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Generalization of the Gauss–Lucas theorem for bicomplex polynomials

Mahmood BIDKHAM*, **Sara AHMADI**
Faculty of Mathematics, Statistics, and Computer Science, Semnan University, Semnan, Iran

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Abstract: The aim of this paper is to extend the domain of the Gauss–Lucas theorem from the set of complex numbers to the set of bicomplex numbers. We also discuss a bicomplex version of another compact generalization of the Gauss–Lucas theorem.

Key words: Bicomplex polynomial, Gauss–Lucas theorem

1. Introduction

Corrado Segre published a paper [13] in 1892, in which he studied an infinite set of algebra whose elements he called bicomplex numbers. The work of Segre remained unnoticed for almost a century, but recently mathematicians have started taking interest in the subject and a new theory of special functions has started coming up[6, 9]. In this paper, we introduce the mathematical tools necessary to investigate the Gauss–Lucas theorem for bicomplex polynomials. We also discuss a bicomplex version of another compact generalization of the Gauss–Lucas theorem proved by Aziz and Rather [1] for complex polynomials.

Let $\mathbb{BC}$ denote the set of bicomplex numbers, i.e.

$$\mathbb{BC} = \{x_1 + ix_2 + j(x_3 + ix_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

with $i^2 = -1$, $j^2 = -1$ and $ij = ji$, and then we can write bicomplex number $Z = x_1 + ix_2 + j(x_3 + ix_4)$ as $z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}$. The addition and the multiplication of two bicomplex numbers are defined in the usual way. If we denote $e_1 = \frac{1+ij}{2}$, $e_2 = \frac{1-ij}{2}$, then the bicomplex number $Z = z_1 + jz_2$, $z_1, z_2 \in \mathbb{C}$, is uniquely represented as $(z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$. It can be easily verified that for every two bicomplex numbers $Z_1 = \alpha_1 e_1 + \beta_1 e_2, Z_2 = \alpha_2 e_1 + \beta_2 e_2$, we can write the following:

$$Z_1 + Z_2 = (\alpha_1 + \alpha_2)e_1 + (\beta_1 + \beta_2)e_2,$$

$$Z_1 Z_2 = (\alpha_1\alpha_2)e_1 + (\beta_1\beta_2)e_2.$$

If $Z_1, Z_2 \in \mathbb{BC}$ and $Z_1Z_2 = 1$, then each of the elements Z_1 and Z_2 is said to be the inverse of the other. An element that has an inverse is said to be invertible. One can easily verify that $Z = \alpha e_1 + \beta e_2 \in \mathbb{BC}$ is invertible iff $\alpha \neq 0$, $\beta \neq 0$; in this case, we have

$$Z^{-1} = \frac{1}{\alpha}e_1 + \frac{1}{\beta}e_2.$$

*Correspondence: mdbidkham@gmail.com
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Also, we can define the usual norm of $Z = z_1 + jz_2$ as

$$|Z| = \sqrt{|z_1|^2 + |z_2|^2}.$$

It is easy to prove that for any bicomplex number $Z = \alpha e_1 + \beta e_2$,

$$|Z| = \sqrt{\frac{|\alpha|^2 + |\beta|^2}{2}},$$

where $\alpha = z_1 - iz_2,\ \beta = z_1 + iz_2$.

Definition 1.1 *Letting $X_1, X_2 \subseteq \mathbb{C}$, we say that $X \subseteq \mathbb{BC}$ is a Cartesian set determined by X_1 and X_2 if*

$$X = X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = \alpha e_1 + \beta e_2, (\alpha, \beta) \in X_1 \times X_2\}.$$

A special Cartesian set in $\mathbb{BC}$, which is called a disk, is defined as follows:

Definition 1.2 *Let $a = a_1 + jb_1 = \alpha_1 e_1 + \beta_1 e_2$ where $a_1, b_1, \alpha_1, \beta_1 \in \mathbb{C}$, be a fixed point in $\mathbb{BC}$. We define the open disk $D(a; r_1, r_2)$ and closed disk $\overline{D}(a; r_1, r_2)$ with center a and radii r_1 and r_2 as follows:*

$$D(a; r_1, r_2) = \{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = \alpha e_1 + \beta e_2, |\alpha - \alpha_1| < r_1, |\beta - \beta_1| < r_2\},$$

$$\overline{D}(a; r_1, r_2) = \{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = \alpha e_1 + \beta e_2, |\alpha - \alpha_1| \leq r_1, |\beta - \beta_1| \leq r_2\}.$$

