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On generalized Kropina change of mth root Finsler metrics with special curvature properties

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Abstract: In the present paper, we consider generalized Kropina change of mth root Finsler metrics and prove that every generalized Kropina change of mth root Finsler metrics with isotropic Berwald curvature, isotropic mean Berwald curvature, relatively isotropic Landsberg curvature, and relatively isotropic mean Landsberg curvature reduces to the Berwald metric, weakly Berwald metric, Landsberg metric, and weakly Landsberg metric, respectively. We also show that every generalized Kropina change of mth root Finsler metrics with almost vanishing $\mathbf{H}$-curvature has vanishing $\mathbf{H}$-curvature.

Key words: Finsler space, Kropina metrics, mth root metrics, Berwald curvature, Landsberg curvature, $\mathbf{H}$-curvature

1. Introduction

The theory of mth root Finsler metrics was developed by Shimada [10] in 1979 as a generalization of the Riemannian metric, applied to ecology by Antonelli [2] and studied by several authors [3–6, 9, 18]. Tiwari and Kumar [16] studied the Randers change of a Finsler space with mth root metric. Tayebi et al. [11–13] studied the mth root Finsler metric with several non-Riemannian quantities of Berwald curvature, Landsberg curvature, $\mathbf{H}$-curvature, etc., and they established a necessary and sufficient condition to be projectively flat and locally dually flat for Kropina change of mth root metrics [15]. Xu and Li [17] studied a class of Finsler metrics called special generalized fourth root metrics and established a necessary and sufficient condition for this Finsler metrics to be projectively flat with constant flag curvature $\mathbf{K}=1$. Recently Tayebi and Shahbazi Nia [14] considered a new class of Finsler metrics, namely Kropina change of generalized mth root metrics, and classified such metrics, which are projectively flat and dually flat.

Let $(M,F)=F^n$ be an n-dimensional Finsler manifold. For a 1-form $\beta(x,y)=b_i(x)y^i$ on M, define a Finsler change as follows:

$$F(x,y)\to\overline{F}(x,y)=f(F,\beta),$$

where $f(F,\beta)$ is a positively homogeneous function of degree one in F and β. A Finsler change is called a Kropina change if $f(F,\beta)=\frac{F^2}{\beta}$ and a generalized Kropina change if $f(F,\beta)=\frac{F^{k+1}}{\beta^k}$, where k is any positive integer. The purpose of this paper is to investigate the generalized Kropina change of mth root metrics, defined

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by

$$\overline{F} = \frac{F^{k+1}}{\beta^k}, \tag{1.1}$$

where $F = \sqrt[m]{A}$ is an mth root metric on the manifold M and 1-form $\beta = b_i(x)y^i$ are positive on the whole manifold M.

A geodesic curve $c = c(t)$ of a Finsler metric $F = F(x,y)$ on a smooth manifold M is given by $\ddot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0$, where the local functions $G^i = G^i(x,y)$ are called the spray coefficients given by

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}. \tag{1.2}$$

Let F^n be a Finsler space. For a tangent vector $y \in T_xM$, in the local coordinate system, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u,v,w) = B^i_{jkl}(x,y)u^j v^k w^l \frac{\partial}{\partial x^i}$ and $E_y(u,v) = E_{jk}(x,y)u^j v^k$, respectively, where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$, $E_{jk} = \frac{1}{2} B^m_{mjk} = \frac{1}{2} \frac{\partial^2}{\partial y^j \partial y^k}[\frac{\partial G^m}{\partial y^m}], u = u^i \frac{\partial}{\partial x^i}, v = v^i \frac{\partial}{\partial x^i}$, and $w = w^i \frac{\partial}{\partial x^i}$. Then $\mathbf{B} = \{B_y | y \in TM_0\}$ and $\mathbf{E} = \{E_y | y \in TM_0\}$ are called the Berwald curvature and mean Berwald curvature, respectively. A Finsler metric is called a Berwald metric and a mean Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively.

