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Modules satisfying double chain condition on nonfinitely generated submodules have Krull dimension

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Abstract: We prove the result in the title. We study submodules N of a module M such that whenever $\frac{M}{N}$ satisfies the double infinite chain condition so does M. Moreover, we observe that an α-atomic module, where $\alpha \geq 2$ is an ordinal number, satisfies the previous chain if and only if it satisfies the descending chain condition on nonfinitely generated submodules.

Key words: Nonfinitely generated modules, Krull dimension, DICC-modules, n.f.g-DICC modules

1. Introduction

Lemonnier [21] introduced the concepts of deviation and codeviation of an arbitrary poset, which, in particular, when applied to the lattice of all submodules of a module M_R give the concepts of the Krull dimension (in the sense of Rentschler and Gabriel), see also [11, 12, 23], and the dual Krull dimension of M, respectively. The dual Krull dimension in [13–15, 17–20], is called Noetherian dimension and in [26] is again called Krull dimension and in [6] is called N-dimension. The name dual Krull dimension is also used by some authors; see [1–4]. The Noetherian dimension of an R-module M is denoted by $n\text{-}\dim M$ and by $k\text{-}\dim M$ we denote the Krull dimension of M. The double infinite chain condition was introduced by Contessa for modules over commutative rings (briefly $DICC$-modules); see [7–9]. Osofsky [24] extended the concept of $DICC$ to objects in $AB5$ category. She characterized $DICC$ objects in this category and obtained some noncommutative generalizations. Karamzadeh and Motamedi [15] undertook a systematic study of the concept of $\alpha - DICC$ modules. Later, Rahimpour [25] studied modules that satisfy the double infinite chain condition on finitely generated submodules, denoted by $f.g. - DICC$-modules. We extensively studied modules with the chain condition on nonfinitely generated submodules. In this article we study modules that satisfy the double infinite chain condition on nonfinitely generated submodules, briefly called $n.f.g - DICC$ modules. We show that if an R-module M satisfies the double infinite chain condition on nonfinitely generated submodules, then it has Krull dimension. We investigate that if N is of finite length submodule of M and $\frac{M}{N}$ is an $n.f.g. - DICC$ module, then so is M. If an R-module M has the Noetherian dimension and α is an ordinal number, then M is called α-atomic if $n\text{-}\dim M = \alpha$ and $n\text{-}\dim N < \alpha$ for all proper submodules N of M. An R-module M is called atomic if M is α-atomic for some ordinal α; see [17] (note, atomic modules are also called conotable, dual critical, and N-critical in some other articles; see for example [4, 22] and [6]). We also observe that an α-

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atomic R-module M is $n.f.g.-DICC$ if and only if M satisfies the descending chain condition on nonfinitely generated submodules, where $\alpha \geq 2$ is an ordinal number. Throughout this paper R will always denote an associative ring with a nonzero identity and M a unital R-module. The notation $N \subseteq M$ (resp. $N \subset M$) means that N is a submodule (resp. proper submodule) of M. The reader is referred to [5, 12, 15, 17], for definitions, concepts, and the necessary background not explicitly given here.

2. Preliminaries

In this section we recall some useful facts about modules with Krull dimension and modules with chain condition on nonfinitely generated submodules.

Let us begin with the following well-known and important result; see [21, Corollary 6] or [17, Proposition 1.1].

Proposition 2.1. *An R-module has Noetherian dimension if and only if it has Krull dimension.*

We should remind the reader that by a quotient finite dimensional module M we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.

Next, we recall the following well-known and important result due to Lemonnier; see [22, Theorem 2.4] and [1, Proposition 2.2].

Proposition 2.2. *The following are equivalent for any R-module M and any ordinal $\alpha \geq 0$.*

1. $n\text{-dim}\, M \leq \alpha$;
2. *M is quotient finite dimensional and for any $N \subset P \subseteq M$, there exists X with $N \subseteq X \subset P$ with* $n\text{-dim}\, \frac{P}{X} \leq \alpha$.

The proof of the next result is similar to the proof of its dual result in [10, Lemma 1.4] and hence it is omitted; see also [15] and [21].

