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On the rate of L_p-convergence of Balakrishnan–Rubin-type hypersingular integrals associated to the Gauss-Weierstrass semigroup

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Abstract: We introduce a family of Balakrishnan–Rubin-type hypersingular integrals depending on a parameter ε and generated by the Gauss–Weierstrass semigroup. Then the connection between the order of L_p–smoothness of a L_p–function φ and the rate of L_p-convergence of these families to φ, as ε tends to 0, is obtained.

Key words: Riesz potentials, Bessel potentials, truncated hypersingular integrals, rate of convergence, Gauss–Weierstrass semigroup

1. Introduction

For a sufficiently good function f on $\mathbb{R}^n$ the Riesz and Bessel potentials of order α are defined by

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,\ \gamma_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \tag{1.1}$$

$$Re\,\alpha > 0, \quad \alpha \neq n, n+2, n+4, ...$$

$$(J^\alpha f)(x) = \frac{1}{\lambda_n(\alpha)} \int_{\mathbb{R}^n} f(y) G_\alpha(x-y)\, dy,\ \lambda_n(\alpha) = 2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right) \tag{1.2}$$

$$G_\alpha(x) = \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-t-\frac{|x|^2}{4t}} \frac{dt}{t},\ Re\,\alpha > 0.$$

These operators can be regarded (in a certain sense) as a negative “fractional” powers of the differential operators, $(-\Delta)$ and $(E-\Delta)$, i.e.

$$I^\alpha = (-\Delta)^{-\frac{\alpha}{2}} \text{ and } J^\alpha = (E-\Delta)^{-\frac{\alpha}{2}},$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2},\ E- \text{ is identity operator.}$$

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If $f \in L_p(\mathbb{R}^n)$, then the integral (1.1) converges a.e. for $1 \le p < \frac{n}{\operatorname{Re}\alpha}$ and the integral (1.2) converges a.e. for $1 \le p < \infty$, and these conditions are accurate.

These potentials are of great importance in harmonic analysis and its applications; we refer to [10, 14, 19–24, 29] for the basic properties and applications of these potentials. We should also mention that some significant generalizations of potentials-type operators on space of homogeneous type and different hypergroups were investigated in [11, 12, 15, 16] and in many others works.

In potentials theory one of the important problems is finding an inversion formula for the potential operator, and hypersingular integrals theory has appeared as a result of these efforts and Stein [25], Lizorkin [13], Wheeden [29], Fisher [10], Samko [21–23], Rubin [17–20] and many other mathematicians have investigated the subject. In particular, the wavelet approach to inversion of the potentials was developed by Rubin [19, 20], Rubin and Aliev [1], and Aliev [2]; see also [3–5, 27].

In the paper [18] Rubin introduced some family of "truncated" integrals $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f, (\varepsilon > 0)$, generated by the Gauss–Weierstrass semigroup, and proved that under some conditions on a function $\varphi \in L_p(\mathbb{R}^n)$ and parameter $\alpha > 0$, the expressions $D_\varepsilon^\alpha I^\alpha \varphi$ and $\mathfrak{D}_\varepsilon^\alpha J^\alpha \varphi$ converge to φ as $\varepsilon \to 0^+$, pointwise (a.e.) and in the L_p-norm. Our main work is to find the relationship between the "order of L_p-smoothness" of a function φ and the "rate of L_p-convergence" of the families $D_\varepsilon^\alpha I^\alpha \varphi$ and $\mathfrak{D}_\varepsilon^\alpha J^\alpha \varphi$ to φ as $\varepsilon \to 0^+$.

