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Evaluation of Euler-like sums via Hurwitz zeta values

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Abstract: In this paper we collect two generalizations of harmonic numbers (namely generalized harmonic numbers and hyperharmonic numbers) under one roof. Recursion relations, closed-form evaluations, and generating functions of this unified extension are obtained. In light of this notion we evaluate some particular values of Euler sums in terms of odd zeta values. We also consider the noninteger property and some arithmetical aspects of this unified extension.

Key words: Harmonic numbers, hyperharmonic numbers, generalized harmonic numbers, Euler sums, multiple zeta functions, Riemann zeta function, Hurwitz zeta function.

1. Introduction

The nth harmonic number is defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}),$$

where the empty sum H_0 is conventionally understood to be zero. Harmonic numbers are a longstanding subject of study and they are significant in various branches of analysis and number theory. These numbers are closely related to the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\mathrm{Re}(s) > 1)$$

and appear in the expressions of miscellaneous special functions.

Among many other generalizations we are interested in two famous generalizations of harmonic numbers, namely generalized harmonic numbers and hyperharmonic numbers.

Generalized harmonic numbers: The generalized nth harmonic number of order m is defined by the nth partial sum of the Riemann zeta function $\zeta(m)$ as:

$$H_n^{(m)} := \sum_{k=1}^{n} \frac{1}{k^m} \quad (m \in \mathbb{N}).$$

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The numerators of the special case $H_n^{(2)}$ are known as Wolstenholme numbers (see OEIS A007406).

Hyperharmonic numbers: Starting with $h_n^{(0)} = \frac{1}{n}$ $(n \in \mathbb{N})$, the nth hyperharmonic number $h_n^{(r)}$ of order r is defined by (see [9])

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)} \quad (r \in \mathbb{N}).$$

The harmonic number H_n is never an integer except for H_1; this is a classical result of Theisinger [19, 21]. Considering hyperharmonic numbers, Mező [19] proved that if $r = 2$ or $r = 3$, these numbers are never integers except the trivial case when $n = 1$. He conjectured that this is always the case, i.e. the hyperharmonic numbers of order r are never integers except when $n = 1$. This conjecture was justified for a class of pairs (n, r) by Ait-Amrane and Belbachir [1, 2] and Cereceda [8]. Very recently Göral and Sertbaş [14] extended the current results for large orders; considering primes in short intervals, they proved that almost all hyperharmonic numbers are not integers.

In this work we define the generalized hyperharmonic numbers as

$$H_n^{(p,r)} := \sum_{k=1}^{n} H_k^{(p,r-1)}$$

where

$$H_n^{(p,1)} := H_n^{(p)}.$$

Observing $H_n^{(1,r)} = h_n^{(r)}$, we see that the generalized hyperharmonic numbers are unified extensions of both generalized harmonic numbers and hyperharmonic numbers. Thus, we collect these two different generalizations under one roof:

$$\begin{tikzcd}
& H_n^{(p,r)} & \\
H_n^{(p)} \arrow[ur] & & h_n^{(r)} \arrow[ul] \\
& H_n \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Unified extension diagram of the generalized hyperharmonic numbers

In what follows we are going to study the properties of these numbers. We find recursion and closed-form formulas for them, and we also study Euler-like sums containing generalized hyperharmonics. For example, using these methods one can evaluate the following two series in terms of odd Riemann zeta values and the powers of π as:

$$\sum_{n=1}^{\infty} H_n^{(2)} \zeta(4,n) = \zeta(3)^2 + \frac{2\pi^2\zeta(3)}{3} - \frac{15\zeta(5)}{2} - \frac{\pi^6}{2835}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n^{(2,2)} \zeta(5,n) \quad &= \quad \frac{9}{4}\zeta(3)^2 + \frac{1}{180}\pi^4(4\zeta(3)+1) \\
&\qquad + \frac{5}{12}\pi^2(\zeta(3)+2\zeta(5)) - 10\zeta(7) - \frac{21}{4}\zeta(5) - \frac{5\pi^6}{1512},
\end{aligned}$$

where $\zeta(s,a)$ is the Hurwitz (or generalized) zeta function defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^{s}}\qquad\left(\mathrm{Re}(s)>1;\ a\notin\mathbb{Z}_0^-\right),$$

where $\mathbb{Z}_0^-:=\{0,-1,-2,...\}$.

Besides, in the final section, we study the noninteger property of these numbers and, more generally, the divisibility properties of them.

2. Some basic recursions

In this section we get some recurrence relations that we use in the next section to compute some special values of the Euler-like sums of generalized hyperharmonic numbers.