Proposition 2.3. *If M is an R-module and for each submodule N of M, either N or $\frac{M}{N}$ has Krull dimension, then so does M.*

We recall that an R-module M is called α-critical, where α is an ordinal number, if $k\text{-dim}\, M = \alpha$ and $k\text{-dim}\, \frac{M}{N} < \alpha$ for all nonzero submodules N of M. An R-module M is called critical if M is α-critical for some ordinal number α.

Note the following well-known result from [12].

Proposition 2.4. *Let M be an R-module with Krull dimension; then it has a critical submodule.*

We recall that a module M is finitely embedded (briefly $f.e.$) if and only if the socle of M is finitely generated and essential in M. Moreover, by [27] a module M is Artinian if and only if every factor module of M is $f.e$.

Next, we recall the following definition from [16].

Definition. *Let M be an R-module. For each ordinal α, we define $S_\alpha = \sum_{i\in I} \oplus C_i$, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of M. S_α is called an α-critical socle of M. Now a critical* *socle of M is defined to be a submodule S of M with $S = \sum_{\alpha<\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \leq \lambda$. If for some ordinal α there is no α-critical submodule, then we put $S_\alpha = 0$. Clearly, the sum of any maximal independent family of critical submodules of M is a critical socle of M.*

Next, we recall the following two well-known and important results (see, [11], [12], and [16]).

Lemma 2.5. *If an R-module M has Krull dimension and $M = \sum_{i\in I} N_i$, then $k\text{-dim}\, M = \sup\{k\text{-dim}\, N_i\}_{i\in I}$.*

Corollary 2.6. *Let M be a quotient finite dimensional R-module. If $M = \sum_{i\in I} N_i$ such that each N_i has Krull dimension, then M has Krull dimension and $k\text{-dim}\, M = \sup\{k\text{-dim}\, N_i\}_{i\in I}$.*

In view of Corollary 2.6, we have the following result.

Corollary 2.7. *Let M be a quotient finite dimensional R-module. If α is an ordinal number, then the α-critical socle of M has Krull dimension. This implies that the critical socle of M has Krull dimension.*

We recall that an R-module M is called λ-finitely embedded (λ-f.e.) if λ is the least ordinal such that each critical submodule of M is α-critical for some $\alpha \leq \lambda$ and M contains a finitely generated essential critical socle (equivalently, M contains an essential critical socle with Krull dimension λ); see [16].

In view of Corollary 2.7, we have the following result.

Corollary 2.8. *Let M be a quotient finite dimensional R-module. If Q is a quotient module of M, then Q is λ-f.e., for some ordinal number λ, if and only if the critical socle of Q is an essential submodule of Q.*

We cite the following result from [16].

Proposition 2.9. *Let M be an R-module; then $k\text{-dim}\, M = \alpha$ if and only if α is the least ordinal such that each factor module of M is λ-f.e. for some $\lambda \leq \alpha$.*

Proof See [16, Proposition 2.20].

In view of Corollary 2.8 and Proposition 2.9, we have the following result.

Corollary 2.10. *Let M be a quotient finite dimensional R-module; then M has Krull dimension if and only if each factor module of M has an essential critical socle.*

Now we have the following results. The proofs are just a minor variant of the familiar argument about Noetherian modules and have been omitted.

Lemma 2.11. *Let an R-module M satisfy the ascending chain condition on nonfinitely generated submodules. If N is a submodule of M, then so is N.*

Lemma 2.12. *Let an R-module M satisfy the ascending chain condition on nonfinitely generated submodules. Let N be any submodule of M; then $\frac{M}{N}$ satisfies the ascending chain condition on nonfinitely generated submodules.*

We continue with the following lemma, whose proof is given for the sake of completeness.

