_{1j}(1,ba)$. Therefore, $h^\sharp$ is independent of j and we have the analogue decomposition lemma.

Lemma 3.4 *We can decompose the Lie superalgebra* $\mathfrak{st}(m,n,A)^\sharp$ *generated by a surjective super 2-cocycle* $\psi\colon \mathfrak{st}(m,n,A) \times \mathfrak{st}(m,n,A) \to \mathcal{W}$ *in the following way:*

$$\mathfrak{st}(m,n,A)^\sharp = \mathcal{W} \,\oplus\, h^\sharp(A,A) \oplus \left(\bigoplus_{j=2}^{m+n} H^\sharp_{1j}(1,A) \right) \bigoplus_{1 \le i \ne j \le m+n} F^\sharp_{ij}(A).$$

4. Universal central extension of $\mathfrak{st}(2,1,A)$

In this section we study the case when $m+n=3$ and prove that $\mathfrak{st}(2,1,A)$ is the universal central extension of $\mathfrak{sl}(2,1,A)$.

Theorem 4.1 *If* $\tau\colon \widetilde{\mathfrak{st}}(2,1,A) \to \mathfrak{st}(2,1,A)$ *is a central extension, then there exists a unique section* $\eta\colon \mathfrak{st}(2,1,A) \to \widetilde{\mathfrak{st}}(2,1,A)$.

Proof We will directly obtain a Lie superalgebra homomorphism $\eta\colon \mathfrak{st}(2,1,A) \to \widetilde{\mathfrak{st}}(2,1,A)$, such that $\tau \circ \eta = \mathrm{id}$ and since $\mathfrak{st}(2,1,A)$ is perfect it must be unique. Let

$$0 \longrightarrow \mathcal{V} \longrightarrow \widetilde{\mathfrak{st}}(2,1,A) \xrightarrow{\tau} \mathfrak{st}(2,1,A) \longrightarrow 0$$

be a central extension. We choose a preimage for $F_{ij}(a)$ denoted by $\widetilde{F}_{ij}(a)$ and extend it by K-linearity to all $a \in A$.

We define $\widetilde{H}_{ij}(a,b) = [\widetilde{F}_{ij}(a), \widetilde{F}_{ji}(b)]$, since it is independent of the choice of $\widetilde{F}_{ij}(a)$. By identity (4) we know that $[\widetilde{H}_{ik}(1,1), \widetilde{F}_{ij}(a)] = \widetilde{F}_{ij}(a) + v_{ij}(a)$, where $v_{ij}(a) \in \mathcal{V}$ and so we will replace $\widetilde{F}_{ij}(a)$ by $\widetilde{F}_{ij}(a) + v_{ij}(a)$.

It suffices to show that these $\widetilde{F}_{ij}(a)$ satisfy relations (1)–(3) because our K-linear section $\eta\colon \mathfrak{st}(2,1,A) \to \widetilde{\mathfrak{st}}(2,1,A)$, $F_{ij}(a) \mapsto \widetilde{F}_{ij}(a)$ will be a Lie superalgebra homomorphism and the result is proved. The first relation is immediate by definition.

To see the second one, we use Jacobi identity and the fact that $\mathcal{V}$ is in the centre of $\widetilde{\mathfrak{st}}(2,1,A)$.

$$\begin{aligned}
\widetilde{F}_{ij}(ab) &= [\widetilde{H}_{ik}(1,1), \widetilde{F}_{ij}(ab)] = \big[\widetilde{H}_{ik}(1,1), [\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)]\big] \\
&= \big[[\widetilde{H}_{ik}(1,1), \widetilde{F}_{ik}(a)], \widetilde{F}_{kj}(b)\big] + \big[\widetilde{F}_{ik}(a), [\widetilde{H}_{ik}(1,1), \widetilde{F}_{kj}(b)]\big] \\
&= [\widetilde{F}_{ik}(a + (-1)^{|i|+|k|}a), \widetilde{F}_{kj}(b)] + [\widetilde{F}_{ik}(a), -(-1)^{(|i|+|k|)(|i|+|k|)} \widetilde{F}_{kj}(b)] \\
&= [\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)].
\end{aligned}$$

Now we check that the remaining brackets vanish.

$$\begin{aligned}
[\widetilde{F}_{ij}(a), \widetilde{F}_{ij}(b)] &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{ik}(b), \widetilde{F}_{kj}(1)]\big] \\
&= \big[[\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)], \widetilde{F}_{kj}(1)\big] \\
&\quad + (-1)^{(|i|+|j|+|a|)(|i|+|k|+|b|)} \big[\widetilde{F}_{ik}(b), [\widetilde{F}_{ij}(a), \widetilde{F}_{kj}(1)]\big] = 0.
\end{aligned}$$

To see that $[\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)] = 0$ we can assume $|i| + |j| = \bar{1}$; then

$$\begin{aligned} 0 &= (-1)^{\bar{1}+|a|}\big[\widetilde{H}_{ij}(1,1), [\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)]\big] \\ &= (-1)^{\bar{1}+|a|}\big[[\widetilde{H}_{ij}(1,1), \widetilde{F}_{ij}(a)], \widetilde{F}_{ik}(b)\big] \\ &\quad + (-1)^{(\bar{1}+|a|)+(|i|+|j|)(|i|+|j|+|a|)}\big[\widetilde{F}_{ij}(a), [\widetilde{H}_{ij}(1,1), \widetilde{F}_{ik}(b)]\big] \\ &= (-1)^{\bar{1}+|a|}[\widetilde{F}_{ij}(a + (-1)^{\bar{1}}a), \widetilde{F}_{ik}(b)] + [\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)] = [\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)]. \end{aligned}$$

If $|i| + |j| = \bar{0}$, we have that $|i| + |k| = \bar{1}$ and the calculation is the same. Therefore, $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = 0$ if $j \neq k$ and $i \neq l$, satisfying relation (3) and completing the proof. □

Corollary 4.2 *The universal central extension of* $\mathfrak{sl}(2,1,A)$ *and* $\mathfrak{st}(2,1,A)$ *is* $\mathfrak{st}(2,1,A)$. *Moreover,* $\mathrm{H}_2\big(\mathfrak{st}(2,1,A)\big) = 0$.