Having the definition of the generalized hyperharmonic numbers, we can find recursions besides the trivial

$$H_n^{(p,r)}=H_{n-1}^{(p,r)}+H_n^{(p,r-1)}. \tag{1}$$

These read as

$$\begin{aligned}H_n^{(p,2)}&=(n+1)H_n^{(p,1)}-H_n^{(p-1,1)}, \qquad (2)\\ H_n^{(p,3)}&=\frac{n(n+1)}{2}H_n^{(p,1)}+H_n^{(p,2)}+\frac{1}{2}H_n^{(p-1,1)}-\frac{1}{2}H_n^{(p-2,1)}-H_n^{(p-1,2)},\\ H_n^{(p,4)}&=\frac{n(n+1)(2n+1)}{12}H_n^{(p,1)}+\frac{n(n+1)}{4}H_n^{(p,1)}+\frac{1}{6}H_n^{(p-1,1)}+H_n^{(p,3)}\\ &\quad+\frac{1}{2}H_n^{(p-1,2)}-\frac{1}{6}H_n^{(p-3,1)}-\frac{1}{2}H_n^{(p-2,2)}-H_n^{(p-1,3)}.\end{aligned}$$

The proofs are easy exercises; for example:

$$\begin{aligned}H_n^{(p,2)}&=H_1^{(p,1)}+\cdots+H_n^{(p,1)}\\ &=\frac{1}{1^p}+\left(\frac{1}{1^p}+\frac{1}{2^p}\right)+\cdots+\left(\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}\right)\\ &=\sum_{k=1}^{n}(n-k+1)\frac{1}{k^p}=nH_n^{(p,1)}-H_n^{p-1}+H_n^{(p,1)}\\ &=(n+1)H_n^{(p,1)}-H_n^{(p-1,1)}.\end{aligned}$$

Proofs of the remaining two recursions are similar but need more computation.

These identities show that for a general r we probably cannot find such a simple expression as the Conway–Guy formula (see [9])

$$H_n^{(1,r)}=h_n^{(r)}=\binom{n+r-1}{r-1}\left(H_{n+r-1}-H_{r-1}\right) \tag{3}$$

for the hyperharmonic numbers, because we cannot reduce the (p,r) parameter pair to one parameter as in (3) where r was reduced to $r=1$.

3. Euler sums of generalized hyperharmonic numbers

Euler discovered the following invaluable formula (see [5] and [13]):

$$2\zeta_H(m) = (m+2)\,\zeta(m+1) - \sum_{n=1}^{m-2} \zeta(m-n)\,\zeta(n+1), \quad m = 2, 3, \ldots \tag{4}$$

where $\zeta_H(m)$ is the harmonic zeta function defined by

$$\zeta_H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \quad (\mathrm{Re}(s) > 1).$$

Recently Alkan [3] showed that real numbers and certain types of log-sine integrals can be strongly approximated by linear combinations of special values of the harmonic zeta function.

Mező and Dil [12] showed that the Euler-like sums of hyperharmonic numbers

$$\zeta_{h^{(r)}}(m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m}$$

can be expressed in terms of series of Hurwitz zeta function values. Later Dil and Boyadzhiev [10] generalized this to the multiple Hurwitz zeta series. Kamano [16] investigated the complex variable function $\zeta_{h^{(r)}}(s)$ and proved that it can be meromorphically continued to the whole complex plane.

In this paper we consider the Euler-like sums of the form

$$\zeta_{H^{(p,r)}}(m) := \sum_{n=1}^{\infty} \frac{H_n^{(p,r)}}{n^m}.$$

We will see in the following sections that such sums are not just connected to the Hurwitz zeta function but they help to find closed form evaluations of some specific Hurwitz zeta sums containing generalized harmonic numbers.

3.1. Asymptotics of $H_n^{(p,r)}$

To be able to determine the parameters m, p, and r for which the series $\zeta_{H^{(p,r)}}(m)$ converges we need the asymptotic behavior of $H_n^{(p,r)}$. In this short subsection we deal with this question.

It is well known that

$$H_n^{(1,1)} = H_n = O(\log n),$$

and it is trivial that

$$H_n^{(p,1)} = O(1) \quad (p > 1). \tag{5}$$

Dil and Mező [12] proved that

$$H_n^{(1,r)} = h_n^{(r)} = O(n^{r-1}\log n) \quad (r \geq 1).$$

The following lemma completes the description of the asymptotic behavior of the sequence $H_n^{(p,r)}$.

Lemma 1 *For all fixed $p>1$ and $r>1$ we have*

$$H_n^{(p,r)} = O(n^{r-1}) \quad \text{as } n \text{ tends to } \infty.$$

Proof Since $p>1$ we can apply (2) and (5) to obtain

$$H_n^{(p,2)} = O\left((n+1)\zeta(p) - H_n^{(p-1,1)}\right) = O(n),$$

as it follows from the above-mentioned asymptotics. For $r=2$ the statement holds. The result now follows by induction on r and using the basic recurrence (1). □

Gathering all the asymptotics together we have the convergence test for the Dirichlet series of generalized hyperharmonics.

Corollary 2 *For all $p \geq 1$ and $r \geq 1$ the series*

$$\zeta_{H^{(p,r)}}(m) = \sum_{n=1}^{\infty} \frac{H_n^{(p,r)}}{n^m}$$

is convergent whenever $m > r$.