Lemma 2.13. *Let M be an R-module. If M satisfies the ascending chain condition on nonfinitely generated submodules, then M has finite Goldie dimension.*

Proof Let $\oplus_{i=1}^{\infty} M_i \subseteq M$. Put $N_0 = \oplus_{i=2k} M_i$, where $k \geq 1$ is an integer number. Now put $N_1 = M_1 \oplus M_2 \oplus (\oplus_{i=2k} M_i)$, where $k \geq 2$ is an integer number. Furthermore, put $N_2 = M_1 \oplus M_2 \oplus M_3 \oplus M_4 \oplus (\oplus_{i=2k} M_i)$, where $k \geq 3$ is an integer number. Similarly we put $N_n = M_1 \oplus M_2 \oplus ... \oplus M_{2n-1} \oplus M_{2n} \oplus (\oplus_{i=2k} M_i)$, where $k \geq n+1$ is an integer number. Therefore we have the following chain

$$N_0 \subset N_1 \subset N_2 \subset ...$$

of nonfinitely generated submodules of M, which is a contradiction.

The following corollary is now immediate.

Corollary 2.14. *Let M be an R-module. If M satisfies the ascending chain condition on nonfinitely generated submodules, then M is quotient finite dimensional.*

Proof We infer that each factor module of M satisfies the ascending chain condition on nonfinitely generated submodules, by Lemma 2.12. Thus by Corollary 2.13, each factor module of M has finite Goldie dimension.

Now we have the following results. The proofs are just a minor variant of the familiar argument about Artinian modules and have been omitted.

Lemma 2.15. *Let an R-module M satisfy the descending chain condition on nonfinitely generated submodules. If N is a submodule of M, then so is N.*

Lemma 2.16. *Let an R-module M satisfy the descending chain condition on nonfinitely generated submodules. If N is a submodule of M, then $\frac{M}{N}$ satisfies the descending chain condition on nonfinitely generated submodules.*

Next we prove an analogue of Lemma 2.13, for modules M satisfying the descending chain condition on nonfinitely generated submodules.

Lemma 2.17. *Let M be an R-module. If M satisfies the descending chain condition on nonfinitely generated submodules, then M has finite Goldie dimension.*

Proof Let $\oplus_{i=1}^{\infty} M_i \subseteq M$. Put $N_0 = \oplus_{i=0}^{\infty} M_i$ and $N_1 = \oplus_{i=1}^{\infty} M_i$ and $N_n = \oplus_{i=n}^{\infty} M_i$. Then the following chain

$$N_0 \supset N_1 \supset N_2 \supset ...$$

of nonfinitely generated submodules of M shows that M does not satisfy the descending chain condition on nonfinitely generated submodules, which is absurd.

In view of the previous corollary we have the following proposition.

Corollary 2.18. *Let M be an R-module. If M satisfies the descending chain condition on nonfinitely generated submodules, then M is quotient finite dimensional.*

Proof We infer that each factor module of M satisfies the descending chain condition on nonfinitely generated submodules, by Lemma 2.16. Thus by Lemma 2.17, each factor module of M has finite Goldie dimension.

We need the following results too.

Proposition 2.19. *Let M be an R-module. If M satisfies the ascending chain condition on nonfinitely generated submodules, then n-dim $M \leq 1$.*

Proof Let P be a submodule of M. By Lemma 2.11 and Corollary 2.14, P is a quotient finite dimensional R-module. Now let N be any proper submodule of P and put $\frac{P}{N} = Q$. By Lemmas 2.11 and 2.12, Q satisfies the ascending chain condition on nonfinitely generated submodules. In view of Proposition 2.2, it is sufficient to show that Q has a nonzero factor module with Noetherian dimension less than or equal to 1. If each proper submodule of Q is finitely generated, then each proper submodule of Q is Noetherian. Therefore $n\text{-dim}\, Q \leq 1$ and we are through; see [17, Proposition 1.4] . Now let Q have a proper nonfinitely generated submodule, N_1 say. Suppose that each proper submodule of $\frac{Q}{N_1}$ is finitely generated; then by the argument we have just given $n\text{-dim}\, \frac{Q}{N_1} \leq 1$ and we are through. Otherwise $\frac{Q}{N_1}$ has a proper nonfinitely generated submodule, $\frac{N_2}{N_1}$ say. By continuing this method for some integer number i, there exists a proper nonfinitely generated submodule N_i of Q, such that $n\text{-dim}\, \frac{Q}{N_i} \leq 1$, or else

$$N_1 \subset N_2 \subset ...$$

is a chain of nonfinitely generated submodules of Q, which is a contradiction.