5. Universal central extension of $\mathfrak{st}(3,1,A)$

In this section we find the universal central extension of $\mathfrak{sl}(3,1,A)$. Let S_4 be the symmetric group of degree 4, i.e. the set of all quadruples (i,j,k,l) where $1 \leq i,j,k,l \leq 4$ distinct. We quotient S_4 by Klein's subgroup, formed by $\{(1,2,3,4),(3,2,1,4),(1,4,3,2),(3,4,1,2)\}$, obtaining 6 cosets denoted by P_m. We have a map θ that sends $(i,j,k,l) \mapsto \theta\big((i,j,k,l)\big) = m$ when $(i,j,k,l) \in P_m$.

Let $\Pi(A_2)$ be the K-supermodule A_2 (see Definition 2.3) with the parity changed, i.e. $\big(\Pi(A_2)\big)_{\bar{0}} = (A_2)_{\bar{1}}$ and $\big(\Pi(A_2)\big)_{\bar{1}} = (A_2)_{\bar{0}}$. Let $\mathcal{W} = \Pi(A_2)^6$ be the K-supermodule formed by the direct sum of six copies of $\Pi(A_2)$ and consider the maps $\epsilon_m \colon \Pi(A_2) \to \mathcal{W}$, $\epsilon_m(\bar{a}) \mapsto (0,\dots,\bar{a},\dots,0)$, in the position m.

Using the decomposition of Lemma 2.2 we consider the K-bilinear map

$$\psi \colon \mathfrak{st}(3,1,A) \times \mathfrak{st}(3,1,A) \to \mathcal{W},$$

where

$$\psi\big(F_{ij}(a), F_{kl}(b)\big) = \epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}),$$
$$\psi(x,y) = 0 \text{ if } x \text{ or } y \text{ belongs to } \mathcal{H}.$$

Lemma 5.1 *The* K*-bilinear map* ψ *is a super 2-cocycle.*

Proof Since the grading in $\mathcal{W}$ is changed and exactly one index is odd, we have that

$$|\psi\big(F_{ij}(a), F_{kl}(b)\big)| = |i| + |j| + |a| + |k| + |l| + |b| = |a| + |b| + \bar{1} = |\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})|,$$

for homogeneous $a, b \in A$ and so ψ is even.

To complete the proof we can just follow the steps of [6, Lemma 2.2] since $\bar{a} = -\bar{a}$ and signs do not play any important role. □

By the previous lemma, we have a central extension

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathfrak{st}(3,1,A)^\sharp \xrightarrow{\pi} \mathfrak{st}(3,1,A) \longrightarrow 0,$$

where $\mathfrak{st}(3,1,A)^\sharp = \mathfrak{st}(3,1,A) \oplus \mathcal{W}$ is the Lie superalgebra constructed by the surjective super 2-cocycle ψ, defined by the following relations

$$a \mapsto F^\sharp_{ij}(a) \text{ is a } K\text{-linear map,} \tag{12}$$

$$[\mathcal{W},\mathcal{W}] = [F^\sharp_{ij}(a),\mathcal{W}] = 0, \tag{13}$$

$$[F^\sharp_{ij}(a), F^\sharp_{jk}(b)] = F^\sharp_{ik}(ab) \text{ for distinct i, j, k,} \tag{14}$$

$$[F^\sharp_{ij}(a), F^\sharp_{ij}(a)] = 0, \tag{15}$$

$$[F^\sharp_{ij}(a), F^\sharp_{ik}(b)] = 0, \tag{16}$$

$$[F^\sharp_{ij}(a), F^\sharp_{kl}(b)] = \epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}) \text{ for distinct } i,j,k,l. \tag{17}$$

Theorem 5.2 *The central extension* $0 \to \mathcal{W} \to \mathfrak{st}(3,1,A)^\sharp \to \mathfrak{st}(3,1,A)$ *is universal.*

Proof Let

$$0 \longrightarrow \mathcal{V} \longrightarrow \widetilde{\mathfrak{st}}(3,1,A) \xrightarrow{\tau} \mathfrak{st}(3,1,A) \longrightarrow 0$$

be a central extension. We need to show that there exists a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(3,1,A)^\sharp \to \widetilde{\mathfrak{st}}(3,1,A)$ such that $\tau \circ \rho = \pi$.

We choose a preimage $\widetilde{F}_{ij}(a)$ of $F_{ij}(a)$ K-linearly for all $a \in A$. Since $\mathcal{V} \subset \mathsf{Z}(\widetilde{\mathfrak{st}}(3,1,A))$, we have that

$$[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)] = \widetilde{F}_{ij}(ab) + v_{ijk}(a,b),$$

for distinct i,j,k, where $v_{ijk}(a,b) \in \mathcal{V}$. Using Jacobi identity we have

$$\begin{aligned}
[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(cb)] &= \big[\widetilde{F}_{ik}(a), [\widetilde{F}_{kl}(c), \widetilde{F}_{lj}(b)]\big] \\
&= \big[[\widetilde{F}_{ik}(a), \widetilde{F}_{kl}(c)], \widetilde{F}_{lj}(b)\big] \\
&\quad + (-1)^{(|i|+|k|+|a|)(|k|+|l|+|c|)}\big[\widetilde{F}_{kl}(c), [\widetilde{F}_{ik}(a), \widetilde{F}_{lj}(b)]\big] \\
&= [\widetilde{F}_{il}(ac), \widetilde{F}_{lj}(b)],
\end{aligned}$$

and so choosing $c = 1$ we have the identities $v_{ijk}(a,b) = v_{ijl}(a,b)$ and $[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)] = [\widetilde{F}_{il}(a), \widetilde{F}_{lj}(b)]$. This means that $v_{ijk}(a,b)$ is independent of the choice of k and so we have

$$[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(b)] = \widetilde{F}_{ij}(ab) + v_{ij}(a,b),$$

and

$$[\widetilde{F}_{ik}(1), \widetilde{F}_{kj}(b)] = \widetilde{F}_{ij}(b) + v_{ij}(1,b).$$

Therefore, we can replace $\widetilde{F}_{ij}(b)$ by $\widetilde{F}_{ij}(b) + v_{ij}(1,b)$. We want to define $\rho\big(F^\sharp_{ij}(a)\big) = \widetilde{F}_{ij}(a)$ and so will see that these elements satisfy relations (12)–(17).