Some comments are in order. In the case of truncated hypersingular integrals generated by the Poisson and metaharmonic semigroups, the analogous problem has been studied in [6] and also the rate of pointwise convergence of the truncated hypersingular integrals generated by the Gauss–Weierstrass semigroup has been studied in [7]. The essential difference between our main result and the analogous statement of the paper [7] is as follows: in the paper [7] the rate of ***pointwise*** convergence of the families $D_\varepsilon^\alpha \phi$ and $\mathfrak{D}_\varepsilon^\alpha \varphi, (\varepsilon > 0)$, to φ as $\varepsilon \to 0^+$, at the some kind of smoothness point of φ, is obtained, whereas in this work we find some relationships between the "order of L_p–smoothness" of function φ and the "rate of L_p-convergence" of the families $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f, (\varepsilon > 0)$, to φ as $\varepsilon \to 0^+$.

2. Auxiliary definitions and lemmas

Let $L_p(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ with the finite norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, 1 \le p < \infty; \;\; \|f\|_\infty = \underset{x \in \mathbb{R}^n}{ess\sup} |f(x)|.$$

The Gauss–Weierstrass semigroup, generated by a function $f(x)$, $x \in \mathbb{R}^n$, is defined by

$$(Uf)(x,t) = \int_{\mathbb{R}^n} W(y;t) \, f(x-y) \, dy, \;\; (t > 0), \tag{2.1}$$

where $W(y;t)$ is the Gauss–Weierstrass kernel,

$$W(y;t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}}, \; t > 0, \text{ which has } \int_{\mathbb{R}^n} W(y;t) dy = 1, \; (\forall t > 0) \tag{2.2}$$

More information about this semigroup, $(Uf)(\cdot, t)$, can be found in [18] and [19, p. 223] (see also [23, 26]).

The modified Gauss-Weierstrass semigroup $U_M f$ is defined as

$$(U_M f)(x,t) = e^{-t}(Uf)(x,t)\,;\ t>0, x\in\mathbb{R}^n. \tag{2.3}$$

For $t=0$ we are assuming that $(Uf)(x,0)=(U_M f)(x,0)=f(x)$.

The finite difference of the function $g(t)$, $\left(t\in\mathbb{R}^1\right)$ with order $l\in\mathbb{N}$ and step $\tau\in\mathbb{R}^1$ is defined by

$$\Delta_\tau^l[g](t)\equiv\sum_{k=0}^{l}\binom{l}{k}(-1)^k g(t+k\tau). \tag{2.4}$$

Using the $(Uf)(x,t)$ and $(U_M f)(x,t)$, we introduce the following Balakrishnan–Rubin-type "truncated" integrals (cf. [19, p.224 and p.262]):

$$(D_\varepsilon^\alpha f)(x)=\frac{1}{\chi_l\left(\frac{\alpha}{2}\right)}\int_\varepsilon^\infty\left[\sum_{k=0}^{l}\binom{l}{k}(-1)^k(Uf)(x,k\tau)\right]\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}; \tag{2.5}$$

$$(\mathfrak{D}_\varepsilon^\alpha f)(x)=\frac{1}{\chi_l\left(\frac{\alpha}{2}\right)}\int_\varepsilon^\infty\left[\sum_{k=0}^{l}\binom{l}{k}(-1)^k(U_M f)(x,k\tau)\right]\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}, \tag{2.6}$$

where the normalized coefficient $\chi_l\left(\frac{\alpha}{2}\right)$ is defined by

$$\chi_l\left(\frac{\alpha}{2}\right)=\int_0^\infty\left(1-e^{-t}\right)^l t^{-1-\frac{\alpha}{2}}dt,\left(0<\frac{\alpha}{2}<l,\ l\in\mathbb{N}\right).$$

As shown in the following lemma, there is a close connection between the constructions (2.5)–(2.6) and the potentials $I^\alpha\varphi$ and $J^\alpha\varphi$.