In linear space L, we call the intersection of all convex sets containing a given set A in L the convex hull of A, denoted by $H(A)$. If $c_k, 1 \leq k \leq n$ are nonnegative real numbers such that $\sum_{k=1}^{n} c_k = 1$, then $\alpha = \sum_{k=1}^{n} c_k \alpha_k$ is called a convex combination of $\alpha_1, ..., \alpha_n \in L$. One can easily verify that $H(A)$ consists precisely of all convex combination of elements of A [5, 12].

The well-known Gauss–Lucas theorem in complex analysis states that every critical point of a complex polynomial $p(z)$ lies in the convex hull of its zeros[8]. As a compact generalization of the Gauss–Lucas theorem, Aziz and Rather [1] proved the following result.

Theorem 1.3 *If all the zeros of complex polynomial $p(z)$ of degree $n \geq 2$ lie in the disk $D := \{z : |z - c| \leq r\}$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq 1$, all the zeros of the polynomial $p(Rz - c(R-1)) - \beta p(z)$ also lie in D.*

If $z_1, ..., z_n$ are n, not necessarily distinct, complex numbers, then the incomplete polynomials of degree $n-1$, associated with $z_1, ..., z_n$, are the polynomials $g_k(z) = \prod_{\substack{m=1 \\ m \neq k}}^{n} (z - z_m)$. In this direction we have the following result due to Díaz-Barrero and Egozcue [4].

Theorem 1.4 *Let $z_1, ..., z_n$ be n, not necessarily distinct, complex numbers and $\lambda_1, ..., \lambda_n$ be nonnegative real numbers such that $\sum_{k=1}^{n} \lambda_k = 1$. Then the polynomial $A_n^{\lambda}(z) = \sum_{k=1}^{n} \lambda_k g_k(z)$ has all its zeros in or on the convex* *hull* $H(\{z_1,...,z_n\})$ *of the zeros of* $A_n(z)=\prod_{k=1}^{n}(z-z_k)$, *where*

$$g_k(z)=\prod_{\substack{m=1\\m\neq k}}^{n}(z-z_m), \qquad 1\le k\le n.$$

2. Main results

To prove our main results, we need the following lemmas.

Lemma 2.1 *If* X_1 *and* X_2 *are convex sets in* $\mathbb{C}$, *then* $X=X_1\times_e X_2$ *is convex in* $\mathbb{BC}$ *[10].*

Lemma 2.2 *Letting*

$$A_1=\{\alpha_1,...,\alpha_n:\alpha_k\in\mathbb{C},1\le k\le n\},$$

$$A_2=\{\beta_1,...,\beta_m:\beta_l\in\mathbb{C},1\le l\le m\},$$

then

(i) $H(A_1\times_e A_2)=H(A_1)\times_e H(A_2)$.

(ii) $H(A_1\times_e \mathbb{C})=H(A_1)\times_e \mathbb{C}$.

(iii) $H(\mathbb{C}\times_e A_2)=\mathbb{C}\times_e H(A_2)$.

Proof (i) By Lemma 2.1, $H(A_1)\times_e H(A_2)$ is convex, and also $A_1\subseteq H(A_1)$ and $A_2\subseteq H(A_2)$; therefore,

$$H(A_1\times_e A_2)\subseteq H(A_1)\times_e H(A_2). \tag{2.1}$$

For the converse, we first show the following:

$$\{\alpha_1,...,\alpha_n\}\times_e H(\{\beta_1,...,\beta_m\})\subseteq H(A_1\times_e A_2). \tag{2.2}$$