Relations (12), (13), and (14) are straightforward by definition. To see relation (15), we choose i, j, k distinct

$$\begin{aligned}[\widetilde{F}_{ij}(a), \widetilde{F}_{ij}(b)] &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{ik}(b), \widetilde{F}_{kj}(1)]\big] \\ &= \big[[\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)], \widetilde{F}_{kj}(1)\big] \\ &\quad + (-1)^{(|i|+|j|+|a|)(|i|+|k|+|b|)}\big[\widetilde{F}_{ik}(b), [\widetilde{F}_{ij}(a), \widetilde{F}_{kj}(1)]\big] \\ &= 0.\end{aligned}$$

For relation (16), taking i, j, k, l distinct, we have

$$\begin{aligned}[\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(b)] &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{il}(b), \widetilde{F}_{ik}(1)]\big] \\ &= \big[[\widetilde{F}_{ij}(a), \widetilde{F}_{il}(b)], \widetilde{F}_{ik}(1)\big] \\ &\quad + (-1)^{(|i|+|j|+|a|)(|i|+|l|+|b|)}\big[\widetilde{F}_{il}(b), [\widetilde{F}_{ij}(a), \widetilde{F}_{ik}(1)]\big] \\ &= 0.\end{aligned}$$

To check relation (17) we define $\widetilde{H}_{ij}(a,b) = [\widetilde{F}_{ij}(a), \widetilde{F}_{ji}(b)]$ and following the steps of Lemma 2.1 we can check that for distinct i, j, k, l,

$$\begin{aligned}\widetilde{H}_{ij}(a,b) &= -(-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)}\widetilde{H}_{ji}(b,a), \\ [\widetilde{H}_{ij}(a,b), \widetilde{F}_{ik}(c)] &= \widetilde{F}_{ik}(abc), \\ [\widetilde{H}_{ij}(a,b), \widetilde{F}_{ki}(c)] &= -(-1)^{(|a|+|b|)(|i|+|k|+|c|)}\widetilde{F}_{ki}(cab), \\ [\widetilde{H}_{ij}(a,b), \widetilde{F}_{kj}(c)] &= (-1)^{(|i|+|j|+|a|)(|i|+|j|+|b|)+(|a|+|b|)(|j|+|k|+|c|)}\widetilde{F}_{kj}(cba), \\ [\widetilde{H}_{ij}(a,b), \widetilde{F}_{ij}(c)] &= \widetilde{F}_{ij}(abc + (-1)^{(|i|+|j|+|a||b|+|b||c|+|c||a|)}cba), \\ [\widetilde{H}_{ij}(a,b), \widetilde{F}_{kl}(c)] &= 0.\end{aligned}$$

When i, j, k, l are distinct we denote

$$[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = v_{ijkl}(a),$$

where $v_{ijkl}(a) \in \mathcal{V}$. We want that $\rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = v_{ijkl}(ab)$, since

$$\begin{aligned}[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= [\rho\big(F^{\sharp}_{ij}(a)\big), \rho\big(F^{\sharp}_{kl}(b)\big)] \\ &= \rho\big([F^{\sharp}_{ij}(a), F^{\sharp}_{kl}(b)]\big) = \rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = v_{ijkl}(ab).\end{aligned}$$

Thus, we have to check that

(R1) $2v_{ijkl}(a) = 0$,

(R2) $v_{ijkl}(a) = v_{kjil}(a) = v_{ilkj}(a) = v_{klij}(a)$, (R3) $v_{ijkl}(a[b,c]) = 0$,

(R4) $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = v_{ijkl}(ab)$.

Assume $|i| + |j| = \bar{0}$,

$$\begin{aligned} 0 &= \big[\widetilde{H}_{ij}(a,b), [\widetilde{F}_{ij}(c), \widetilde{F}_{kl}(1)]\big] \\ &= \big[[\widetilde{H}_{ij}(a,b), \widetilde{F}_{ij}(c)], \widetilde{F}_{kl}(1)\big] - \big[\widetilde{F}_{ij}(a), [\widetilde{H}_{ij}(1,1), \widetilde{F}_{kl}(1)]\big] \\ &= [\widetilde{F}_{ij}(abc + (-1)^{|i|+|j|+|a||b|+|b||c|+|c||a|}cba), \widetilde{F}_{kl}(1)] \\ &= v_{ijkl}(abc + (-1)^{|a||b|+|b||c|+|c||a|}cba). \end{aligned}$$

If $b = c = 1$, we have that

$$v_{ijkl}(2a) = 2v_{ijkl}(a) = 0,$$

proving (R1).

If $c = 1$, we have that

$$v_{ijkl}(ab - (-1)^{|a||b|}ba) = 0,$$

and so

$$0 = v_{ijkl}(abc + (-1)^{|a||b|+|b||c|+|c||a|}cba) = v_{ijkl}\big((ab + (-1)^{|a||b|}ba)c\big),$$

implying (R2). If $|k| + |l| = \bar{0}$, the calculation is the same.