Proposition 2.20. *Let an R-module M satisfy the descending chain condition on nonfinitely generated submodules; then it has Krull dimension.*

Proof We infer that M is quotient finite dimensional; see Corollary 2.18. Let Q be any nonzero factor module of M and X be a nonzero submodule of Q. By Lemmas 2.15 and 2.16, X satisfies the descending chain condition on nonfinitely generated submodules. According to Corollary 2.10 and Proposition 2.4, it suffices to show that X has a nonzero submodule with Krull dimension. If each proper submodule of X is finitely generated, then each proper submodule of X is Noetherian. Therefore $n\text{-dim}\, X \leq \sup\{n\text{-dim}\, N : N \subset X\}+1 \leq 1$; see [17, Proposition 1.4]. It follows that X is Noetherian or 1-atomic. Thus, by Proposition 2.1, X has Krull dimension and we are through. Now let X have a proper nonfinitely generated submodule, X_1 say. If each proper submodule of X_1 is finitely generated, then by what we have already shown X_1 is Noetherian or 1-atomic, and thus it has Krull dimension and we are through. Otherwise X_1 has a proper nonfinitely generated submodule, X_2 say. By continuing this method X has a nonzero submodule with Krull dimension; otherwise there exists the following chain

$$X_0 \supset X_1 \supset X_2 \supset X_3 \supset ...$$

of nonfinitely generated submodules of Q, which is a contradiction.

Recall that the converse of the previous proposition is not true in general. Let $\mathbb{Z}$ be the ring of integers. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ has Noetherian dimension and $n\text{-dim}\, \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty} = 1$, but the following chain

$$< \frac{1}{p} + \mathbb{Z} > \oplus \mathbb{Z}_{p^\infty} \subset < \frac{1}{p^2} + \mathbb{Z} > \oplus \mathbb{Z}_{p^\infty} \subset ...$$

of nonfinitely generated submodules of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ shows that it does not satisfy the ascending chain condition on nonfinitely generated submodules.

3. Double chain condition on nonfinitely generated submodules

In this section we study modules that satisfy the double infinite chain condition on nonfinitely generated submodules, briefly called $n.f.g - DICC$ modules. Next, we give our definition of $n.f.g. - DICC$-modules.

Definition. *An R-module M is said to be $n.f.g. - DICC$, if given any doubly infinite chain*

$$... \subset M_{-2} \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset ...$$

of nonfinitely generated submodules of M, there exists an integer k, such that $M_i = M_{i+1}$ for each $i \geq k$ or $M_i = M_{i+1}$ for each $i \leq k$.

We continue with the following lemma, whose proof is given for the sake of completeness.

Lemma 3.1. *If M is an n.f.g.-DICC module, then given any infinite descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq ...$ of nonfinitely generated submodules of M either $\frac{N_{i+1}}{N_i}$ satisfies the ascending chain condition on nonfinitely generated submodules for all i or there exists an integer k such that $N_{i+1} = N_i$ for each $i \geq k$.*

Proof Let $\frac{N_r}{N_{r+1}}$ not satisfy the ascending chain condition on nonfinitely generated submodules, for some r. Thus there exists an infinite chain $\frac{N'_1}{N_{r+1}} \subset \frac{N'_2}{N_{r+1}} \subset ....$ of nonfinitely generated submodules of $\frac{N_r}{N_{r+1}}$. Thus

$$... \subseteq N_{r+2} \subseteq N_{r+1} \subset N'_1 \subset N'_2 \subset ...$$

is a doubly infinite chain of nonfinitely generated submodules of M. It follows that there exists an integer $k > r$ such that $N_m = N_{m+1}$, for all $m \geq k$.

The proof of the next lemma is similar to the proof of Lemma 3.1, and it is therefore omitted.