For a tangent vector $y \in T_xM$, in the local coordinate system, define $L_y(u,v,w) = L_{ijk}(x,y)u^i v^j w^k$, where $L_{ijk} = -\frac{1}{2} F F_{y^s}[G^s]_{y^i y^j y^k}$, $u = u^i \frac{\partial}{\partial x^i}, v = v^i \frac{\partial}{\partial x^i}$, and $w = w^i \frac{\partial}{\partial x^i}$. Then $\mathbf{L} = \{L_y | y \in TM_0\}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

In this paper, we consider the generalized Kropina change of mth root Finsler metrics and, more precisely, we establish the following theorems:

Theorem 1.1 *Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be a generalized Kropina change of mth root Finsler metrics on the manifold, where $m \geq 2$. Suppose that $\overline{F}$ has isotropic Berwald curvature (isotropic mean Berwald curvature). Then $\overline{F}$ reduces to a Berwald metric (weakly Berwald metric).*

Theorem 1.2 *Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be a generalized Kropina change of mth root Finsler metrics on the manifold, where $m \geq 2$. Suppose that $\overline{F}$ is of relatively isotropic Landsberg curvature. Then $\overline{F}$ reduces to a Landsberg metric.*

Theorem 1.3 *Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be a generalized Kropina change of mth root Finsler metrics on the manifold, where $m \geq 2$. Suppose that $\overline{F}$ is of relatively isotropic mean Landsberg curvature. Then $\overline{F}$ reduces to a weakly Landsberg metric.*

In [1], Akbar-Zadeh introduced the non-Riemannian quantity $\mathbf{H}$, which is obtained from the mean Berwald curvature by covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity $\mathbf{H} = H_{ij} dx^i \otimes dx^j$ is defined by $H_{ij} := E_{ij|s} y^s$. It is proved that for a Finsler manifold of scalar flag curvature $\mathbf{K}$ with dimension $n \geq 3, \mathbf{K} = \mathit{constant}$ if and only if $\mathbf{H} = 0$. The notion of almost vanishing $\mathbf{H}$-curvature was first introduced by Shen et al. [8]. They investigated that the non-Riemannian quantity $\mathbf{H}$ is closely related to the flag curvature.

In the present paper, we prove the following theorem:

Theorem 1.4 *Let* $\overline{F} = \frac{F^{k+1}}{\beta^k}$ *be a generalized Kropina change of* m*th root Finsler metrics on the manifold, where* $m \geq 2$. *Suppose that* $\overline{F}$ *has almost vanishing* $\boldsymbol{H}$*-curvature. Then* $\boldsymbol{H} = 0$.

Throughout the paper we call the Finsler metric $\overline{F}$ a generalized Kropina change of mth root metric and $\overline{F}^n = (M, \overline{F})$ a generalized Kropina transformed Finsler space. We restrict ourselves for $m \geq 2$ throughout the paper and the quantities corresponding to the generalized Kropina transformed Finsler space $\overline{F}^n$ will be denoted by putting a bar on the top of that quantity.

2. Preliminaries

Let M be an n-dimensional C^∞-manifold; T_xM denotes the tangent space of M at x. The tangent bundle TM is the union of tangent spaces, $TM := \bigcup_{x \in M} T_xM$. We denote the elements of TM by (x, y), where $x = (x^i)$ is a point of M and $y \in T_xM$. We denote $TM_0 = TM \setminus \{0\}$.

Definition: A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) F is C^∞ on TM_0,

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and

(iii) the Hessian of F^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called the fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n.

The normalized supporting element l_i and angular metric tensor h_{ij} of F are defined respectively as:

$$l_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}. \tag{2.1}$$

Let F be a Finsler metric defined by $F = \sqrt[m]{A}$, where A is given by $A = a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ with $a_{i_1 \ldots i_m}$ symmetric in all its indices [10]. Then F is called an mth root Finsler metric. Clearly, A is homogeneous of degree m in y. Let

$$A_i = \frac{\partial A}{\partial y^i}, A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, A_{x^i} = \frac{\partial A}{\partial x^i}, A_0 = A_{x^i} y^i. \tag{2.2}$$

Then the following holds:

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (2-m)A_iA_j\right],$$

$$y^iA_i = mA, y^iA_{ij} = (m-1)A_j, y_i = \frac{1}{m}A^{\frac{2}{m}-1}A_i,$$

$$A^{ij}A_{jk} = \delta^i_k, A^{ij}A_i = \frac{1}{m-1}y^j, A_iA_jA^{ij} = \frac{m}{m-1}A.$$

3. Fundamental metric tensors and geodesic sprays of generalized Kropina changed mth root metrics

The differentiation of equation (1.1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = \overline{F}\left[\frac{k+1}{mA}A_i - \frac{k}{\beta}b_i\right] \tag{3.1}$$

and the angular metric tensor

$$\overline{h}_{ij} = (k+1)\overline{F}^2 \left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right]. \tag{3.2}$$