On the other hand,

$$\begin{aligned} [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= \big[[\widetilde{F}_{ik}(a), \widetilde{F}_{kj}(1)], \widetilde{F}_{kl}(b)\big] \\ &= \big[\widetilde{F}_{ik}(a), [\widetilde{F}_{kj}(1), \widetilde{F}_{kl}(b)]\big] \\ &\quad - (-1)^{(|i|+|k|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), [\widetilde{F}_{ik}(a), \widetilde{F}_{kl}(b)]\big] \\ &= (-1)^{(|l|+|k|+|b|)(|k|+|j|)}[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] \\ &= v_{ilkj}(ab), \end{aligned}$$

since the sign does not play any role. Choosing $b = 1$ and using (R1), we have that

$$v_{ijkl}(a) = v_{ilkj}(a).$$

Doing the same but changing the indexes we have relations (R3) and (R4).

Thus, the morphism $\rho\colon \mathfrak{st}(3,1,A)^\sharp \to \widetilde{\mathfrak{st}}(3,1,A)$ defined by

$$\rho\big(F^\sharp_{ij}(a)\big) = \widetilde{F}_{ij}(a) \qquad \text{and} \qquad \rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = v_{ijkl}(ab)$$

is actually a Lie superalgebra homomorphism completing the proof. □

Corollary 5.3 *The universal central extension of* $\mathfrak{sl}(3,1,A)$ *is* $\mathfrak{st}(3,1,A)^\sharp \cong \mathfrak{sl}(3,1,A) \oplus \Pi(A_2)^6$. *Moreover,* $\mathrm{H}_2\left(\mathfrak{st}(3,1,A)\right) \cong \mathcal{W} \cong \Pi(A_2)^6$.

6. Universal central extension of $\mathfrak{st}(2,2,A)$

In this section we find the universal central extension of $\mathfrak{st}(2,2,A)$. As in the previous section we consider the partition of S_4 but with a small difference. Not all the cosets will be considered as equals. The coset formed by

$$\{(1,3,2,4),(1,4,2,3),(2,3,1,4),(2,4,1,3)\},$$

is named P_5 and the one formed by

$$\{(3,1,4,2),(3,2,4,1),(4,1,3,2),(4,2,3,1)\},$$

is named P_6. The order of the other cosets $P_1,\dots,P_4$, will not be relevant. Note that all the elements of P_5 and P_6 have the property that $|i|=|k|,|j|=|l|$ and $|i|+|j|=|k|+|l|=\bar{1}$.

Let $\sigma\colon S_4\to\{-1,1\}$ be a map defined by

$$\sigma\big((i,j,k,l)\big)=1 \text{ if } (i,j,k,l)\in P_1,P_2,P_3 \text{ or } P_4,$$

in P_5,

$$\begin{aligned}
\sigma\big((i,j,k,l)\big)&=1 && \text{if } (i,j,k,l)=(1,3,2,4) \text{ or } (2,4,1,3),\\
\sigma\big((i,j,k,l)\big)&=-1 && \text{if } (i,j,k,l)=(1,4,2,3) \text{ or } (2,3,1,4),
\end{aligned}$$

and in P_6,

$$\begin{aligned}
\sigma\big((i,j,k,l)\big)&=1 && \text{if } (i,j,k,l)=(3,1,4,2) \text{ or } (4,2,3,1),\\
\sigma\big((i,j,k,l)\big)&=-1 && \text{if } (i,j,k,l)=(3,2,4,1) \text{ or } (4,1,3,2).
\end{aligned}$$

Furthermore, let $\mathcal{W}=A_2^4\oplus A_0^2$ be K-supermodule formed by the direct sum of four copies of A_2 and two copies of A_0 and the maps $\epsilon_m(\bar{a})=(0,\dots,\bar{a},\dots,0)$ in position m.

Using the decomposition of Lemma 2.2 we consider the K-bilinear map

$$\psi\colon \mathfrak{st}(2,2,A)\times\mathfrak{st}(2,2,A)\to\mathcal{W},$$

where

$$\begin{aligned}
\psi\big(F_{ij}(a),F_{kl}(b)\big)&=\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}), && \text{if } (i,j,k,l)\in P_1,P_2,P_3,P_4\\
\psi\big(F_{ij}(a),F_{kl}(b)\big)&=(-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}), && \text{if } (i,j,k,l)\in P_5 \text{ or } P_6,\\
\psi(x,y)&=0 && \text{if } x \text{ or } y \text{ belongs to } \mathcal{H}.
\end{aligned}$$

Lemma 6.1 *The K-bilinear map ψ is a super 2-cocycle.*

Proof The map is even since $|i|+|j|+|k|+|l|=\bar{0}$. To check antisymmetry, it suffices to see what happens when $(i,j,k,l)\in P_5$ or P_6 since in the other cases the signs do not make any difference since A_2 and A_0 are commutative. Let $(i,j,k,l)\in P_5$, and we know that $|i|+|j|=|k|+|l|=\bar{1}$,

$$
\begin{aligned}
-(-1)^{|F_{ij}(a)||F_{kl}(b)|}\psi\big(F_{kl}(b),F_{ij}(a)\big) &= -(-1)^{(|i|+|j|+|a|)(|k|+|l|+|b|)}\psi\big(F_{kl}(b),F_{ij}(a)\big)\\
&= -(-1)^{(\bar{1}+|a|)(\bar{1}+|b|)}(-1)^{|a|}\sigma\big((k,l,i,j)\big)\epsilon_5(\overline{ba})\\
&= (-1)^{|b|+|a||b|}\sigma\big((k,l,i,j)\big)\epsilon_5((-1)^{|a||b|}\overline{ab})\\
&= (-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})\\
&= \psi\big(F_{ij}(a),F_{kl}(b)\big),
\end{aligned}
$$

since $\sigma\big((i,j,k,l)\big)=\sigma\big((k,l,i,j)\big)$ and $\overline{ab}=(-1)^{|a||b|}\overline{ba}$. If (i,j,k,l) belongs to P_6 it is analogue.

The identity $\psi(x_{\bar{0}},x_{\bar{0}})=0$ where $x_{\bar{0}}\in\big(\mathfrak{st}(2,2,A)\big)_{\bar{0}}$ is straightforward by definition. The last step is to check Jacobi identity. In order to ease notation, we denote by $J(x,y,z)$ the expression

$$
(-1)^{|x||z|}\psi([x,y],z)+(-1)^{|x||y|}\psi([y,z],x)+(-1)^{|y||z|}\psi([z,x],y).
$$

We have to check that $J(x,y,z)=0$ for all $x,y,z\in\mathfrak{st}(2,2,A)$.