Letting $Z^*\in\{\alpha_1,...,\alpha_n\}\times_e H(\{\beta_1,...,\beta_m\})$, then there exist nonnegative real numbers $c_l, 1\le l\le m$ with $\sum_{l=1}^{m}c_l=1$ such that

$$Z^*=\alpha_k e_1+(\sum_{l=1}^{m}c_l\beta_l)e_2,$$

for some $1\le k\le n$. Therefore,

$$Z^*=\sum_{l=1}^{m}c_l(\alpha_k e_1+\beta_l e_2)\in H(A_1\times_e A_2),$$

and hence

$$\{\alpha_1,...,\alpha_n\}\times_e H(\{\beta_1,...,\beta_m\})\subseteq H(A_1\times_e A_2).$$

Now, letting $Z \in H(A_1) \times_e H(A_2)$, one can find nonnegative real numbers $c_1, ..., c_n, d_1, ..., d_m$, with $\sum_{k=1}^{n} c_k = 1$ and $\sum_{l=1}^{m} d_l = 1$, such that

$$Z = (\sum_{k=1}^{n} c_k \alpha_k) e_1 + (\sum_{l=1}^{m} d_l \beta_l) e_2.$$

By (2.2),

$$\alpha_k e_1 + (\sum_{l=1}^{m} d_l \beta_l) e_2 \in H(A_1 \times_e A_2),$$

for all $1 \leq k \leq n$, and hence

$$\begin{aligned} \sum_{k=1}^{n} c_k (\alpha_k e_1 + (\sum_{l=1}^{m} d_l \beta_l) e_2) &= (\sum_{k=1}^{n} c_k \alpha_k) e_1 + (\sum_{l=1}^{m} d_l \beta_l) e_2 \\ &= Z \in H(A_1 \times_e A_2). \end{aligned}$$

Therefore,

$$H(A_1) \times_e H(A_2) \subseteq H(A_1 \times_e A_2), \tag{2.3}$$

and the result follows from (2.1) and (2.3).
(ii) It is obvious by Lemma 2.1 that $H(A_1) \times_e \mathbb{C}$ is convex and $A_1 \times_e \mathbb{C} \subseteq H(A_1) \times_e \mathbb{C}$; hence,

$$H(A_1 \times_e \mathbb{C}) \subseteq H(A_1) \times_e \mathbb{C}. \tag{2.4}$$

Letting $Z^* \in H(A_1) \times_e \mathbb{C}$, then it can be easily shown that there exist nonnegative real numbers $c_1, ..., c_n$, with $\sum_{k=1}^{n} c_k = 1$, and a complex number β such that

$$Z^* = (\sum_{k=1}^{n} c_k \alpha_k) e_1 + \beta e_2,$$

or

$$Z^* = \sum_{k=1}^{n} c_k (\alpha_k e_1 + \beta e_2) \in H(A_1 \times_e \mathbb{C}),$$

so we have

$$H(A_1) \times_e \mathbb{C} \subseteq H(A_1 \times_e \mathbb{C}), \tag{2.5}$$

and the result follows from (2.4) and (2.5). Using a similar argument, we can easily verify (iii). □

Lemma 2.3 *Let $Z_1, ..., Z_n$ be n bicomplex numbers and $Z_k = \alpha_k e_1 + \beta_k e_2$, $1 \leq k \leq n$; then $H(\{\alpha_1, ..., \alpha_n\}) \times_e H(\{\beta_1, ..., \beta_n\})$ is the smallest convex Cartesian set that contains $Z_1, ..., Z_n$.*

Proof Let $X = H(\{\alpha_1, ..., \alpha_n\}) \times_e H(\{\beta_1, ..., \beta_n\})$; then $Z_1, ..., Z_n \in X$, and by Lemma 2.1, X is convex. If $T = T_1 \times_e T_2$ is a convex Cartesian set that includes $Z_1, ..., Z_n$, then T_1 and T_2 are convex sets and

$$\alpha_1, ..., \alpha_n \in T_1 \qquad , \qquad \beta_1, ..., \beta_n \in T_2,$$

and hence

$$H(\{\alpha_1, ..., \alpha_n\}) \subseteq T_1 \qquad , \qquad H(\{\beta_1, ..., \beta_n\}) \subseteq T_2,$$

and it follows that $X \subseteq T$.