Lemma 3.2. *If M is an n.f.g.-DICC module, then given any infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ of nonfinitely generated submodules of M either $\frac{M_{i+1}}{M_i}$ satisfies the descending chain condition on nonfinitely generated submodules for all i or there exists an integer k such that $M_i = M_{i+1}$ for each $i \geq k$.*

The proof of the next lemma is elementary and is omitted.

Lemma 3.3. *An R-module M is an $n.f.g. - DICC$-module if and only if for any nonfinitely generated submodule A of M either A satisfies the descending chain condition on nonfinitely generated submodules or $\frac{M}{A}$ satisfies the ascending chain condition on nonfinitely generated submodules.*

In view of the previous lemma we have the following proposition.

Proposition 3.4. *If M is an $n.f.g. - DICC$ module, then M has Krull dimension.*

Proof It suffices to show that M is satisfied in Proposition 2.3. Let X be a nonfinitely generated submodule of M. By Lemma 3.3, either X satisfies the descending chain condition on nonfinitely generated submodules or $\frac{M}{X}$ satisfies the ascending chain condition on nonfinitely generated submodules. Hence by Propositions 2.20, 2.19, and 2.1, either X or $\frac{M}{X}$ has Krull dimension. Let N be any proper finitely generated submodule of M; then either N is Noetherian or N has a proper nonfinitely generated submodule, X say. By the argument we have just given either X or $\frac{M}{X}$ has Krull dimension. If $\frac{M}{X}$ has Krull dimension, then $\frac{M}{N}$ has Krull dimension; see [12, Lemma 1.1], (note, $\frac{M/X}{N/X} = \frac{M}{N}$). This implies that for each proper finitely generated submodule N of M and any nonfinitely generated submodule X of N either X or $\frac{M}{N}$ has Krull dimension. We claim that for each proper finitely generated submodule N of M either N or $\frac{M}{N}$ has Krull dimension. Suppose that there exists a proper finitely generated submodule N' of M such that $\frac{M}{N'}$ does not have Krull dimension. We are to show that N' has Krull dimension. If each proper submodule of N' is finitely generated, then N' is Noetherian and we are through; see Proposition 2.1. Otherwise N' has a proper nonfinitely generated submodule, X' say. However, we have already shown that if $X \subset N \subset M$, where N is finitely generated and X is nonfinitely generated, then either X or $\frac{M}{N}$ has Krull dimension; therefore X' has Krull dimension (note, by our assumption $\frac{M}{N'}$ does not have Krull dimension). This implies that any nonfinitely generated submodule X of N' has Krull dimension. Now let P be a finitely generated submodule of N'. If P is contained in a nonfinitely generated submodule X of N', then X and therefore P has Krull dimension; see [12, Lemma 1.1]. Otherwise $\frac{N'}{P}$ is Noetherian and by Proposition 2.1 it has Krull dimension. Thus for each submodule X of N', either X or $\frac{N'}{X}$ has Krull dimension; hence by Proposition 2.3 N' has Krull dimension. However, at the beginning of the proof we have shown that for each nonfinitely generated submodule X of M, either X or $\frac{M}{X}$ has Krull dimension. It follows that for each submodule P of M, either P or $\frac{M}{P}$ has Krull dimension and we are done.

The next example shows that the converse of the previous proposition is not true in general.

Example 1. *Let $\mathbb{Z}$ be the ring of integers; then the $\mathbb{Z}$-module $M = \mathbb{Z}\oplus\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_{p^\infty}$ has Krull dimension; see [12, Lemma 1.1]. The following chain*

$$\mathbb{Z}\oplus\mathbb{Z}_{p^\infty} \supset 2\mathbb{Z}\oplus\mathbb{Z}_{p^\infty} \supset 4\mathbb{Z}\oplus\mathbb{Z}_{p^\infty} \supset ...$$

of nonfinitely generated submodules of the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}_{p^\infty}$ shows that it does not satisfy the descending chain condition on nonfinitely generated submodules. The following chain

$$< \frac{1}{p} + \mathbb{Z} > \oplus\mathbb{Z}_{p^\infty} \subset < \frac{1}{p^2} + \mathbb{Z} > \oplus\mathbb{Z}_{p^\infty} \subset < \frac{1}{p^3} + \mathbb{Z} > \oplus\mathbb{Z}_{p^\infty} \subset ...$$

of nonfinitely generated submodules of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}\oplus\mathbb{Z}_{p^\infty}$ shows that it does not satisfy the ascending chain condition on nonfinitely generated submodules. Since $\frac{M}{\mathbb{Z}\oplus Z_{p^\infty}} \simeq \mathbb{Z}_{p^\infty}\oplus Z_{p^\infty}$, we infer that M is not an $n.f.g$-DICC module by Lemma 3.3.