This knowledge permits us to generalize the result of Dil and Boyadzhiev [10] as:

Theorem 3 *For a positive integer p and r and a positive real number m such that $m>r$, we have*

$$\zeta_{H^{(p,r)}}(m) = \sum_{n=1}^{\infty} H_n^{(p,r-1)} \zeta(m,n). \tag{6}$$

This result can also be extended to the multiple sums as

$$\zeta_{H^{(p,r)}}(m) = \sum_{n=1}^{\infty} H_n^{(p,r-k-1)} \sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_k < \infty} \zeta(m, i_k), \tag{7}$$

where

$$\sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_k < \infty} \zeta(m, i_k) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_k=i_{k-1}}^{\infty} \zeta(m, i_k).$$

Proof From Lemma 1 we can infer, when $m>r$, that

$$\sum_{n=1}^{\infty} \frac{H_n^{(p,r)}}{n^m} < \infty.$$

This convergence implies the absolute convergence. Hence, we can reorder the terms of the series $\zeta_{H^{(p,r)}}(m)$ as

$$\begin{aligned}
\zeta_{H^{(p,r)}}(m) &= \frac{H_1^{(p,r-1)}}{1^m} + \frac{H_1^{(p,r-1)} + H_2^{(p,r-1)}}{2^m} + \frac{H_1^{(p,r-1)} + H_2^{(p,r-1)} + H_3^{(p,r-1)}}{3^m} + \\
&\quad \cdots + \frac{H_1^{(p,r-1)} + H_2^{(p,r-1)} + H_3^{(p,r-1)} + \cdots + H_k^{(p,r-1)}}{k^m} + \cdots \\
&= \sum_{n=1}^{\infty} H_n^{(p,r-1)} \sum_{s=0}^{\infty} \frac{1}{(n+s)^m},
\end{aligned}$$

and this completes the proof of the first equation. Similarly, after one more step we get

$$\zeta_{H^{(p,r)}}(m)=\sum_{n=1}^{\infty}H_n^{(p,r-2)}\sum_{i_1=n}^{\infty}\zeta(m,i_1).$$

Using a similar argument, after k steps we obtain the second equation. □

In particular, we have the following result.

Corollary 4 *For the case $r=1$ in (6) we have*

$$\zeta_{H^{(p,1)}}(m)=\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}\right)}{n^m}=\sum_{n=1}^{\infty}\frac{\zeta(m,n)}{n^p}.$$

Also, for the case $r=2$, we have

$$\zeta_{H^{(p,2)}}(m)=\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}H_k^{(p)}}{n^m}=\sum_{n=1}^{\infty}H_n^{(p)}\zeta(m,n).$$

In general, we can give the following equalities:

Setting $k=r-1$ in (7) gives

$$\zeta_{H^{(p,r)}}(m)=\sum_{n=1}^{\infty}\frac{\sum_{n\leq i_1\leq i_2\leq\cdots\leq i_{r-1}<\infty}\zeta(m,i_{r-1})}{n^p}$$

and also by setting $k=r-2$ we get

$$\zeta_{H^{(p,r)}}(m)=\sum_{n=1}^{\infty}H_n^{(p)}\sum_{n\leq i_1\leq i_2\leq\cdots\leq i_{r-2}<\infty}\zeta(m,i_{r-2}).$$

3.2. Some closed-form evaluations of $\zeta_{H^{(p,r)}}(m)$

In this section we show how to evaluate sums of the form

$$\zeta_{H^{(p,r)}}(m)=\sum_{n=1}^{\infty}\frac{H_n^{(p,r)}}{n^m}$$

at least for small r. Recall that such sums are convergent when $m\geq r+1$. We use the set of recurrences (2) and the multiple zeta functions

$$\zeta(m|p)=\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}}{n^m}.$$

(We remark that in place of $\zeta(m|p)$ the notation $\zeta(m,p)$ is the standard. However, this would lead to confusion here since this notation coincides with that of the Hurwitz zeta function.)

The theory of multiple zeta functions is studied by a large number of authors; see the website "References on multiple zeta values and Euler sums by M. Hoffman" (http://www.usna.edu/Users/math/meh/biblio.html) for a set of articles. For comprehensive literature see [22].

3.2.1. Some concrete values of $\zeta_{H^{(p,1)}}(m)$ and $\zeta_{H^{(p,2)}}(m)$

Considering Corollary 4 we have the case $r=1$ as

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^m}=\sum_{n=1}^{\infty}\frac{\zeta(m,n)}{n^p}.$$

The left-hand side is a multiple zeta sum, namely

$$\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^m}=\sum_{n=1}^{\infty}\frac{H_{n-1}^{(p)}}{n^m}+\sum_{n=1}^{\infty}\frac{1}{n^{m+p}}=\zeta(m|p)+\zeta(m+p).$$

Hence, $\zeta_{h^{(p,1)}}(m)$ can be written in terms of multiple zeta and zeta values as

$$\zeta_{H^{(p,1)}}(m)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^m}=\sum_{n=1}^{\infty}\frac{\zeta(m,n)}{n^p}=\zeta(m|p)+\zeta(m+p). \tag{8}$$