The fundamental metric tensor $\overline{g}_{ij}$ of Finsler space $\overline{F}^n$ is given by

$$\overline{g}_{ij} = \overline{h}_{ij} + \overline{l}_i\overline{l}_j.$$

Therefore, using equations (3.1) and (3.2), we obtain metric tensor $\overline{g}_{ij}$ as

$$\begin{aligned}\overline{g}_{ij} \;=\; & \overline{F}^2 \left[\frac{k+1}{mA}A_{ij} + \frac{(k+1)(2k+2-m)}{m^2A^2}A_iA_j - \frac{2k(k+1)}{mA\beta}(A_ib_j + A_jb_i)\right. \\ & \left. + \frac{k(2k+1)}{\beta^2}b_ib_j\right].\end{aligned} \tag{3.3}$$

The above equation can be written as

$$\begin{aligned}\overline{g}_{ij} \;=\; & \overline{F}^2 \left[\frac{k+1}{mA}A_{ij} + \frac{(k+1)(2k+2-(2k+1)m)}{(2k+1)m^2A^2}A_iA_j\right. \\ & \left. + \left(\frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_i - \frac{\sqrt{k(2k+1)}}{\beta}b_i\right)\left(\frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_j - \frac{\sqrt{k(2k+1)}}{\beta}b_j\right)\right].\end{aligned} \tag{3.4}$$

Let

$$X_{ij} = \frac{k+1}{mA}A_{ij} + \frac{(k+1)(2k+2-(2k+1)m)}{(2k+1)m^2A^2}A_iA_j. \tag{3.5}$$

We know that [7] $A_{ij} = B_{ij} + \delta C_iC_j$, and then
$A^{ij} = B^{ij} - \frac{\delta C^iC^j}{1+\delta C^2}$, where $C^2 = B^{ij}C_iC_j$ and $C^i = B^{ij}C_j$.
Using the above results, we obtain

$$X^{ij} = \frac{mA}{k+1}A^{ij} - \frac{(2k+2-(2k+1)m)}{(k+1)(m-1)}y^iy^j. \tag{3.6}$$

Thus, in view of equations (3.4) and (3.5), $\overline{g}_{ij}$ can be written as $\overline{g}_{ij} = \overline{F}^2\left[X_{ij} + Y_iY_j\right]$, where $Y_i = \left(\frac{2(k+1)\sqrt{k}}{\sqrt{2k+1}mA}A_i - \frac{\sqrt{k(2k+1)}}{\beta}b_i\right)$. By direct computation, we have the contravariant metric tensor

$$\overline{g}^{ij} = \frac{1}{\overline{F}^2}\left[a_0A^{ij} + a_1y^iy^j + a_2B^iB^j + a_3(y^iB^j + y^jB^i)\right], \tag{3.7}$$

where

$$\begin{aligned} B^i &= A^{ij}b_j, B^2 = B^i b_i, a_0 = \frac{mA}{k+1}, a_1 = -\Bigg[\frac{(2k+2)-(2k+1)m}{(k+1)(m-1)} \\ &\quad + \frac{k(2k+1)m^2\beta^2}{(k+1)(m-1)\left[\beta^2\left\{k(2m-3)+(m-1-2k^2)\right\}+km(m-1)(2k+1)AB^2\right]}\Bigg], \\ a_2 &= -\left[\frac{k(2k+1)m^2(m-1)^2A^2}{(k+1)(m-1)\left[\beta^2\left\{k(2m-3)+(m-1-2k^2)\right\}+km(m-1)(2k+1)AB^2\right]}\right], \\ a_3 &= -\left[\frac{k(2k+1)m^2(m-1)^2A\beta}{(k+1)(m-1)\left[\beta^2\left\{k(2m-3)+(m-1-2k^2)\right\}+km(m-1)(2k+1)AB^2\right]}\right]. \end{aligned} \tag{3.8}$$

Thus, we obtain the following proposition:

Proposition 3.1 The covariant metric tensor $\overline{g}_{ij}$ and the contravariant metric tensor $\overline{g}^{ij}$ of generalized Kropina transformed Finsler space $\overline{F}^n$ are given by the following:

$$\overline{g}_{ij} = \overline{F}^2\left[\frac{k+1}{mA}A_{ij} + \frac{(k+1)(2k+2-m)}{m^2A^2}A_iA_j - \frac{2k(k+1)}{mA\beta}(A_ib_j + A_jb_i) + \frac{k(2k+1)}{\beta^2}b_ib_j\right]$$

and

$$\overline{g}^{ij} = \frac{1}{\overline{F}^2}\left[a_0A^{ij} + a_1y^iy^j + a_2B^iB^j + a_3(y^iB^j + y^jB^i)\right],$$

where a_0, a_1, a_2, a_3, B^i, and B^2 are given by equation (3.8).