It is clear that if an R-module M satisfies the ascending or the descending chain condition on nonfinitely generated submodules, then it is $n.f.g - DICC$. Also it is evident that any $DICC$ module is $n.f.g - DICC$, but the converse is not true in general. For example, it is clear that the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}_{p^\infty}$ satisfies the ascending chain condition on nonfinitely generated submodules, and thus it is an $n.f.g. - DICC$-module. Clearly $\mathbb{Z}\oplus\mathbb{Z}_{p^\infty}$ is not $DICC$; see the comment that follows [15, Definition 1.1].

The following result is clear and its proof omitted.

Corollary 3.5. *Let M be an R-module and N be a proper submodule of M. If M is an $n.f.g. - DICC$ module, then so are N and $\frac{M}{N}$.*

We recall that a composition series for a module M is a chain of submodules

$$0 = M_0 \subset M_1 \subset ... \subset M_n = M$$

such that each of the factors $\frac{M_i}{M_{i-1}}$ is a simple module. A module of finite length is any module that has a composition series. Moreover, it is well known that a module M has finite length if and only if M is both Noetherian and Artinian.

Note the following result. The proof is standard but we include it for completeness.

Corollary 3.6. *Let M be an R-module and let N be of finite length submodule of M. If $\frac{M}{N}$ is $n.f.g. - DICC$, then so is M.*

Proof Let $... \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ be a double infinite chain of nonfinitely generated submodules of M. Then $... \subseteq \frac{M_{-2}+N}{N} \subseteq \frac{M_{-1}+N}{N} \subseteq \frac{M_0+N}{N} \subseteq \frac{M_1+N}{N} \subseteq \frac{M_2+N}{N} \subseteq ...$ is a double infinite chain of nonfinitely generated submodules of $\frac{M}{N}$. Thus there exists an integer number i such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \geq i$, or there exists an integer number i_1 such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \leq i_1$. Without less of generality we consider that there exists an integer number i such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \geq i$. Thus we get $M_k + N = M_{k+1} + N$ for each $i \geq k$. However, it is clear that $... \subseteq M_{-2} \cap N \subseteq M_{-1} \cap N \subseteq M_0 \cap N \subseteq M_1 \cap N \subseteq ...$ is a double infinite chain of submodules of N. Since N has finite length, it follows that there exists an integer number i_2 such that $M_k \cap N = M_{k+1} \cap N$ for each $k \geq i_2$. Put $n = \max\{i_1, i_2\}$. Thus $M_k + N = M_n + N$ and $M_k \cap N = M_n \cap N$ for each $k \geq n$. For each $k \geq n$, we conclude that $M_k = M_k \cap (M_k + N) = M_k \cap (M_n + N) = M_n + (M_k \cap N) = M_n + (M_n \cap N) = M_n$ and we are through.

Finally, we investigate when atomic modules are $n.f.g.$-DICC modules.

Proposition 3.7. *Let $\alpha \geq 2$ be an ordinal number. An α-atomic R-module M is $n.f.g.-DICC$ if and only if M satisfies the descending chain condition on nonfinitely generated submodules.*

Proof The sufficiency is obvious. Conversely, since M is α-atomic, we infer that for each proper submodule N of M, n-dim $\frac{M}{N} = \alpha$. By Proposition 2.19, $\frac{M}{N}$ does not satisfy the ascending chain condition on nonfinitely generated submodules. Now by Lemma 3.3, N satisfies the descending chain condition on nonfinitely generated submodules and so does M.

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