Again the case $r=2$ can be seen from Corollary 4 as

$$\zeta_{H^{(p,2)}}(m)=\sum_{n=1}^{\infty}\frac{\sum_{k=1}^{n}H_k^{(p)}}{n^m}=\sum_{n=1}^{\infty}H_n^{(p)}\zeta(m,n).$$

This also reduces to multiple zeta values if we use identity (2):

$$\sum_{k=1}^{n}H_k^{(p)}=H_n^{(p,2)}=nH_n^{(p)}+H_n^{(p)}-H_n^{(p-1)}.$$

From this it follows that

$$\zeta_{H^{(p,2)}}(m)=\sum_{n=1}^{\infty}H_n^{(p)}\zeta(m,n)=\zeta_{H^{(p,1)}}(m-1)+\zeta_{H^{(p,1)}}(m)-\zeta_{H^{(p-1,1)}}(m),$$

or

$$\zeta_{H^{(p,2)}}(m)=\sum_{n=1}^{\infty}H_n^{(p)}\zeta(m,n)=\zeta(m-1|p)-\zeta(m|p-1)+\zeta(m|p)+\zeta(m+p)$$

in terms of multiple zeta values. The Euler sum $\zeta(m|p)$ can be evaluated in terms of the classical zeta function for some particular values [4, 6, 18]. For example, taking $(m,p)=(4,2)$, we have

$$\zeta_{H^{(2,2)}}(4)=\sum_{n=1}^{\infty}H_n^{(2)}\zeta(4,n)=\zeta(3)^2+\frac{2\pi^2\zeta(3)}{3}-\frac{15\zeta(5)}{2}-\frac{\pi^6}{2835},$$

and for $(m,p)=(4,3)$ we have

$$\zeta_{H^{(3,2)}}(4)=\sum_{n=1}^{\infty}H_n^{(3)}\zeta(4,n)=18\zeta(7)+\frac{\pi^6}{1134}-\frac{\zeta(3)^2}{2}-\frac{5\pi^2\zeta(5)}{3}.$$

3.2.2. Some concrete values of $\zeta_{H^{(p,3)}}(m)$

Now we express $\zeta_{H^{(p,3)}}(m)$ in terms of multiple zeta values. By using our recurrence (2) we easily find that

$$\zeta_{H^{(p,3)}}(m) = \frac{1}{2}\left[\zeta_{H^{(p,1)}}(m-2) + \zeta_{H^{(p,1)}}(m-1)\right] + \zeta_{H^{(p,2)}}(m)$$
$$+\frac{1}{2}\left[\zeta_{H^{(p-1,1)}}(m) - \zeta_{H^{(p-2,1)}}(m)\right] - \zeta_{H^{(p-1,2)}}(m).$$

This can be written in terms of multiple zeta values via the corresponding formulas for $\zeta_{H^{(p,1)}}(m)$ and for $\zeta_{H^{(p,2)}}(m)$. For any $m \geq r+1$ we have

$$\zeta_{H^{(p,3)}}(m) = \frac{1}{2}\zeta(m-2|p) + \frac{3}{2}\zeta(m-1|p) + \zeta(m|p) + \frac{1}{2}\zeta(m|p-2) - \frac{3}{2}\zeta(m|p-1)$$
$$-\zeta(m-1|p-1) + \zeta(m+p).$$

We give two explicit evaluations:

$$\zeta_{H^{(2,3)}}(4) = \zeta(3)^2 + \left(\frac{1}{2}+\pi^2\right)\zeta(3) - \frac{45}{4}\zeta(5) - \frac{\pi^4}{240} - \frac{\pi^6}{2835},$$

$$\zeta_{H^{(2,3)}}(5) = \frac{9}{4}\zeta(3)^2 + \frac{1}{180}\pi^4(4\zeta(3)+1) + \frac{5}{12}\pi^2(\zeta(3)+2\zeta(5))$$
$$-10\zeta(7) - \frac{21}{4}\zeta(5) - \frac{5\pi^6}{1512}.$$

Some of the necessary multiple zeta values can be found in [4, 6, 7, 18, 22] or can be deduced by standard techniques [19].

4. Closed-form evaluations and combinatorial expressions

The well-known generating function of the generalized harmonic numbers is given by the polylogarithm function $Li_p(x)$ (see [22]):

$$\frac{Li_p(x)}{1-x} = \sum_{n=1}^{\infty} H_n^{(p)} x^n.$$

From this we obtain the ordinary generating function of the generalized hyperharmonic numbers as:

$$\frac{Li_p(x)}{(1-x)^r} = \sum_{n=1}^{\infty} H_n^{(p,r)} x^n. \tag{9}$$

The closed form of the nth derivative of the polylogarithm in terms of the Stirling numbers of the first kind is evaluated as follows:

Proposition 5 *We have*

$$z^n \frac{d^n}{dz^n} Li_p(z) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] (-1)^{n-k} Li_{p-k}(z),$$

where $\left[\begin{array}{c} n \\ k \end{array}\right]$ *denotes the Stirling numbers of the first kind.*

Proof Considering the equation

$$\frac{d}{dz}Li_p(z) = \frac{1}{z}Li_{p-1}(z),$$

the proof can be completed by induction on n. □

Now we give a recursion formula for $H_n^{(p,r)}$ in terms of r.