In view of equation (1.2), to calculate spray coefficients $\overline{G}^i$ of $\overline{F} = \frac{F^{k+1}}{\beta^k}$, we need the following:

$$\left[\overline{F}^2\right]_{x^t} = \overline{F}^2\left[\frac{2(k+1)A_{x^t}}{mA} - \frac{2k\beta_{x^t}}{\beta}\right] \tag{3.9}$$

and

$$\begin{aligned} \left[\overline{F}^2\right]_{x^ty^l}y^t &= \overline{F}^2\Bigg[\frac{2(k+1)(A_l)_0}{mA} + \frac{(k+1)(4(k+1)-2m)}{m^2A^2}A_lA_0 - \frac{2k}{\beta}(b_l)_0 \\ &\quad + \frac{2k(2k+1)}{\beta^2}b_l\beta_0 - \frac{4k(k+1)}{mA\beta}(A_l\beta_0 + A_0b_l)\Bigg]. \end{aligned} \tag{3.10}$$

In view of equations (1.2), (3.9), and (3.10) and Proposition (3.1), we have

$$\begin{aligned} \overline{G}^i &= \frac{1}{4}\left[a_0A^{il} + a_1y^iy^l + a_2B^iB^l + a_3(y^iB^l + y^lB^i)\right] \\ &\quad \times \Bigg[\frac{2(k+1)(A_l)_0}{mA} + \frac{(k+1)(4(k+1)-2m)}{m^2A^2}A_lA_0 - \frac{2k}{\beta}(b_l)_0 + \frac{2k(2k+1)}{\beta^2}b_l\beta_0 \\ &\quad - \frac{4k(k+1)}{mA\beta}(A_l\beta_0 + A_0b_l)\Bigg]. \end{aligned} \tag{3.11}$$

Proposition 3.2 Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be a generalized Kropina change of mth root Finsler metric F on a manifold, with $m \geq 2$. Then the spray coefficients $\overline{G}^i$ of $\overline{F}^n$ are given by equation (3.11).

Remark 3.1 It is remarkable to note that the metric tensors $\overline{g}_{ij}$ and $\overline{g}^{ij}$ of $\overline{F}^n$ are not necessarily rational functions in y but spray coefficients $\overline{G}^i$ of $\overline{F}^n$ are rational functions in y.

4. Berwald curvature

A Finsler metric F is said to be an isotropic Berwald metric if its Berwald curvature is in the following form:

$$B^i_{jkl} = c(F_{y^j y^k}\delta^i_l + F_{y^k y^l}\delta^i_j + F_{y^l y^j}\delta^i_k + F_{y^j y^k y^l}y^i), \tag{4.1}$$

and an isotropic mean Berwald metric if its mean curvature is in the form $E_{ij} = \frac{n+1}{2F}ch_{ij}$, where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric tensor. A Finsler metric F is said to be weakly Berwald metric if its mean curvature $E_{ij} = 0$.

Proof of Theorem 1.1 Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be a generalized Kropina change of the mth root Finsler metric. Suppose that $\overline{F}$ has isotropic Berwald curvature given by equation (4.1). By Remark (3.1), the left-hand side of equation (4.1) is a rational function in y, while the right-hand side is an irrational function. This implies that $c = 0$ and hence $\overline{F}$ reduces to a Berwald metric.

Moreover, $\overline{F} = \frac{F^{k+1}}{\beta^k}$ is of isotropic mean Berwald curvature, that is,

$$\overline{E}_{ij} = \frac{n+1}{2\overline{F}}c\overline{h}_{ij}, \tag{4.2}$$

where $c = c(x)$ is a scalar function on M and $\overline{h}_{ij}$ is the angular metric.

By putting the angular metric $\overline{h}_{ij}$ given by equation (3.2) in (4.2), we have

$$\overline{E}_{ij} = \frac{(n+1)(k+1)c}{2}\overline{F}\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right]. \tag{4.3}$$

The left side of equation (4.3) is a rational function in y while its right side is an irrational function in y. Thus, equation (4.3) implies either $c = 0$ or

$$\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right] = 0. \tag{4.4}$$

If equation (4.4) holds, then $\overline{h}_{ij} = 0$, which is not possible. Hence, $c = 0$ and $\overline{E}_{ij} = 0$. This completes the proof of Theorem 1.1.