Let X be a set in $\mathbb{BC}$ and define functions $h_1 : X \to \mathbb{C}$ and $h_2 : X \to \mathbb{C}$ as follows:

$$\begin{aligned} h_1(z_1 + jz_2) &= z_1 - iz_2, \qquad z_1 + jz_2 \in X, \\ h_2(z_1 + jz_2) &= z_1 + iz_2, \qquad z_1 + jz_2 \in X. \end{aligned} \tag{2.6}$$

□

Lemma 2.4 *Let X be a set in $\mathbb{BC}$, and let h_1 and h_2 map X into X_1 and X_2, respectively. If X is an open set in $\mathbb{BC}$, then X_1 and X_2 are open sets in $\mathbb{C}$ [10].*

Lemma 2.5 *Let X be the open Cartesian set in $\mathbb{BC}$, which is determined by X_1 and X_2. Also let α_1, β_1 be points respectively in the closure of X_1, X_2. If $f_{e_1} : X_1 \to \mathbb{C}$, $f_{e_2} : X_2 \to \mathbb{C}$ are two complex functions such that*

$$\lim_{\alpha \to \alpha_1} f_{e_1}(\alpha) = a_1 \quad \text{and} \quad \lim_{\beta \to \beta_1} f_{e_2}(\beta) = b_1,$$

then $F : X \to \mathbb{BC}$ is defined by

$$F(Z) = F(\alpha e_1 + \beta e_2) := f_{e_1}(\alpha) e_1 + f_{e_2}(\beta) e_2, \quad \text{for } \alpha e_1 + \beta e_2 \in X,$$

which has the limit $A := a_1 e_1 + b_1 e_2$ at $Z_1 := \alpha_1 e_1 + \beta_1 e_2$.

Proof It is easy to verify that Z_1 is a point in the closure of X (see [10]). For $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that for $\alpha \in X_1$ and $\beta \in X_2$, the conditions $0 < |\alpha - \alpha_1| < \delta_1$ and $0 < |\beta - \beta_1| < \delta_2$ imply that $|f_{e_1}(\alpha) - a_1| < \varepsilon$ and $|f_{e_2}(\beta) - b_1| < \varepsilon$, respectively. Let

$$\delta := Min\{\delta_1, \delta_2\},$$

and $Z = \alpha e_1 + \beta e_2 \in X$ with $0 < |Z - Z_1| < \frac{\delta}{\sqrt{2}}$; then

$$|F(Z) - A| = \sqrt{\frac{|f_{e_1}(\alpha) - a_1|^2 + |f_{e_2}(\beta) - b_1|^2}{2}} < \varepsilon,$$

and it follows that $\lim_{Z \to Z_1} F(Z)$ exists and $\lim_{Z \to Z_1} F(Z) = A$. □

By using a similar argument as used in the proof of Lemma 2.5, we can prove the following lemma:

Lemma 2.6 *Let X be the open set in $\mathbb{R}$, R_1 be a point in the closure of X, and $f_{e_1} : X \to \mathbb{C}, f_{e_2} : X \to \mathbb{C}$ such that*

$$\lim_{R \to R_1} f_{e_1}(R) = a_1 \quad \text{and} \quad \lim_{R \to R_1} f_{e_2}(R) = b_1.$$

If $F : X \to \mathbb{BC}$ *is defined by*

$$F(R) = F(Re_1 + Re_2) := f_{e_1}(R)e_1 + f_{e_2}(R)e_2, \quad \text{for} \ \ R \in X,$$

then $\lim_{R \to R_1} F(Z)$ *exists and*

$$\lim_{R \to R_1} F(Z) = a_1 e_1 + b_1 e_2.$$

Let $P(Z) = \sum_{k=0}^{n} A_k Z^k$ be a bicomplex polynomial of degree n, with $Z = z_1 + jz_2 = \alpha e_1 + \beta e_2$ and bicomplex coefficients $A_k = \gamma_k e_1 + \delta_k e_2$, for $k = 0, 1, ..., n$. Then $Z^k = \alpha^k e_1 + \beta^k e_2$ and we can rewrite $P(Z)$ as

$$P(Z) = \sum_{k=0}^{n} (\gamma_k \alpha^k) e_1 + \sum_{k=0}^{n} (\delta_k \beta^k) e_2 =: \phi(\alpha) e_1 + \psi(\beta) e_2,$$

where $\phi(\alpha)$ and $\psi(\beta)$ are complex polynomials of degree at most n. For bicomplex polynomials we have the following result [7]:

Lemma 2.7 *(Analogue of the fundamental theorem of algebra for bicomplex polynomials) Consider a bicomplex polynomial* $P(Z) = \sum_{k=0}^{n} A_k Z^k$. *If all the coefficients* A_k *with the exception of the free term* $A_0 = \gamma_0 e_1 + \delta_0 e_2$ *are complex multiples of* e_1 *(respectively of* e_2*), but* A_0 *has* $\delta_0 \neq 0$ *(respectively* $\gamma_0 \neq 0$*), then* $P(Z)$ *has no roots. In all other cases,* $P(Z)$ *has at least one root.*

Lemma 2.8 *Let* X_1 *and* X_2 *be open sets in* $\mathbb{C}$. *If* $f_{e_1} : X_1 \longrightarrow \mathbb{C}$ *and* $f_{e_2} : X_2 \longrightarrow \mathbb{C}$ *are holomorphic functions in* $\mathbb{C}$ *on domains* X_1 *and* X_2*, respectively, then the function* $f : X_1 \times_e X_2 \longrightarrow \mathbb{BC}$ *defined as*

$$f(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2, \quad \forall z_1 + jz_2 \in X_1 \times_e X_2,$$

is $\mathbb{BC}$*-holomorphic on the open set* $X_1 \times_e X_2$ *and*

$$f'(z_1 + jz_2) = f'_{e_1}(z_1 - iz_2)e_1 + f'_{e_2}(z_1 + iz_2)e_2, \quad \forall z_1 + jz_2 \in X_1 \times_e X_2.$$

This lemma was proved by Charak et al. [2] (see also [3] and [11]).

Remark 2.9 *Let* $P(Z) = \sum_{k=0}^{n} A_k Z^k = \phi(\alpha)e_1 + \psi(\beta)e_2$ *be a bicomplex polynomial. In the above lemma, if we take* $X_1 = X_2 = \mathbb{BC}$*, then* $P(Z)$ *is* $\mathbb{BC}$*-holomorphic on* $\mathbb{BC}$ *and*

$$P'(Z) = P'(z_1 + jz_2) = \phi'(z_1 - iz_2)e_1 + \psi'(z_1 + iz_2)e_2 =: \phi'(\alpha)e_1 + \psi'(\beta)e_2. \tag{2.7}$$

Now we first prove the analogue of the Gauss–Lucas theorem and Theorem 1.4 for bicomplex polynomials, respectively.

Theorem 2.10 *(Analogue of Gauss–Lucas theorem) Let* $P(Z)$ *be a nonconstant bicomplex polynomial with at least one zero. Then every critical point of* $P(z)$ *lies in the convex hull of its zeros.*

Proof Let $P(Z) = \sum_{k=0}^{n} A_k Z^k =: \phi(\alpha)e_1 + \psi(\beta)e_2$. If at least one of the ϕ or ψ is a complex polynomial of degree one, then by using (2.7) and Lemma 2.7, $P'(Z)$ has no zeros and so we have nothing to prove. Assume that neither ϕ nor ψ is a complex polynomial of degree one and A is the set of distinct roots of $P(Z)$. By Lemma 2.7, $P(Z)$ has at least one zero and hence we should consider the following two cases:

Case 1. Let $\phi(\alpha)$ and $\psi(\beta)$ be complex polynomials of degree at least two. Let $A_1 = \{\alpha_1, ..., \alpha_k\}$ and $A_2 = \{\beta_1, ..., \beta_l\}$, with $k, l \leq n$, be the sets of distinct roots of ϕ and ψ, respectively. If $A = A_1 \times_e A_2$, then by Lemma 2.2,

$$H(A) = H(A_1) \times_e H(A_2).$$

If $Z^* = \alpha^* e_1 + \beta^* e_2 \in \mathbb{BC}$ such that $P'(Z^*) = 0$, then by (2.7),

$$\phi'(\alpha^*) = 0 \quad \text{and} \quad \psi'(\beta^*) = 0,$$

and hence, by applying the Gauss–Lucas theorem for ϕ and ψ, we have

$$\alpha^* \in H(A_1) \quad \text{and} \quad \beta^* \in H(A_2);$$

therefore, $Z^* \in H(A)$.
Case 2. Let $\phi \equiv 0$ (respectively $\psi \equiv 0$), and $A_1 = \mathbb{C}$, $A_2 = \{\beta_1, ..., \beta_l\}$, with $l \leq n$, be the sets of distinct roots of ϕ and ψ, respectively. Then $P'(Z) = \psi'(\beta)e_2$. If $Z^* = \alpha^* e_1 + \beta^* e_2 \in \mathbb{BC}$ such that $P'(Z^*) = 0$, then $\psi'(\beta^*) = 0$ and by the Gauss–Lucas theorem for ψ, we have $\beta^* \in H(A_2)$; hence, $Z^* \in \mathbb{C} \times_e H(A_2)$.

□

Theorem 2.11 *Let $Z_1, ..., Z_n$ be n, not necessarily distinct, bicomplex numbers where $Z_k = \alpha_k e_1 + \beta_k e_2$, for $k = 1, ..., n$, and $\lambda_1, ..., \lambda_n$ be nonnegative real numbers such that $\sum_{k=1}^{n} \lambda_k = 1$. Then the polynomial $A_n^{\lambda}(Z) = \sum_{k=1}^{n} \lambda_k G_k(Z)$ has all its zeros in or on $H(A)$, where $A := \{\alpha_k e_1 + \beta_l e_2 : 1 \leq k \leq n, 1 \leq l \leq n\}$, and*

$$G_k(Z) = \prod_{\substack{m=1 \\ m \neq k}}^{n} (Z - Z_m), \qquad 1 \leq k \leq n.$$

Proof Letting $Z = z_1 + jz_2 = \alpha e_1 + \beta e_2$ be a bicomplex number, we have

$$\begin{aligned}
\lambda_k G_k(Z) &= \lambda_k \prod_{\substack{m=1 \\ m \neq k}}^{n} (Z - Z_m) \\
&= (\lambda_k e_1 + \lambda_k e_2) \prod_{\substack{m=1 \\ m \neq k}}^{n} \big((\alpha e_1 + \beta e_2) - (\alpha_m e_1 + \beta_m e_2)\big) \qquad (e_1 + e_2 = 1) \\
&= \big(\lambda_k \prod_{\substack{m=1 \\ m \neq k}}^{n} (\alpha - \alpha_m)\big)e_1 + \big(\lambda_k \prod_{\substack{m=1 \\ m \neq k}}^{n} (\beta - \beta_m)\big)e_2 \\
&= \lambda_k g_k(\alpha)e_1 + \lambda_k h_k(\beta)e_2,
\end{aligned}$$

where $g_k(\alpha) = \prod_{\substack{m=1 \\ m\neq k}}^{n} (\alpha - \alpha_m)$ and $h_k(\beta) = \prod_{\substack{m=1 \\ m\neq k}}^{n} (\beta - \beta_m)$.

Hence,

$$\begin{aligned} A_n^{\lambda}(Z) &= \sum_{k=1}^{n} \lambda_k G_k(Z) \\ &= \big(\sum_{k=1}^{n} \lambda_k g_k(\alpha)\big)e_1 + \big(\sum_{k=1}^{n} \lambda_k h_k(\beta)\big)e_2 \\ &= \phi_n^{\lambda}(\alpha)e_1 + \psi_n^{\lambda}(\beta)e_2, \end{aligned} \tag{2.8}$$

where $\phi_n^{\lambda}(\alpha) = \sum_{k=1}^{n} \lambda_k g_k(\alpha)$ and $\psi_n^{\lambda}(\beta) = \sum_{k=1}^{n} \lambda_k h_k(\beta)$.