Lemma 2.1 *(Rubin [18], [19, p.224 and p.262]).*

(a) Let $\varphi\in L_p(\mathbb{R}^n),(1\le p<\infty)$ and $0<\alpha<\frac{n}{p}$. Then for any $\varepsilon>0$ and for a.e. $x\in\mathbb{R}^n$,

$$(D_\varepsilon^\alpha I^\alpha\varphi)(x)=\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)(U\varphi)(x,\varepsilon\eta)\,d\eta; \tag{2.7}$$

(b) Let $\varphi\in L_p(\mathbb{R}^n),(1\le p\le\infty)$ and $0<\alpha<\infty$. Then for any $\varepsilon>0$ and for a.e. $x\in\mathbb{R}^n$,

$$(\mathfrak{D}_\varepsilon^\alpha J^\alpha\varphi)(x)=\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)(U_M\varphi)(x,\varepsilon\eta)\,d\eta. \tag{2.8}$$

Here the function $K_{\frac{\alpha}{2}}^{(l)}(\eta)$ is defined as

$$K_{\frac{\alpha}{2}}^{(l)}(\eta)=\left[\Gamma\left(1+\frac{\alpha}{2}\right)\chi_l\left(\frac{\alpha}{2}\right)\right]^{-1}\eta^{-1}\Delta_{-1}^l\left[\eta_+^{\frac{\alpha}{2}}\right], \tag{2.9}$$

where in accordance (2.4),

$$\Delta_{-1}^l\left[\eta_+^{\frac{\alpha}{2}}\right]=\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\eta-k)_+^{\frac{\alpha}{2}}\ \ and\ a_+^{\frac{\alpha}{2}}=\left\{\begin{array}{ll}a^{\frac{\alpha}{2}}, & if\ a>0\\ 0, & if\ a\le 0\end{array}\right\}.$$

The following lemma gives some properties of the function $K_{\frac{\alpha}{2}}^{(l)}(\eta)$ that will be used later.

Lemma 2.2 *(see [23, p.125] and [19, p.158])*

(a) $K_{\frac{\alpha}{2}}^{(l)}(\eta) \in L_1(0,\infty)$ *and* $\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)\,d\eta = 1.$

(b) $K_{\frac{\alpha}{2}}^{(l)}(\eta) = \left\{ \begin{array}{ll} O(\eta^{\frac{\alpha}{2}-1}), & if\ \eta \to 0^+ \\ O(\eta^{\frac{\alpha}{2}-l-1}), & if\ \eta \to \infty \end{array} \right\}.$

The following definition and the subsequent lemmas play crucial roles in the sequel.

Definition 2.3 *Let* $\rho \in (0,1)$ *be a fixed parameter and the function* $\mu(r)$, $(0 \le r \le \rho)$ *be continuous on* $[0,\rho]$, *positive on* $(0,\rho]$ *and* $\mu(0) = 0$.

We say that a function $\varphi \in L_p(\mathbb{R}^n), (1 \le p < \infty)$ *has* μ*-smoothness property in* L_p-*sense if*

$$\mathcal{M}_\mu = \mathcal{M}_\mu(\varphi) \equiv \sup_{0<r\le\rho} \frac{1}{r^n \mu(r)} \int_{|x|\le r} \|\varphi(\cdot - x) - \varphi(\cdot)\|_p \,dx < \infty. \tag{2.10}$$

(here, as usual, $|x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$ *and* $dx = dx_1...dx_n$*).*

Remark 2.4 *Let the function* μ *be defined as in Definition 2.3 and* μ_φ *be the* L_p*–modulus of continuity of function* $\varphi \in L_p(\mathbb{R}^n)$, *i.e.*

$$\mu_\varphi(r) = \sup_{|x|\le r} \|\varphi(\cdot - x) - \varphi(\cdot)\|_p \ ,\left(|x| = \sqrt{x_1^2 + \cdots x_n^2}\ \right).$$

It is clear that if $\mu_\varphi(r) \le \mu(r)$, $(0 \le r \le \rho)$ *then the expression* M_μ *in (2.10) is finite.*