5. Landsberg curvature

For a nonzero vector $y \in T_pM$, the Cartan torsion $C_y = C_{ijl}dx^i \otimes dx^j \otimes dx^l : T_pM \otimes T_pM \otimes T_pM \to \mathbb{R}$ is defined by $C_{ijl} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^l}$.

Proof of Theorem 1.2 Let $\overline{F} = F^{k+1}/\beta^k$ be a relatively isotropic Landsberg metric, that is,

$$\overline{L}_{ijl} = c\overline{F}\overline{C}_{ijl}, \tag{5.1}$$

where $c = c(x)$ is a scalar function on M. The Cartan torsion $\overline{C}_{ijl}$ of $\overline{F}$ is given by the following:

$$\begin{aligned}\overline{C}_{ijl} &= \frac{(k+1)\overline{F}^2}{2}\left[\frac{1}{mA}A_{ijl} + \frac{2(2k+2-m)(k+1-m)}{m^3A^3}A_iA_jA_l\right.\\ &+ \frac{(2k+2-m)}{m^2A^2}\sum_{(i,j,l)}A_{ij}A_l - \frac{2k}{mA\beta}\sum_{(i,j,l)}A_{ij}b_l - \frac{2k(2k+2-m)}{m^2A^2\beta}\sum_{(i,j,l)}A_iA_jb_l\\ &\left.+ \frac{2k(2k+1)}{mA\beta^2}\sum_{(i,j,l)}A_ib_jb_l - \frac{2k(2k+1)}{\beta^3}b_ib_jb_l\right],\end{aligned} \tag{5.2}$$

where $\sum_{(i,j,l)}$ denotes the cyclic interchange of suffices i,j,l and summation, for instance $\sum_{(i,j,l)} A_{ij}A_l = A_{ij}A_l + A_{jl}A_i + A_{li}A_j$. Also, we know that

$$\overline{L}_{ijl} = -\frac{1}{2}\overline{F}\overline{F}_{y^s}\overline{G}^s_{y^iy^jy^l}. \tag{5.3}$$

In our case,

$$\overline{L}_{ijl} = -\frac{\overline{F}^2}{2}\left[\frac{(k+1)}{mA}A_s - \frac{k}{\beta}b_s\right]\overline{G}^s_{y^iy^jy^l}. \tag{5.4}$$

Using equations (5.1), (5.2), and (5.4), we obtain

$$\begin{aligned}\left(\frac{2A_s}{mA} - \frac{b_s}{\beta}\right)G^s_{y^iy^jy^l} &= -c\overline{F}\left[\frac{1}{mA}A_{ijl} + \frac{2(2k+2-m)(k+1-m)}{m^3A^3}A_iA_jA_l\right.\\ &+ \frac{(2k+2-m)}{m^2A^2}\sum_{(i,j,l)}A_{ij}A_l - \frac{2k}{mA\beta}\sum_{(i,j,l)}A_{ij}b_l - \frac{2k(2k+2-m)}{m^2A^2\beta}\sum_{(i,j,l)}A_iA_jb_l\\ &\left.+ \frac{2k(2k+1)}{mA\beta^2}\sum_{(i,j,l)}A_ib_jb_l - \frac{2k(2k+1)}{\beta^3}b_ib_jb_l\right].\end{aligned} \tag{5.5}$$

The left side of equation (5.5) is a rational function in y while its right side is an irrational function in y. Therefore, either $c = 0$ or A satisfies the following PDEs:

$$\begin{aligned}&\left[\frac{1}{mA}A_{ijl} + \frac{2(2k+2-m)(k+1-m)}{m^3A^3}A_iA_jA_l + \frac{(2k+2-m)}{m^2A^2}\sum_{(i,j,l)}A_{ij}A_l\right.\\ &- \frac{2k}{mA\beta}\sum_{(i,j,l)}A_{ij}b_l - \frac{2k(2k+2-m)}{m^2A^2\beta}\sum_{(i,j,l)}A_iA_jb_l + \frac{2k(2k+1)}{mA\beta^2}\sum_{(i,j,l)}A_ib_jb_l\\ &\left.- \frac{2k(2k+1)}{\beta^3}b_ib_jb_l\right] = 0.\end{aligned}$$

If the above equation holds, then $\overline{C}_{ijl=0}$. Hence, $\overline{F}$ is a Riemannian metric, which contradicts our assumption. Therefore, $c = 0$. This completes the proof of Theorem (1.2).