Proposition 6 *For a nonnegative integer r and a positive integer s we have*

$$H_n^{(p,r+s)} = \sum_{j=1}^{n}\binom{n-j+s-1}{s-1}H_j^{(p,r)}.$$

Proof Using the generating function (9) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}H_n^{(p,r+s)}z^n &= \frac{Li_p(z)}{(1-z)^r}\times\frac{1}{(1-z)^s} = \sum_{n=1}^{\infty}H_n^{(p,r)}z^n\times\sum_{n=0}^{\infty}\binom{n+s-1}{s-1}z^n \\ &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\binom{n-j+s-1}{s-1}H_j^{(p,r)}\right)z^n.\end{aligned}$$

Comparison of coefficients gives the desired result. □

If we consider the special case $r=0$ we get

$$H_n^{(p,s)} = \sum_{j=1}^{n}\binom{n-j+s-1}{s-1}\frac{1}{j^p}$$

where $H_n^{(p,0)} = \frac{1}{n^p}$. The case $r=0$ and $p=1$ is given in [11].

Proposition 7 *We have the following expression for $n>1$:*

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{k+n}{k}H_{k+n}^{(p,r)}z^k &= \frac{1}{(1-z)^{r+n}}\sum_{j=1}^{n}\binom{r+n-j-1}{r-1}\frac{\left(1-\frac{1}{z}\right)^j}{j!} \\ &\quad\times\sum_{s=1}^{j}\left[\begin{matrix}j\\s\end{matrix}\right](-1)^s Li_{p-s}(z).\end{aligned}$$

Proof The Leibnitz formula yields that

$$\begin{aligned}&\frac{d^n}{dz^n}\left(Li_p(z)\times\frac{1}{(1-z)^r}\right) \\ &= \sum_{j=1}^{n}\binom{n}{j}\frac{1}{z^j}\sum_{s=1}^{j}\left[\begin{matrix}j\\s\end{matrix}\right](-1)^{j-s}Li_{p-s}(z)\binom{r+n-j-1}{r-1}\frac{(n-j)!}{(1-z)^{r+n-j}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{d^n}{dz^n}\left(\frac{Li_p(z)}{(1-z)^r}\right) &= \sum_{k=n}^{\infty} k(k-1)(k-2)\ldots(k-n+1)\,H_k^{(p,r)}z^{k-n} \\ &= \sum_{k=0}^{\infty}\binom{k+n}{k} n!H_{k+n}^{(p,r)}z^k.\end{aligned}$$

These two evaluations yield the result. □

Remark 8 *As a special case for $r=1$ we get the following expression:*

$$\sum_{k=0}^{\infty}\binom{k+n}{k}H_{k+n}^{(p)}z^k = \frac{1}{(1-z)^{n+1}}\sum_{j=1}^{n}\frac{\left(1-\frac{1}{z}\right)^j}{j!}\sum_{s=1}^{j}\left[\begin{matrix} j \\ s\end{matrix}\right](-1)^s Li_{p-s}(z).$$

4.1. A connection with Bernoulli polynomials

In this short subsection we are going to present the generating function for the Bernoulli polynomials $B_p(x)$ where the argument is the index of the summation. More concretely, we prove the following theorem.

Theorem 9 *For any positive integer a and any integer $p>1$ we have*

$$\sum_{n=1}^{\infty}\frac{B_p(n+1)}{n^a}x^n = B_pLi_a(x)+$$

$$p\sum_{k=1}^{p-1}\frac{x^k}{k^a}A(p-1,p-1-k)\ {}_{a+1}F_a\left(\begin{matrix} p+2 & k & k & \ldots & k & k \\ k+1 & k+1 & k+1 & \ldots & k+1 & \end{matrix}, x\right),$$

where $A(n,k)$ is an Eulerian number giving the number of permutations of length n with k runs, and ${}_{a+1}F_a$ is the hypergeometric sum. In the argument of the hypergeometric function the parameter k occurs a times in the upper line and all the lower parameters (there are a occurrences) are $k+1$.