Remark 2.5 *From now on it will be assumed that* $\mu(t) \ge at, (0 \le t \le \rho)$, *for some* $a > 0$ *and* $\mu(t) = \mu(\rho)$ *for* $\rho \le t < \infty$. *It is well known that if* μ *is modulus of continuity then* $\mu(\lambda t) \le (\lambda + 1)\mu(t)$ *for* $\lambda \ge 0$ *(see, for instance [9, p. 41]).*

Lemma 2.6 *(cf.[7]; see also [8] and [28]) Let a function* $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$ *have* μ*-smoothness property in* L_p-*sense. Let, further, the function* $\psi(r), (0 \le r \le \rho)$ *be decreasing, nonnegative, and continuously differentiable on* $[0,\rho]$. *Then*

$$\int_{|x|\le\rho} \|\varphi(t-x) - \varphi(t)\|_p \,\psi(|x|)\,dx \le \mathcal{M}_\mu\big[\rho^n \mu(\rho)\,\psi(\rho) + \int_0^\rho r^n \mu(r)\,(-\psi'(r))\,dr\big]. \tag{2.11}$$

Proof

We give here a short proof of (2.11). Set $g(x) = \|\varphi(t-x) - \varphi(t)\|_p$ and $x = r\theta$, where $r = |x|$ $,\theta \in S^{n-1}$. Then

$$I \equiv \int_{|x|\le\rho} \|\varphi(t-x) - \varphi(t)\|_p \,\psi(|x|)\,dx = \int_{|x|\le\rho} g(x)\psi(|x|)\,dx$$

$$= \int_0^\rho r^{n-1}\psi(r)\,(\int_{|\theta|=1} g(r\theta)d\sigma(\theta))dr.$$

By introducing the functions

$$\lambda(r)=\int_{|\theta|=1} g(r\theta)d\sigma(\theta) \text{ and } \Omega(r)=\int_0^r \lambda(r)t^{n-1}dt \tag{2.12}$$

we have

$$I\equiv\int_0^\rho \psi(r)\,\lambda(r)r^{n-1}dr=\int_0^\rho \psi(r)\,d\Omega(r)=\psi(r)\,\Omega(r)\,|_0^\rho-\int_0^\rho \Omega(r)\psi'(r)\,dr$$
$$=\psi(\rho)\,\Omega(\rho)+\int_0^\rho \Omega(r)(-\psi'(r))dr.$$

The condition (2.10) yields that

$$\Omega(r)=\int_0^r \lambda(r)t^{n-1}dt=\int_{|x|\le r} g(x)dx=\int_{|x|\le\rho}\|\varphi(t-x)-\varphi(t)\|_p\,dx\le r^n\mu(r)M_\mu.$$

Hence,

$$I\le M_\mu\left[\rho^n\mu(\rho)\Psi(\rho)+\int_0^\rho r^n\mu(r)(-\psi'(r))dr\right].$$

□

Lemma 2.7 *Let a function φ have μ–smoothness property and $W(x;\varepsilon)$ be the Gauss–Weierstrass kernel with parameter $\varepsilon>0$ (cf(2.2)):*

$$W(x;\varepsilon)=(4\pi\varepsilon)^{-\frac{n}{2}}\,e^{-\frac{|x|^2}{4\varepsilon}}\ \ (x\in\mathbb{R}^n).$$

Then

$$\int_{|x|\le\rho}\|\varphi(t-x)-\varphi(t)\|_p\,W(x;\varepsilon)\,dx\le c\mu\left(\sqrt{\varepsilon}\right),\ \forall\varepsilon\in(0,\rho) \tag{2.13}$$

where $c>0$ does not depend on $\varepsilon\ll 1$.