If $W = w_1 + jw_2 = ae_1 + be_2$ is a zero of $A_n^{\lambda}(Z)$, then by (2.8) we have

$$\phi_n^{\lambda}(a) = 0, \qquad \psi_n^{\lambda}(b) = 0,$$

and by Theorem 1.4,

$$a \in H(\{\alpha_1, ..., \alpha_n\}), \qquad b \in H(\{\beta_1, ..., \beta_n\}),$$

and hence $W \in H(\{\alpha_1, ..., \alpha_n\}) \times_e H(\{\beta_1, ..., \beta_n\})$, but by using (i) of Lemma 2.2, we have

$$H(\{\alpha_1, ..., \alpha_n\}) \times_e H(\{\beta_1, ..., \beta_n\}) = H(\{\alpha_1, ..., \alpha_n\}) \times_e \{\beta_1, ..., \beta_n\}),$$

and this completes the proof of Theorem 2.11. □

Next, as an extension of Theorem 1.3 for bicomplex polynomials, we prove the following result.

Theorem 2.12 *If all the zeros of bicomplex polynomial* $P(Z) = \sum_{k=0}^{n} A_k Z^k =: \phi(\alpha)e_1 + \psi(\beta)e_2$ *of degree* n *lie in the disk* $\overline{D}(C; r_1, r_2)$ *where* $C = c_1e_1 + c_2e_2 \in \mathbb{BC}$ *and* A_k *is invertible for some* $2 \leq k \leq n$, *then for any bicomplex number* $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \in D(0; 1, 1)$ *and* $R \geq 1$, *all the zeros of the polynomial* $P(RZ - C(R-1)) - \lambda P(Z)$ *also lie in* $\overline{D}(C; r_1, r_2)$.

Proof Since A_k is invertible for some $2 \leq k \leq n$, it follows that ϕ and ψ are polynomials of degree at least 2. Let $D_1 = \{\alpha \in \mathbb{C} : |\alpha - c_1| \leq r_1\}$ and $D_2 = \{\beta \in \mathbb{C} : |\beta - c_2| \leq r_2\}$. Since $P(Z)$ has all its zeros in $\overline{D}(C; r_1, r_2) = D_1 \times_e D_2$, hence ϕ and ψ have all their zeros in D_1 and D_2, respectively. For any $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \in D(0; 1, 1)$ and $R \geq 1$, by applying Theorem 1.3, all the zeros of $\phi(R\alpha - c_1(R-1)) - \lambda_1\phi(\alpha)$ and $\psi(R\beta - c_2(R-1)) - \lambda_2\psi(\beta)$ lie in D_1 and D_2, respectively; hence,

$$\begin{aligned} &P(RZ + C(R-1)) - \lambda P(Z) = \\ &(\phi(R\alpha - c_1(R-1)) - \lambda_1\phi(\alpha))e_1 + (\psi(R\beta - c_2(R-1)) - \lambda_2\psi(\beta))e_2 \end{aligned}$$

has all its zeros in $\overline{D}(C; r_1, r_2)$. This completes the proof of Theorem 2.12. □

By Theorem 2.12, for $\lambda = e_1 + e_2$, we can obtain the following result.

Proposition 2.13 *If all the zeros of bicomplex polynomial* $P(Z)=\sum_{k=0}^{n}A_kZ^k=:\phi(\alpha)e_1+\psi(\beta)e_2$ *of degree* n *lie in the disk* $\overline{D}(C;r_1,r_2)$ *where* $C=c_1e_1+c_2e_2\in\mathbb{BC}$ *and* A_k *is invertible for some* $2\leq k\leq n$, *then all the zeros of* $P'(Z)$ *also lie in* $\overline{D}(C;r_1,r_2)$.