Let $\psi([x,y],z)\neq 0$. Using the decomposition of Lemma 2.2 we see that at most one of x,y belongs to $\mathcal{H}$. We can assume that $x\in\mathcal{H}$. To exclude trivial cases we need that $y=F_{ij}(a)$ and $z=F_{kl}(b)$, where i,j,k,l are distinct. If $(i,j,k,l)\in P_1,\dots,P_4$, the signs do not make any difference and so the proof is the same as in [6, Lemma 2.2]. Therefore, we just need to check when $(i,j,k,l)=(1,3,2,4)\in P_5$ since the other cases are similar.

If $x=h(c,d)$, then

$$
\begin{aligned}
J(x,y,z) &= (-1)^{(|c|+|d|)(|b|+\bar{1})}\psi\big([h(c,d),F_{13}(a)],F_{24}(b)\big)\\
&\quad+(-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24},h(c,d)],F_{13}(b)\big)\\
&= (-1)^{(|c|+|d|)(|b|+\bar{1})}\psi\big(F_{13}\big((ab-(-1)^{|a||b|}ba)c\big),F_{24}(b)\big)+0\\
&= (-1)^{(|c|+|d|)(|b|+\bar{1})+|b|}\sigma\big((1,3,2,4)\big)\epsilon_5\big(\overline{(ab-(-1)^{|a||b|}ba)cb}\big)\\
&= 0.
\end{aligned}
$$

If $x = H_{12}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|a|+\bar{1})}\psi\big([H_{12}(1,c),F_{13}(a)],F_{24}(b)\big)\\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24}(b),H_{12}(1,c)],F_{13}(a)\big)\\
&= (-1)^{|c|(|a|+\bar{1})}\psi\big(F_{13}(ca),F_{24}(b)\big)\\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|(|b|+\bar{1})}\psi\big(F_{24}(cb),F_{13}(a)\big)\\
&= (-1)^{|c|(|b|+\bar{1})+|a|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{cab})\\
&\quad + (-1)^{(|a|+|c|+\bar{1})(|b|+\bar{1})+|b|}\sigma\big((2,4,1,3)\big)\epsilon_5(\overline{cba})\\
&= (-1)^{|c|(|b|+\bar{1})}\big((-1)^{|a|}\epsilon_5(\overline{cab}) + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|b|}\epsilon_5(\overline{cba})\big)\\
&= (-1)^{|c|(|b|+\bar{1})+|a|}\big(\epsilon_5(\overline{cab-(-1)^{|a||b|}cba})\big)\\
&= 0.
\end{aligned}$$

If $x = H_{13}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|a|+\bar{1})}\psi\big([H_{13}(1,c),F_{13}(a)],F_{24}(b)\big)\\
&= \psi\big(F_{13}(ca+(-1)^{\bar{1}+|a||c|}ac),F_{24}(b)\big)\\
&= (-1)^{|b|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{(ca-(-1)^{|a||c|}ac)b})\\
&= 0.
\end{aligned}$$

If $x = H_{14}(1,c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|c|(|b|+\bar{1})}\psi\big([H_{14}(1,c),F_{13}(a)],F_{24}(b)\big)\\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})}\psi\big([F_{24}(b),H_{14}(1,c)],F_{13}(a)\big)\\
&= (-1)^{|c|(|b|+\bar{1})}\psi\big(F_{13}(ca),F_{24}(b)\big)\\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|}\psi\big(F_{24}(bc),F_{13}(a)\big)\\
&= (-1)^{|c|(|b|+\bar{1})+|b|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{cab})\\
&\quad + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+|c|+|a|}\sigma\big((2,4,1,3)\big)\epsilon_5(\overline{bca})\\
&= (-1)^{|b|+|c|}\big((-1)^{|c||b|}\epsilon_5(\overline{cab}) - (-1)^{|a||b|}\epsilon_5(\overline{bca})\big)\\
&= (-1)^{|b|+|c|}\big((-1)^{|c||b|+|b||c|+|a||b|}\epsilon_5(\overline{bca}) - (-1)^{|a||b|}\epsilon_5(\overline{bca})\big)\\
&= 0.
\end{aligned}$$

Assume now that neither $x,y,z \in \mathcal{H}$. If $\psi([x,y],z) \neq 0$ we must have $\psi\big([F_{ik}(a),F_{kj}(b)],F_{kl}(c)\big)$ or $\psi\big([F_{il}(a),F_{lj}(b)],F_{kl}(c)\big)$. Again, if $(i,j,k,l) \in P_1,\dots,P_4$, the sign does not matter and so the proof is the same as in [6]. Assume that $(i,j,k,l) = (1,3,2,4) \in P_5$.

If $x = F_{12}(a)$, $y = F_{23}(b)$, and $z = F_{24}(c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{|a|(|c|+\bar{1})}\psi\big(F_{13}(ab), F_{24}(c)\big) \\
&\quad - (-1)^{(|b|+\bar{1})(|c|+\bar{1})+|a|(|c|+\bar{1})}\psi\big(F_{14}(ac), F_{23}(b)\big) \\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{abc}) \\
&\quad - (-1)^{(|a|+|b|+\bar{1})(|c|+\bar{1})+|b|}\sigma\big((1,4,2,3)\big)\epsilon_5(\overline{acb}) \\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\big(\epsilon_5(\overline{abc}) + (-1)^{|b||c|+\bar{1}}\epsilon_5(\overline{acb})\big) \\
&= (-1)^{|a|(|c|+\bar{1})+|c|}\epsilon_5\big(\overline{a(bc-(-1)^{|b||c|}cb)}\big) \\
&= 0.
\end{aligned}$$