Proof We set $\psi(|x|)=W(x;\varepsilon)$ in (2.11). Since $\psi(r)=(4\pi\varepsilon)^{-\frac{n}{2}}\,e^{-\frac{r^2}{4\varepsilon}}$, after simple calculation we get

$$-\psi'(r)=c_1r\varepsilon^{-1-\frac{n}{2}}e^{-\frac{r^2}{4\varepsilon}};\ c_1=2^{-(n+1)}\pi^{-\frac{n}{2}}.$$

Putting this value of $(-\psi'_\varepsilon(r))$ into (2.11), we have for $\rho<1$

$$\int_{|x|\le\rho}\|\varphi(t-x)-\varphi(t)\|_p\,\psi(|x|)\,dx\le\mathcal{M}_\mu\big[\rho^n\mu(\rho)\,(4\pi\varepsilon)^{-\frac{n}{2}}\,e^{-\frac{\rho^2}{4\varepsilon}}$$
$$+\int_0^\rho c_1r^{n+1}\mu(r)\,\varepsilon^{-1-\frac{n}{2}}e^{-\frac{r^2}{4\varepsilon}}dr\big]\le c_2\mathcal{M}_\mu\varepsilon^{-1-\frac{n}{2}}\left[\int_0^\rho r^{n+1}\mu(r)\,e^{-\frac{r^2}{4\varepsilon}}dr+\varepsilon^{-\frac{n}{2}}e^{-\frac{\rho^2}{4\varepsilon}}\right].$$

By changing variables as $r=\sqrt{\varepsilon}t$, we get

$$\int_{|x|\le\rho}\|\varphi(t-x)-\varphi(t)\|_p\,\psi(|x|)\,dx\le c_3\left[\int_0^{\frac{\rho}{\sqrt{\varepsilon}}} t^{n+1}\mu\left(\sqrt{\varepsilon}t\right)e^{-\frac{t^2}{4}}dt+\varepsilon^{-\frac{n}{2}}e^{-\frac{\rho^2}{4\varepsilon}}\right]. \tag{2.14}$$

From the inequality $\mu(\sqrt{\varepsilon}t) \leq (1+t)\,\mu(\sqrt{\varepsilon})$ it follows that

$$\int_0^{\frac{\rho}{\sqrt{\varepsilon}}} t^{n+1}\mu\left(\sqrt{\varepsilon}t\right)e^{-\frac{t^2}{4}}dt \leq \mu\left(\sqrt{\varepsilon}\right)\int_0^{\infty} t^{n+1}(1+t)\,e^{-\frac{t^2}{4}}dt \leq c_4\mu\left(\sqrt{\varepsilon}\right).$$

On the other hand, since for any $k>0$, $\lim\limits_{\varepsilon\to 0}\varepsilon^{-k}e^{-\frac{\rho^2}{4\varepsilon}}=0,$ we have

$$\varepsilon^{-\frac{n}{2}}e^{-\frac{\rho^2}{4\varepsilon}} \leq c_5\sqrt{\varepsilon} \text{ for some } c_5>0.$$

By making use of these in (2.14) we have

$$\int_{|x|\leq\rho}\|\varphi(t-x)-\varphi(t)\|_p\,\psi(|x|)\,dx \leq c_6\left(\mu\left(\sqrt{\varepsilon}\right)+\sqrt{\varepsilon}\right).$$

The condition $\mu(t)\geq at, (0\leq t\leq\rho,\ a>0)$ yields $(\mu(\sqrt{\varepsilon})+\sqrt{\varepsilon})\leq c_7\mu(\sqrt{\varepsilon})$, and the desired result

$$\int_{|x|\leq\rho}\|\varphi(t-x)-\varphi(t)\|_p\,W(x;\varepsilon)\,dx \leq c\mu\left(\sqrt{\varepsilon}\right)$$