Proof We begin with the generating function

$$\sum_{n=1}^{\infty}(1^p+\cdots+n^p)x^n = \frac{Li_{-p}(x)}{1-x}. \tag{10}$$

It is known that $Li_{-p}(x)$ is a rational function, namely

$$Li_{-p}(x) = \sum_{n=1}^{\infty} n^p x^n = \frac{1}{(1-x)^{p+1}}\sum_{k=1}^{p}A(p,p-k)x^k.$$

Dividing (10) by x and integrating we can see that

$$\sum_{n=1}^{\infty}(1^p+\cdots+n^p)\frac{x^n}{n} = \int_0^x \frac{Li_{-p}(t)}{1-t}dt = \sum_{k=1}^{p}A(p,p-k)\int_0^x\frac{t^k}{t(1-t)^{p+2}}dt.$$

The integral on the rightmost expression can be integrated term by term:

$$\int_0^x \frac{t^k}{t(1-t)^{p+2}}dt = \sum_{m=0}^{\infty}\binom{m+p+1}{p+1}\frac{x^{m+k}}{m+k}.$$

This can be continued to increase the power of n in the denominator of the leftmost sum. Thus, in general, we have

$$\sum_{n=1}^{\infty}(1^p+\cdots+n^p)\frac{x^n}{n^a} = \sum_{k=1}^{p} x^k A(p,p-k)\sum_{m=0}^{\infty}\binom{m+p+1}{p+1}\frac{x^m}{(m+k)^a}.$$

The inner sum on m is a hypergeometric sum ${}_aF_a$, denoted simply as F_a:

$$\sum_{m=0}^{\infty}\binom{m+p+1}{p+1}\frac{x^m}{(m+k)^a} = \frac{1}{k^a}F_a\left(\begin{array}{ccccc} p+1 & k & k & \ldots & k \\ k+1 & k+1 & k+1 & \ldots & k \end{array}, x\right).$$

Therefore, we have

$$\sum_{n=1}^{\infty}(1^p+\cdots+n^p)\frac{x^n}{n^a} = \sum_{k=1}^{p}\frac{x^k}{k^a}A(p,p-k)F_a\left(\begin{array}{ccccc} p+1 & k & k & \ldots & k \\ k+1 & k+1 & k+1 & \ldots & k \end{array}, x\right). \tag{11}$$

Using this result, the statement of the theorem immediately follows once we take into account the well-known Faulhaber's formula:

$$1^p+\cdots+n^p = \frac{1}{(p+1)}\left(B_{p+1}(n+1)-B_{p+1}\right),$$

where B_{p+1} is the $(p+1)$th Bernoulli number. Substituting this into (11) and writing $p-1$ instead of p, the result follows. □

5. On the noninteger property

It was proven in 1915 by Theisinger [21] that the harmonic numbers H_n are never integers. Mező [19] proved that the second and third order hyperharmonics $H_n^{(1,2)}$ and $H_n^{(1,3)}$ are never integers. Later this result was improved by Amrane and Belbachir [1, 2] (see also Cereceda [8]) to some general class of the parameter r. All these results were further sharpened by Göral and Sertbaş [14].

It was conjectured by Mező [19] that the hyperharmonic numbers $H_n^{(1,r)}$ are never integers if $n>1$.

We can ask the same question for the generalized hyperharmonic numbers: Are there some indices n,s and r for which $H_n^{(s,r)}$ is an integer? By obvious reasons and by Theisinger's theorem, if $r=1$, then $H_n^{(s,1)}$ is never integer except $n=1$. We prove the following theorem.

Theorem 10 *For any $n,s\geq 1$ the number $H_n^{(s,2)}$ is not an integer.*

Proof Following verbatim the proof of Theorem 1 in [19], we can verify the useful fact that

$$|H_n^{(s,1)}|_2 = 2^{\mathrm{ord}_2(n)}. \tag{12}$$

Here $|\cdot|_2$ is the 2-adic valuation [17], and $\mathrm{ord}_2(n)$ is a unique integer such that

$$2^{\mathrm{ord}_2(n)} \leq n < 2^{\mathrm{ord}_2(n)+1}.$$

Statement (12) simply means that

$$H_n^{(s,1)} = 2^{-\mathrm{ord}_2(n)} \frac{a}{b}, \tag{13}$$

where a and b are odd integers.

Now applying our recursion (2), which reads as

$$H_n^{(s,2)} = (n+1) H_n^{(s,1)} - H_n^{(s-1,1)},$$

and substituting (13) into this recursion, we can easily find that

$$H_n^{(s,2)} = \frac{1}{2^{(s-1)\mathrm{ord}_2(n)}} \left(\frac{ad\|n+1\|_2 - bc \cdot 2^{\mathrm{ord}_2(n)}}{bd \cdot 2^{\mathrm{ord}_2(n)}} \right),$$

where a, b, c, d are positive odd integers, and $\|n+1\|_2$ is the even part of $n+1$, that is $\frac{1}{|n+1|_2}$. From these considerations we get an interesting formula for the 2-adic value of $H_n^{(s,2)}$:

$$|H_n^{(s,2)}|_2 = 2^{s\mathrm{ord}_2(n) - \min\{\log_2(\|n+1\|_2),\, \mathrm{ord}_2(n)\}}.$$

Here $\log_2(\cdot)$ is the logarithm function of base 2. Since $\log_2(\|n+1\|_2) \leq \mathrm{ord}_2(n) + 1$ this yields that

$$|H_n^{(s,2)}|_2 \geq 2^{(s-1)\mathrm{ord}_2(n) - 1} > 1 \quad (s \geq 2, n \geq 4),$$

and we get, in particular, that the denominator is always even and the nominator is always odd of $H_n^{(s,2)}$ if we write it in reduced fraction form (in the other cases when $n = 2$ and $n = 3$ the 2-adic norm of $H_n^{(s,2)}$ is $2^s > 1$ and $2^{s-1} > 1$, respectively). Our theorem then follows. $\square$