If $x = F_{14}(a)$, $y = F_{43}(b)$, and $z = F_{24}(c)$, then

$$\begin{aligned}
J(x,y,z) &= (-1)^{(|a|+\bar{1})(|c|+\bar{1})}\psi\big(F_{13}(ab), F_{24}(c)\big) \\
&\quad - (-1)^{|b|(|a|+\bar{1})+|b|(|c|+\bar{1})}\psi\big(F_{23}(ac), F_{14}(b)\big) \\
&= (-1)^{(|a|+\bar{1})(|c|+\bar{1})+|c|}\sigma\big((1,3,2,4)\big)\epsilon_5(\overline{abc}) \\
&\quad - (-1)^{|b|(|a|+|c|)+|a|}\sigma\big((2,3,1,4)\big)\epsilon_5(\overline{cba}) \\
&= -(-1)^{|a|}\epsilon_5\big(\overline{(-1)^{|a||c|}abc - (-1)^{|a||b|+|b||c|}cba}\big) \\
&= -(-1)^{|a|}\epsilon_5\big(\overline{(-1)^{|a||c|}abc - (-1)^{|a||b|+|b||c|+|c|(|b|+|a|)}bac}\big) \\
&= -(-1)^{|a|+|a||c|}\epsilon_5\big(\overline{(ab-(-1)^{|a||b|})c}\big) \\
&= 0.
\end{aligned}$$

□

We have a central extension

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathfrak{st}(2,2,A)^\sharp \xrightarrow{\pi} \mathfrak{st}(2,2,A) \longrightarrow 0,$$

where $\mathfrak{st}(2,2,A)^\sharp = \mathfrak{st}(2,2,A) \oplus \mathcal{W}$ is the Lie superalgebra constructed by the surjective super 2-cocycle ψ, defined by the following relations

$$a \mapsto F^\sharp_{ij}(a) \text{ is a } K\text{-linear map,} \tag{18}$$

$$[\mathcal{W},\mathcal{W}] = [F^\sharp_{ij}(a),\mathcal{W}] = 0, \tag{19}$$

$$[F^\sharp_{ij}(a), F^\sharp_{jk}(b)] = F^\sharp_{ik}(ab) \text{ for distinct i, j, k,} \tag{20}$$

$$[F^\sharp_{ij}(a), F^\sharp_{ij}(a)] = 0, \tag{21}$$

$$[F^\sharp_{ij}(a), F^\sharp_{ik}(b)] = 0, \tag{22}$$

$$[F^\sharp_{ij}(a), F^\sharp_{kl}(b)] = \epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}) \text{ if } (i,j,k,l) \in P_1, P_2, P_3, P_4 \tag{23}$$

$$[F^\sharp_{ij}(a), F^\sharp_{kl}(b)] = (-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab}) \text{ if } (i,j,k,l) \in P_5, P_6. \tag{24}$$

Theorem 6.2 *The central extension* $0 \to \mathcal{W} \to \mathfrak{st}(2,2,A)^\sharp \to \mathfrak{st}(2,2,A) \to 0$ *is universal.*

Proof Let

$$0 \longrightarrow \mathcal{V} \longrightarrow \widetilde{\mathfrak{st}}(2,2,A) \xrightarrow{\ \tau\ } \mathfrak{st}(2,2,A) \longrightarrow 0$$

be a central extension. As done in Theorem 4.1, we need a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(2,2,A)^\sharp \to \widetilde{\mathfrak{st}}(2,2,A)$ such that $\tau \circ \rho = \pi$. Choosing preimages $\widetilde{F}_{ij}(a)$ of τ, we have to check they satisfy relations (18)–(24). Doing the analogue computations as in Theorem 4.1 it is obvious that relations (18)–(22) are satisfied. We have to check that the $\widetilde{F}_{ij}(a)$ follow (23) and (24) to complete the proof.

As in the previous section, when i,j,k,l are distinct, denote

$$[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = v_{ijkl}(a).$$

To satisfy relations (23) and (24) we want to define the homomorphism from $\mathcal{W}$ by the expression $\rho(\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})) = \sigma\big((i,j,k,l)\big)v_{ijkl}(ab)$. If $(i,j,k,l) \in P_1, \dots, P_4$, we have to check the conditions

($\mathcal{R}1$) $2v_{ijkl}(a) = 0$,

($\mathcal{R}2$) $v_{ijkl}(a) = v_{kjil}(a) = v_{ilkj}(a) = v_{klij}(a)$,

($\mathcal{R}3$) $v_{ijkl}(a[b,c]) = 0$,

($\mathcal{R}4$) $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = v_{ijkl}(ab)$.

Note that every permutation in $P_1, \dots, P_4$, has an element such that $|i| + |j| = \bar{0}$. Thus, recovering some computations of the previous section we have that

$$\begin{aligned} 0 &= \big[\widetilde{H}_{ij}(1,1), [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)]\big] \\ &= [\widetilde{F}_{ij}(a + (-1)^{|i|+|j|}a), \widetilde{F}_{kl}(1)] \\ &= v_{ijkl}(a + (-1)^{|i|+|j|}a). \end{aligned}$$

If $|i| + |j| = \bar{0}$, we have that $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] = -[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)]$. Then

$$[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -(-1)^{(|k|+|l|)(|k|+|j|+|b|)}[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)],$$

and so $[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)]$. Changing the indexes we obtain ($\mathcal{R}1$) and ($\mathcal{R}2$), and proceeding as in the proof of Theorem 5.2, conditions ($\mathcal{R}3$) and ($\mathcal{R}4$) are satisfied.