The mean Cartan torsion $I_y = I_i(x,y)dx^i : T_pM \to \mathbb{R}$ is defined by

$$I_i = g^{jl}C_{ijl}. \tag{5.6}$$

Thus, the mean Cartan torsion of $\overline{F}$ is given by

$$\begin{aligned}\overline{I}_i \;=\; & \frac{1}{2}\left[a_0A^{jl} + a_1y^jy^l + a_2B^jB^l + a_3(y^jB^l + y^lB^j)\right] \times \left[\frac{1}{mA}A_{ijl}\right.\\ & + \frac{2(2k+2-m)(k+1-m)}{m^3A^3}A_iA_jA_l + \frac{(2k+2-m)}{m^2A^2}\sum_{(i,j,l)}A_{ij}A_l - \frac{2k}{mA\beta}\sum_{(i,j,l)}A_{ij}b_l\\ & \left.-\frac{2k(2k+2-m)}{m^2A^2\beta}\sum_{(i,j,l)}A_iA_jb_l + \frac{2k(2k+1)}{mA\beta^2}\sum_{(i,j,l)}A_ib_jb_l - \frac{2k(2k+1)}{\beta^3}b_ib_jb_l\right].\end{aligned} \tag{5.7}$$

Lemma 5.1 *Let $\overline{F}^n$ be a generalized Kropina change of an mth root Finsler manifold. Then mean Cartan torsions $\overline{I}_i$ of $\overline{F}^n$ are rational functions in y.*

The horizontal covariant derivative of $\mathbf{I}$ along a vector $u \in T_xM$ gives rise to the mean Landsberg curvature $J_y(u) := J_i(y)u^i$, where $J_i = I_{i|s}y^s$. A Finsler metric with $\mathbf{J} = 0$ is called a weakly Landsberg metric.

The mean Landsberg curvature of $\overline{F}$ is given by

$$\overline{J}_i = \overline{g}^{jl}\overline{L}_{ijl}. \tag{5.8}$$

In view of equations (3.7), (5.4), and (5.8), we have

$$\overline{J}_i = -\frac{1}{2}\left[a_0A^{jl} + a_1y^jy^l + a_2B^jB^l + a_3(y^jB^l + y^lB^j)\right]\left[\frac{(k+1)}{mA}A_s - \frac{k}{\beta}b_s\right]\overline{G}^s_{y^iy^jy^l}. \tag{5.9}$$

Lemma 5.2 *Let $\overline{F}^n$ be a generalized Kropina change of an mth root Finsler manifold. Then mean Landsberg curvatures $\overline{J}_i$ of $\overline{F}^n$ are rational functions in y.*

In view of equations (5.1), (5.6), and (5.8), we have

$$\overline{J}_i = c\overline{F}\,\overline{I}_i. \tag{5.10}$$

Proof of Theorem 1.3 Using Lemma 5.1 and Lemma 5.2, the left side of equation (5.10) is a rational function in y while its right side is an irrational function in y. Thus, if equation (5.10) holds, then either $c = 0$ or $\overline{I}_i = 0$. If $\overline{I}_i = 0$, $\overline{F}$ is a Riemannian metric, which contradicts our assumption. Therefore, $c = 0$. This completes the proof of Theorem 1.3.

6. H-curvature

A Finsler metric on an n-dimensional manifold M is said to be of almost vanishing **H**-curvature if $H_{ij} = \frac{n+1}{2F}\theta h_{ij}$, for some 1-form θ on M, where h_{ij} is the angular metric.

Proof of Theorem 1.4 Let $\overline{F} = \frac{F^{k+1}}{\beta^k}$ be of almost vanishing $\overline{\mathbf{H}}$-curvature, that is,

$$\overline{H}_{ij} = \frac{n+1}{2\overline{F}}\theta\overline{h}_{ij}, \tag{6.1}$$

for some 1-form θ on M, where $\overline{h}_{ij}$ is the angular metric.

Plugging the angular metric $\overline{h}_{ij}$ given by equation (3.2) into (6.1), we have

$$\overline{H}_{ij} = \frac{(n+1)(k+1)}{2}\theta\overline{F}\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right]. \tag{6.2}$$

The left side of equation (6.2) is a rational function in y while its right side is an irrational function in y. Thus, equation (6.2) implies either $\theta = 0$ or

$$\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right] = 0. \tag{6.3}$$

If equation (6.3) holds, then $\overline{h}_{ij} = 0$, which is not possible. Hence, $\theta = 0$ and $\overline{H}_{ij} = 0$. This completes the proof of Theorem (1.4).

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