The rest of our investigation will depend on the concept of p-integers (cf. [15]). The unique integer $\mathrm{ord}_2(n)$ defined above can be stated for a general prime number p as $\mathrm{ord}_p(n) = k$ whenever $n = p^k m$ with $\gcd(m, p) = 1$, and $\mathrm{ord}_p(0) = \infty$. The map ord_p can be generalized to rational numbers by $\mathrm{ord}_p\left(\frac{m}{n}\right) = \mathrm{ord}_p(m) - \mathrm{ord}_p(n)$. A rational number a is said to be a p-integer if $\mathrm{ord}_p(a) \geq 0$; in other words, a is a p-integer if $a = \frac{m}{n}$, $m, n \in \mathbb{Z}$ and $p \nmid n$. ord_p further satisfies

$$\begin{aligned} \mathrm{ord}_p(ab) &= \mathrm{ord}_p(a) + \mathrm{ord}_p(b), \\ \mathrm{ord}_p(a+b) &\geq \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\} \end{aligned}$$

for p-integers a and b, and in particular

$$\mathrm{ord}_p(a+b) = \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\} \tag{14}$$

if $\mathrm{ord}_p(a) \neq \mathrm{ord}_p(b)$. It is a classical result due to Legendre that

$$\mathrm{ord}_p(n!) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right],$$

where $[x]$ denotes the integer part of the real number x, and the series on the right is in fact a finite sum. We also have (see [20]) that

$$\operatorname{ord}_p((m+n)!) \geq \operatorname{ord}_p(m!) + \operatorname{ord}_p(n!), \tag{15}$$

with equality if and only if $p \nmid \binom{m+n}{n}$. The set of p-integers forms a ring, which is denoted by $\mathbb{Z}_p$.

Theorem 11 *For a prime number p and positive integers s and r, $p^s H_p^{(s,r)}$ is a p-integer.*

Proof We write

$$H_p^{(s,r)} = \sum_{j=1}^{p} \binom{p-j+r-1}{r-1} \frac{1}{j^s} = \frac{1}{p^s} + \sum_{j=1}^{p-1} \binom{p-j+r-1}{r-1} \frac{1}{j^s}.$$

Let

$$c = c(j,s,r,p) = \binom{p-j+r-1}{r-1} \frac{1}{j^s} = \frac{(p-j+r-1)!}{(r-1)!\,(p-j)!} \frac{1}{j^s},$$

for $j = 1, 2, \ldots, p-1$. Then

$$\operatorname{ord}_p(c) = \operatorname{ord}_p((p-j+r-1)!) - \operatorname{ord}_p((r-1)!) - \operatorname{ord}_p((p-j)!) - s\operatorname{ord}_p(j).$$

We observe that $\operatorname{ord}_p((p-j)!) = s\operatorname{ord}_p(j) = 0$. Thus, by (15),

$$\operatorname{ord}_p(c) = \operatorname{ord}_p((p-j+r-1)!) - \operatorname{ord}_p((r-1)!) \geq 0$$

for all $j = 1, 2, \ldots, p-1$, which implies that the summation

$$\sum_{j=1}^{p-1} \binom{p-j+r-1}{r-1} \frac{1}{j^s}$$

is a p-integer. On the other hand, since $\operatorname{ord}_p\left(\frac{1}{p^s}\right) = -s$, we see that

$$\operatorname{ord}_p\left(H_p^{(s,r)}\right) = -s$$

by (14), from which we deduce that $p^s H_p^{(s,r)} - 1 \in \mathbb{Z}_p$. Now the result follows since $\mathbb{Z}_p$ is a ring including -1. □

Corollary 12 *$H_p^{(s,r)}$ is not an integer for prime p and positive integers s and r.*

We now consider a more general case.

Theorem 13 *For a prime number p, $p^s H_{mp}^{(s,kp)}$ is a p-integer, where $1 < m < p$ and $k \geq 1$.*

Proof We may write the sum

$$H_{mp}^{(s,kp)} = \sum_{j=1}^{mp} \binom{mp-j+kp-1}{kp-1} \frac{1}{j^s}$$

as

$$H_{mp}^{(s,kp)} = \sum_{i=0}^{m-1}\binom{(k+m-i-1)\,p-1}{kp-1}\frac{1}{(p+ip)^{s}} + \sum_{i=0}^{m-1}\sum_{j=1}^{p-1}\binom{(k+m-i)\,p-j-1}{kp-1}\frac{1}{(j+ip)^{s}}.$$

Consider the double sum on the right, and let

$$\begin{aligned} c &= c(k,m,j,i,s,p) = \binom{(k+m-i)\,p-j-1}{kp-1}\frac{1}{(j+ip)^{s}} \\ &= \frac{[(k+m-i)\,p-j-1]!}{(kp-1)!\,[(m-i)\,p-j]!}\frac{1}{(j+ip)^{s}}. \end{aligned}$$