If $(i,j,k,l) \in P_5, P_6$, we have that

$$\begin{aligned} [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= [\rho\big(F^\sharp_{ij}(a)\big), \rho\big(F^\sharp_{kl}(b)\big)] \\ &= \rho\big([F^\sharp_{ij}(a), F^\sharp_{kl}(b)]\big) \\ &= \rho\big((-1)^{|b|}\sigma\big((i,j,k,l)\big)\epsilon_{\theta\left((i,j,k,l)\right)}(\overline{ab})\big) \\ &= (-1)^{|b|}v_{ijkl}(ab) \\ &= (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)]. \end{aligned}$$

Thus, we have to check the following conditions:

(C1) $v_{ijkl}(a) = -v_{kjil}(a) = -v_{ilkj}(a) = v_{klij}(a)$,

(C2) $[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] = (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)]$,

(C3) $v_{ijkl}(a[b,c]) = 0$.

To see (C1),

$$\begin{aligned} v_{ijkl}(a) &= [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] \\ &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{ki}(1), \widetilde{F}_{il}(1)]\big] = \big[[\widetilde{F}_{ij}(a), \widetilde{F}_{ki}(1)], \widetilde{F}_{il}(1)\big] \\ &= -(-1)^{(|i|+|j|+|a|)(|k|+|i|)}[\widetilde{F}_{kj}(a), \widetilde{F}_{il}(1)] \\ &= -[\widetilde{F}_{kj}(a), \widetilde{F}_{il}(1)] = -v_{kjil}(a), \end{aligned}$$

and

$$\begin{aligned} v_{ijkl}(a) &= [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(1)] \\ &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{kj}(1), \widetilde{F}_{jl}(1)]\big] = \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), [\widetilde{F}_{ij}(a), \widetilde{F}_{jl}(1)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}[\widetilde{F}_{kj}(1), \widetilde{F}_{il}(a)] = \\ &= -(-1)^{(|k|+|j|)(|l|+|j|)}[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = \\ &= -[\widetilde{F}_{il}(a), \widetilde{F}_{kj}(1)] = -v_{ilkj}(a). \end{aligned}$$

To check (C2),

$$\begin{aligned} [\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(b)] &= \big[\widetilde{F}_{ij}(a), [\widetilde{F}_{kj}(1), \widetilde{F}_{jl}(b)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), [\widetilde{F}_{ij}(a), \widetilde{F}_{jl}(b)]\big] \\ &= (-1)^{(|i|+|j|+|a|)(|k|+|j|)}\big[\widetilde{F}_{kj}(1), \widetilde{F}_{il}(ab)\big] \\ &\quad - (-1)^{(|j|+|l|+|b|)(|k|+|j|)}[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] \\ &= -(-1)^{|b|}[\widetilde{F}_{il}(ab), \widetilde{F}_{kj}(1)] = (-1)^{|b|}[\widetilde{F}_{ij}(ab), \widetilde{F}_{kl}(1)] \\ &= (-1)^{|b|}v_{ijkl}(ab), \end{aligned}$$

by part (C1).

Using (C2) and the fact that $|k| + |l| = \bar{1}$,

$$\begin{aligned} v_{ijkl}(a[b,c]) &= [\widetilde{F}_{ij}(a[b,c]), \widetilde{F}_{kl}(1)] \\ &= (-1)^{|b|+|c|}\sigma\big((i,j,k,l)\big)[\widetilde{F}_{ij}(a), \widetilde{F}_{kl}(bc - (-1)^{|b||c|}cb)] \\ &= (-1)^{|b|+|c|}\sigma\big((i,j,k,l)\big)\big[\widetilde{F}_{ij}(a), [\widetilde{H}_{kl}(b,c), \widetilde{F}_{kl}(1)]\big] \\ &= 0, \end{aligned}$$

by Jacobi identity, we have that (C3) is satisfied.

Thus, we obtained a Lie superalgebra homomorphism $\rho\colon \mathfrak{st}(2,2,A)^\sharp \to \widetilde{\mathfrak{st}}(2,2,A)$ completing the proof. □

Corollary 6.3 *The universal central extension of* $\mathfrak{sl}(2,2,A)$ *is* $\mathfrak{st}(2,2,A)^\sharp \cong \mathfrak{sl}(2,2,A) \oplus A_2^4 \oplus A_0^2$. *Moreover,* $\mathrm{H}_2\left(\mathfrak{st}(2,2,A)\right) \cong \mathcal{W} \cong A_2^4 \oplus A_0^2$.

7. Nonabelian tensor product and cyclic homology

In this section, we will consider the associative superalgebra A free as a K-supermodule. This assumption is needed in the definition of cyclic homology via a complex.

Definition 7.1 ([10]) *Let* $\mathrm{C}_n(A) = A^{\otimes n}/I_n$ *be the chain complex for* $n \geq 0$ *where* I_n *is the submodule generated by the relations*

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n - (-1)^{n+|a_0|\sum_{i=0}^{n-1}|a_i|} a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

for $a_i \in A$ *homogeneous. The boundary maps* d_n *are defined on generators by*

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{i+1} \otimes \cdots a_n$$
$$+ (-1)^{n+|a_n|\sum_{i=0}^{n-1}|a_i|} a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

for $a_i \in A$ *homogeneous. The* cyclic homology $\mathrm{HC}_n(A)$ *of the associative superalgebra* A *is the homology of the chain complex* $\mathrm{C}_*(A)$.

In [7], the nonabelian tensor product of two Lie superalgebras acting on each other is introduced. For the sake of simplicity, we recover the definition in the particular case of a Lie superalgebra L acting on itself by the canonical action, also called nonabelian tensor square.

Definition 7.2 *Let* L *be a Lie superalgebra. The* nonabelian tensor square $L \widehat{\otimes} L$ *is the tensor product of supermodules* $L \otimes L$ *quotient by the submodule generated by the relations*

(i) $[x,y] \otimes z = x \otimes [y,z] - (-1)^{|x||y|} y \otimes [x,z]$,

(ii) $x \otimes [y,z] = (-1)^{|z|(|x|+|y|)}([z,x] \otimes y) - (-1)^{|x||y|}([y,x] \otimes z)$,

for all $x,y,z \in L$. *It has a Lie superalgebra structure with bracket*

$$[x \otimes y, z \otimes w] = [x,y] \otimes [z,w].$$

It is shown in [7] that if L is perfect, the homomorphism $\mathfrak{u}\colon L \widehat{\otimes} L \to L$, $x \otimes y \mapsto [x,y]$, is the universal central extension of L and $\mathsf{Ker}\,\mathfrak{u} = \mathrm{H}_2(L)$. Therefore, the universal central extension of $\mathfrak{sl}(m,n,A)$ is the same as the universal central extension of $\mathfrak{st}(m,n,A)$, which is $\mathfrak{st}(m,n,A) \widehat{\otimes} \mathfrak{st}(m,n,A)$. Additionally, we know that the universal central extension of $\mathfrak{st}(m,n,A)$ is just itself plus a K-supermodule, which will be denoted by $\mathcal{W}(m,n,A)$, possibly zero.