follows. □

3. Formulation and proof of the main theorem

Theorem 3.1 *Let $\varphi\in L_p(\mathbb{R}^n), (1\leq p<\infty)$ has the μ-smoothness property in the L_p-sense, i.e. the condition (2.10) is satisfied. Further, let $\mu(r)$ be a L_p-modulus of continuity of φ that satisfies the inequality $\mu(r)\geq ar, (0\leq r\leq\rho)$ for some $a>0$. Assume that the operators D_ε^α and $\mathfrak{D}_\varepsilon^\alpha$ are defined as in (2.5) – (2.6) and the parameter $l\in\mathbb{N}$ satisfies the condition $l>\frac{\alpha}{2}+1$. Then*

$$(a)\ \ \|D_\varepsilon^\alpha I^\alpha\varphi-\varphi\|_p = O\left(\mu\left(\sqrt{\varepsilon}\right)\right)\ \ \text{as } \varepsilon\to 0^+, \tag{3.1}$$

$$(b)\ \ \|\mathfrak{D}_\varepsilon^\alpha J^\alpha\varphi-\varphi\|_p = O\left(\mu\left(\sqrt{\varepsilon}\right)\right)\ \ \text{as } \varepsilon\to 0^+. \tag{3.2}$$

Proof

By making use of the formula (2.7), Lemma 2.2 (a), and the Minkowski inequality, we have:

$$\begin{aligned}\|D_\varepsilon^\alpha I^\alpha\varphi-\varphi\|_p &= \left\|\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)\,(U\varphi)(\cdot,\varepsilon\eta)\,d\eta-\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)\,\varphi(\cdot)\,d\eta\right\|_p\\ &= \left\|\int_0^\infty K_{\frac{\alpha}{2}}^{(l)}(\eta)\,((U\varphi)(\cdot,\varepsilon\eta)-\varphi(\cdot))\,d\eta\right\|_p\\ &\leq \int_0^\infty \left|K_{\frac{\alpha}{2}}^{(l)}(\eta)\right|\,\|(U\varphi)(\cdot,\varepsilon\eta)-\varphi(\cdot)\|_p\,d\eta. \end{aligned}\tag{3.3}$$

Further, since the integral of the Gauss–Weierstrass kernel is $\int_{\mathbb{R}^n} W(y;t)\,dy = 1$, it follows that

$$\|(U\varphi)(\cdot,\varepsilon\eta) - \varphi(\cdot)\|_p = \left\|\int_{\mathbb{R}^n} W(y;\varepsilon\eta)\left[\varphi(t-y) - \varphi(t)\right]dy\right\|_p$$

$$\leq \int_{\mathbb{R}^n} W(y;\varepsilon\eta)\,\|\varphi(t-y) - \varphi(t)\|_p\,dy$$

$$= \int_{|y|\leq\rho} W(y;\varepsilon\eta)\,\|\varphi(t-y) - \varphi(t)\|_p\,dy$$

$$+ \int_{|y|>\rho} W(y;\varepsilon\eta)\,\|\varphi(t-y) - \varphi(t)\|_p\,dy = i_1(\varepsilon\eta) + i_2(\varepsilon\eta). \tag{3.4}$$

Now we estimate the $i_1(\varepsilon\eta)$ and $i_2(\varepsilon\eta)$, separately.

As a result of (2.13) we have $i_1(\varepsilon\eta) \leq c_1\mu(\sqrt{\varepsilon\eta})$. On the other hand, denoting by $d\sigma(\theta)$ the area element of the unit n–sphere, S^{n-1}, we have

$$i_2(\varepsilon\eta) = \int_{|y|>\rho} W(y;\varepsilon\eta)\,\|\varphi(t-y) - \varphi(t)\|_p\,dy$$

$$\leq 2\,\|\varphi\|_p \int_{|y|>\rho} W(y;\varepsilon\eta)\,dy$$

$$\overset{(2.2)}{=} 2\,\|\varphi\|_p \int_{|y|>\rho} (4\pi\varepsilon\eta)^{-\frac{n}{2}}\, e^{-\frac{|y|^2}{4\varepsilon\eta}}\,dy, (t>0)$$