Using Legendre's formula we find that

$$\begin{aligned} \mathrm{ord}_p\left(((k+m-i)\,p-j-1)!\right) &= k+m-i-1 \\ &\quad+\left[\frac{k+m-i}{p}-\frac{j+1}{p^2}\right]+\left[\frac{k+m-i}{p^2}-\frac{j+1}{p^3}\right]+\cdots, \end{aligned}$$

$$\begin{aligned} \mathrm{ord}_p\left((kp-1)!\right) &= k-1 \\ &\quad+\left[\frac{k}{p}-\frac{1}{p^2}\right]+\left[\frac{k}{p^2}-\frac{1}{p^3}\right]+\cdots, \end{aligned}$$

and

$$\begin{aligned} \mathrm{ord}_p\left(((m-i)\,p-j)!\right) &= m-i-1 \\ &\quad+\left[\frac{m-i}{p}-\frac{j}{p^2}\right]+\left[\frac{m-i}{p^2}-\frac{j}{p^3}\right]+\cdots. \end{aligned}$$

Employing the well-known fact that $[a]-[b]\geqslant[a-b]$, we then obtain

$$\begin{aligned} &\mathrm{ord}_p\left(((k+m-i)\,p-j-1)!\right)-\mathrm{ord}_p\left((kp-1)!\right)-\mathrm{ord}_p\left(((m-i)\,p-j)!\right) \\ = \;& 1+\left[\frac{k+m-i}{p}-\frac{j+1}{p^2}\right]+\left[\frac{k+m-i}{p^2}-\frac{j+1}{p^3}\right]+\cdots \\ &-\left[\frac{k}{p}-\frac{1}{p^2}\right]-\left[\frac{k}{p^2}-\frac{1}{p^3}\right]-\cdots \\ &-\left[\frac{m}{p}-\frac{i}{p}-\frac{j}{p^2}\right]-\left[\frac{m-i}{p^2}-\frac{j}{p^3}\right]-\cdots \\ \geq \;& 1. \end{aligned}$$

Moreover, since $\gcd(p,j+ip)=1$, we have $\mathrm{ord}_p\left((j+ip)^{s}\right)=s\,\mathrm{ord}_p\,(j+ip)=0$. Therefore, we obtain $\mathrm{ord}_p\,(c)\geq 1$, which yields that the sum

$$\sum_{i=0}^{m-1}\sum_{j=1}^{p-1}\binom{(k+m-i)\,p-j-1}{kp-1}\frac{1}{(j+ip)^{s}}$$

is a p-integer.

Now consider the sum

$$\sum_{i=0}^{m-1}\binom{(k+m-i-1)\,p-1}{kp-1}\frac{1}{(p+ip)^s}=\frac{1}{p^s}\sum_{i=0}^{m-1}\binom{(k+m-i-1)\,p-1}{kp-1}\frac{1}{(i+1)^s},$$

and let

$$d=d\,(k,m,i,s,p)=\binom{(k+m-i-1)\,p-1}{kp-1}\frac{1}{(i+1)^s}.$$

Then the sum $\sum_{i=0}^{m-1} d$ is a p-integer, since $1<m<p$.

In conclusion, we find that

$$\begin{aligned} p^s H_{mp}^{(s,kp)} &= \sum_{i=0}^{m-1}\binom{(k+m-i-1)\,p-1}{kp-1}\frac{1}{(i+1)^s} \\ &\quad +p^s\sum_{i=0}^{m-1}\sum_{j=1}^{p-1}\binom{(k+m-i)\,p-j-1}{kp-1}\frac{1}{(j+ip)^s} \end{aligned}$$

is a p-integer, since it is just a sum of two p-integers. □

As a final result we consider $H_{p-1}^{(s,r)}$.

Theorem 14 *For an odd prime p, $H_{p-1}^{(s,r)}$ is a p-integer for $2\le r\le p$.*

Proof We note that

$$H_{p-1}^{(s,r)}=\sum_{j=1}^{p-1}\binom{p-1-j+r-1}{r-1}\frac{1}{j^s}=\sum_{j=1}^{p-1}\frac{(p-1-j+r-1)!}{(r-1)!\,(p-1-j)!}\frac{1}{j^s}.$$

Let

$$c=c\,(j,s,r,p)=\frac{(p-1-j+r-1)!}{(r-1)!\,(p-1-j)!}\frac{1}{j^s},$$

where $j=1,2,\ldots,p-1$. Then

$$\operatorname{ord}_p(c)=\operatorname{ord}_p((p-1-j+r-1)!)-\operatorname{ord}_p((r-1)!)-\operatorname{ord}_p((p-1-j)!)-s\operatorname{ord}_p(j)\,,$$

and since $2\le r\le p$, we obtain that

$$\operatorname{ord}_p((p-1-j+r-1)!)\ge\operatorname{ord}_p((r-1)!)=0,$$

which implies that $H_{p-1}^{(s,r)}$ is a p-integer. □

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