Theorem 7.3 *Let $m+n \geq 3$. Then there is an isomorphism of K-supermodules*

$$\mathrm{H}_2(\mathfrak{sl}(m,n,A)) \cong \mathrm{HC}_1(A) \oplus \mathcal{W}(m,n,A).$$

Proof We consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{HC}_1(A) \oplus \mathcal{W}(m,n,D) & \longrightarrow & \dfrac{A \otimes A}{\mathsf{Im}\, d_2} \oplus \mathcal{W}(m,n,A) & \xrightarrow{d_1} & [A,A] & \longrightarrow & 0 \\
 & & & & \mathrm{Str}_2 \uparrow \downarrow \mu & & \mathrm{Str}_1 \uparrow \downarrow E_{11}(-) & & \\
0 & \longrightarrow & \mathrm{H}_2\big(\mathfrak{sl}(m,n,A)\big) & \longrightarrow & \mathfrak{st}(m,n,A)\widehat{\otimes}\mathfrak{st}(m,n,A) & \xrightarrow{\omega} & \mathfrak{sl}(m,n,A) & \longrightarrow & 0,
\end{array}$$

where $\mu(a \otimes b) = F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{1j}(ba) \otimes F_{j1}(1)$, $\mu\big(v_{ijkl}(a)\big) = F_{ij}(a) \otimes F_{kl}(b)$ and

$$\mathrm{Str}_2\left(F_{ij}(a) \otimes F_{kl}(b)\right) = \begin{cases} a \otimes b, & \text{if } i=j \text{ and } k=l, \\ v_{ijkl}(ab), & \text{when it makes sense depending on } m,n, \\ 0, & \text{otherwise.} \end{cases}$$

It is a straightforward computation that $\mu \circ \mathrm{Str}_2$ and $E_{11}(-) \circ \mathrm{Str}_1$ are the identity maps and that the diagram is commutative. Then the restriction of Str_2 to the kernel of ω is also a split epimorphism, with μ restricted to the kernel of d_1 as section. Let us see that these restrictions are indeed isomorphisms. An element in the kernel of ω is a sum of elements of the form $F_{ij}(a) \otimes F_{ji}(b)$ plus the elements of $\mathcal{W}(m,n,D)$. Any element of $\mathsf{Ker}\,\omega$ can be written as an element of $\mathsf{Im}\,\mu$ plus $\sum_{i=2}^{m+n} F_{1i}(a_i) \otimes F_{i1}(1)$, since

$$F_{ij}(a) \otimes F_{ji}(b) = F_{i1}(a) \otimes F_{1i}(b) - (-1)^{(|F_{ij}(a)|)(|F_{ji}(b)|)} F_{j1}(ba) \otimes F_{1j}(1),$$

and

$$F_{1j}(a) \otimes F_{j1}(b) = F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{j1}(ba) \otimes F_{1j}(1) + (-1)^{|a||b|} F_{j1}(ba) \otimes F_{1j}(1).$$

Furthermore, if it is in the kernel of ω, all the a_i must be zero. Then the restriction of μ to the kernel of d_1 is surjective. □

8. Concluding remarks

Combining the main theorems presented here with the main theorems of [3, 7] we have a complete characterization of $\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right)$ and $\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right)$ for $m+n \geq 3$.

Theorem 8.1 *Let K be a unital commutative ring and A an associative unital K-superalgebra. Then*

$$\mathrm{H}_2\left(\mathfrak{st}(m,n,A)\right) = \begin{cases} 0 & \textit{for } m+n \geq 5 \textit{ or } m=2, n=1, \\ A_3^6 & \textit{for } m=3, n=0, \\ A_2^6 & \textit{for } m=4, n=0, \\ \Pi(A_2)^6 & \textit{for } m=3, n=1, \\ A_2^4 \oplus A_0^2 & \textit{for } m=2, n=2, \end{cases}$$

where A_m is the quotient of A by the ideal $mA + A[A,A]$ (Definition 2.3) and Π is the parity change functor.

Theorem 8.2 *Let K be a unital commutative ring and A an associative unital K-superalgebra with a K-basis containing the identity. Then*

$$\mathrm{H}_2\left(\mathfrak{sl}(m,n,A)\right) = \begin{cases} \mathrm{HC}_1(A) & \text{for } m+n \geq 5 \text{ or } m=2, n=1, \\ \mathrm{HC}_1(A) \oplus A_3^6 & \text{for } m=3, n=0, \\ \mathrm{HC}_1(A) \oplus A_2^6 & \text{for } m=4, n=0, \\ \mathrm{HC}_1(A) \oplus \Pi(A_2)^6 & \text{for } m=3, n=1, \\ \mathrm{HC}_1(A) \oplus A_2^4 \oplus A_0^2 & \text{for } m=2, n=2, \end{cases}$$

where A_m is the quotient of A by the ideal $mA + A[A,A]$ (Definition 2.3) and Π is the parity change functor.

Acknowledgments

The authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-79661-P, and by Xunta de Galicia, grant GRC2013-045 (European FEDER support included). The first author was also supported by an FPU scholarship, Ministerio de Educación, Cultura y Deporte (Spain) and a Fundación Barrié scholarship.

The authors would like to express their gratitude to the anonymous referee for his help in improving the presentation of this paper.

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