$$(\text{set } y = r\theta, \rho < r < \infty, \theta \in S^{n-1}; dy = r^{n-1}dr d\sigma(\theta))$$

$$= c_1(\varepsilon\eta)^{-\frac{n}{2}} \int_{\rho}^{\infty} r^{n-1} e^{-\frac{r^2}{4\varepsilon\eta}}\,dr$$

$$= c_2 \int_{\frac{\rho}{2\sqrt{\varepsilon\eta}}}^{\infty} t^{n-1} e^{-t^2}\,dt = c_2 \int_{\frac{\rho}{2\sqrt{\varepsilon\eta}}}^{\infty} t^{n-1} e^{-\frac{t^2}{2}} e^{-\frac{t^2}{2}}\,dt$$

$$\leq c_3 e^{-\frac{\rho^2}{8\varepsilon\eta}}.$$

The equality $\inf\limits_{\tau>0}\left(\tau e^{\frac{\delta}{\tau}}\right) = e\delta$ yields that $e^{-\frac{\delta}{\tau}} \leq \frac{1}{e\delta}\tau$, and therefore $i_2(\varepsilon\eta) \leq c_4\varepsilon\eta$, where c_4 does not depend on ε and η. Then we have,

$$\|(U\varphi)(\cdot,\varepsilon\eta) - \varphi(\cdot)\|_p \overset{(3.4)}{\leq} c_1\mu(\sqrt{\varepsilon\eta})) + c_4\varepsilon\eta$$

and hence,

$$\|D_\varepsilon^\alpha I^\alpha\varphi - \varphi\|_p \overset{(3.3)}{\leq} \int_0^\infty \left|K_{\frac{\alpha}{2}}^{(l)}(\eta)\right| \left(c\mu(\sqrt{\varepsilon\eta}) + c_4\varepsilon\eta\right)d\eta$$

$$(\text{we use } \mu(\sqrt{\varepsilon\eta}) \leq (1+\sqrt{\eta})\,\mu\left(\sqrt{\varepsilon}\right)\ )$$

$$\leq c_5\mu\left(\sqrt{\varepsilon}\right) \int_0^\infty \left|K_{\frac{\alpha}{2}}^{(l)}(\eta)\right| (\eta + \sqrt{\eta} + 1)\,d\eta. \tag{3.5}$$

Finally, using the condition $l > \frac{\alpha}{2} + 1$ and keeping in mind the asymptotic behavior of the function $K^{(l)}_{\frac{\alpha}{2}}(\eta)$ as $\eta \to \infty$ (see Lemma 2.2(b)) we get that the integral at the right-hand side of (3.5) converges, that is,

$$\|D^\alpha_\varepsilon I^\alpha \varphi - \varphi\|_p = O\left(\mu\left(\sqrt{\varepsilon}\right)\right) \text{ as } \varepsilon \to 0^+.$$

The proof of part (a) is complete. The proof of (3.2) follows in a similar way and is based on the following inequalities:

$$\|\mathfrak{D}^\alpha_\varepsilon J^\alpha \varphi - \varphi\|_p \le \int_0^\infty |K^{(l)}_{\alpha/2}(\eta)| \|(\mathrm{U}_{\mathrm{M}}\varphi)(\cdot;\varepsilon\eta) - \varphi(\cdot)\|_p dt$$

and

$$\|(\mathrm{U}_{\mathrm{M}}\varphi)(\cdot;\varepsilon\eta) - \varphi(\cdot)\|_p \le (1 - e^{-\varepsilon\eta})\|(\mathrm{U}\varphi)(\cdot;\varepsilon\eta)\|_p + \|(\mathrm{U}\varphi)(\cdot;\varepsilon\eta) - \varphi(\cdot)\|_p$$

□

Remark 3.2 *It is very interesting to solve an analogous problem by using the wavelet measure instead of the finite difference; see, e.g. [1, 4], where wavelet-like transforms are used in inversion formulas for potentials.*

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