Proof For each $\alpha\neq c_1,\beta\neq c_2$, and $R\neq 1$, we have

$$\begin{aligned}\frac{P(RZ-C(R-1))-P(Z)}{(R-1)(Z-C)}&=\frac{\phi(R\alpha-c_1(R-1))-\phi(\alpha)}{(R-1)(\alpha-c_1)}e_1\\&+\frac{\psi(R\beta-c_2(R-1))-\psi(\beta)}{(R-1)(\beta-c_2)}e_2,\end{aligned}\tag{2.9}$$

and also

$$\lim_{R\to 1}\frac{\phi(R\alpha-c_1(R-1))-\phi(\alpha)}{(R-1)(\alpha-c_1)}=\phi'(\alpha),$$
$$\lim_{R\to 1}\frac{\psi(R\beta-c_2(R-1))-\psi(\beta)}{(R-1)(\beta-c_2)}=\psi'(\beta),$$

and hence by Lemma 2.6 and (2.9) we have

$$\begin{aligned}\lim_{R\to 1}\frac{P(RZ-C(R-1))-P(Z)}{(R-1)(Z-C)}&=\phi'(\alpha)e_1+\psi'(\beta)e_2\\&=P'(Z).\end{aligned}$$

Also, by Theorem 2.12, for $\lambda=e_1+e_2$ all the zeros of $P(RZ-C(R-1))-P(Z)$ lie in $\overline{D}(C;r_1,r_2)$; therefore, $P'(Z)$ has all its zeros in $\overline{D}(C;r_1,r_2)$, and this completes the proof of Proposition 2.13.

□

Taking $C=0$ in Theorem 2.12, we have the following result.

Corollary 2.14 *If all the zeros of bicomplex polynomial* $P(Z)=\sum_{k=0}^{n}A_kZ^k=:\phi(\alpha)e_1+\psi(\beta)e_2$ *of degree* n *lie in the disk* $\overline{D}(0;r_1,r_2)$ *and* A_k *is invertible for some* $2\leq k\leq n$, *then for every bicomplex number* $\lambda=\lambda_1e_1+\lambda_2e_2\in D(0;1,1)$ *and* $R\geq 1$, *all the zeros of the polynomial* $P(RZ)-\lambda P(Z)$ *also lie in* $\overline{D}(0;r_1,r_2)$.

References

[1] Aziz A, Rather NA. On an inequality of S. Bernstein and the Gauss-Lucas theorem. In: Rassias TM, editor. Analytic and Geometric Inequalities and Applications. Dordrecht, the Netherlands: Kluwer Academic Publishers, 1999, pp. 29-35.

[2] Charak KS, Rochon D. On factorization of bicomplex meromorphic functions. In: Sabadini I, Shapiro M, Sommen F, editors. Hypercomplex Analysis. Trends in Mathematics. Basel, Switzerland: Birkhäuser Verlag, 2008, pp. 55-68.

[3] Charak KS, Rochon D, Sharma N. Normal families of bicomplex holomorphic functions. Fractals 2009; 17: 257-268.

[4] Díaz-Barrero JL, Egozcue JJ. A generalization of the Gauss-Lucas theorem. Czech Math J 2008; 58: 481-486.

[5] Frenk JBG, Kassay G. Handbook of Generalized Convexity and Generalized Monotonicity. New York, NY, USA: Springer-Verlag, 2005.

[6] Luna-Elizarraras ME, Shapiro M, Struppa DC. On Clifford analysis for holomorphic mappings. Adv Geom 2014; 14: 413-426.

[7] Luna-Elizarraras ME, Shapiro M, Struppa DC, Vajiac A. Bicomplex numbers and their elementary functions. CUBO 2012; 14: 61-80.

[8] Marden M. Geometry of Polynomials. Providence, RI, USA: American Mathematical Society, 1966.

[9] Pogorui AA, Rodriguez-Dagnino RM. On the set of zeros of bicomplex polynomials. Complex Var Elliptic 2006; 51: 725-730.

[10] Price GB. An Introduction to Multicomplex Spaces and Functions. Monographs and Textbooks in Pure and Applied Mathematics. New York, NY, USA: Marcel Dekker, 1991.

[11] Riley JD. Contributions to the theory of functions of a bicomplex variable. Tohoku Math J 1953; 2: 132-165.

[12] Roberts AW, Varberg DE. Convex Functions. New York, NY, USA: Academic Press, 1973.

[13] Segre C. Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. Math Ann 1892; 40: 413-467 (in Italian).