Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1656 – 1672
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doi:10.3906/mat-1408-11

Piecewise asymptotically almost periodic solution of neutral Volterra integro-differential equations with impulsive effects

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Received: 06.08.2014 • **Accepted/Published Online:** 15.02.2017 • **Final Version:** 23.11.2017

Abstract: In this paper, we investigate the existence and uniqueness of a piecewise asymptotically almost periodic mild solution to nonautonomous neutral Volterra integro-differential equations with impulsive effects in Banach space. The working tools are based on the Krasnoselskii's fixed point theorem and semigroup theory. In order to illustrate our main results, we study the piecewise asymptotically almost periodic solution of the impulsive partial differential equations with Dirichlet conditions.

Key words: Neutral Volterra integro-differential equations, impulsive effects, asymptotically almost periodicity, Krasnoselskii's fixed point theorem

1. Introduction

In this paper, we investigate the existence and uniqueness of a piecewise asymptotically almost periodic mild solution of neutral Volterra integro-differential equations with impulsive effects:

$$\begin{cases} \dfrac{d}{dt}D(t,u(t)) = A(t)D(t,u(t)) + \displaystyle\int_{-\infty}^{t} k(t-s)g(s,u(s))ds + h(t,u(t)),\ t\in\mathbb{R},\ t\neq t_i,\ i\in\mathbb{Z}, \\ \Delta u(t_i) = u(t_i^+) - u(t_i^-) = \gamma_i u(t_i) + \delta_i, \end{cases} \tag{1.1}$$

where $A(t) : \mathcal{D} \subset X \to X$ are a family of closed linear operators on Banach space X, $D(t,u(t)) = u(t) + f(t,u(t))$, $f,g,h : \mathbb{R}\times X \to X$ are piecewise asymptotically almost periodic functions in $t\in\mathbb{R}$ uniformly in the second variable, γ_i, δ_i are asymptotically almost periodic sequences, and $u(t_i^+), u(t_i^-)$ represent the right-hand side and the left-hand side limits of $u(\cdot)$ at t_i, respectively.

There are many physical phenomena that are described by means of integro-differential equations with impulsive effects, for instance, biological systems, electrical engineering, and chemical reactions. For more details about this topic, one can see [6, 7, 9, 16, 23, 24, 26], where the authors have given an important overview about the theory of impulsive differential and integro-differential equations. On the other hand, the asymptotic properties of solutions of impulsive differential equations have been studied from different points, such as almost periodicity [8, 17, 18, 20, 21, 29], almost automorphy [1, 30], asymptotic stability [19, 28], asymptotic equivalence [5], and oscillation[15]. However, the existence and uniqueness of a piecewise asymptotically almost periodic

*Correspondence: xiazn299@zjut.edu.cn
2010 *AMS Mathematics Subject Classification:* 35B15, 35R12
This research is supported by the National Natural Science Foundation of China (Grant No. 11501507).

mild solution for neutral Volterra integro-differential equations with impulsive effects in the form (1.1) is an untreated topic in the literature and this fact is the motivation of the present work.

The paper is organized as follows. In Section 2, we recall some fundamental results about the notion of piecewise asymptotically almost periodic function including composition theorem. Section 3 is devoted to the existence and uniqueness of a mild solution to nonautonomous neutral Volterra integro-differential equations with impulsive effects in Banach space. In Section 4, an application of impulsive partial differential equations with Dirichlet conditions is given.

2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces, Ω be a subset of X, and $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For A being a linear operator on X, $\mathcal{D}(A)$, $\rho(A)$, $R(\lambda, A)$, $\sigma(A)$ stand for the domain, the resolvent set, the resolvent, and spectrum of A. Let T be the set consisting of all real sequences $\{t_i\}_{i\in\mathbb{Z}}$ such that $\alpha = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to-\infty} t_i = -\infty$.

In order to facilitate the discussion below, we further introduce the following notations:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times\Omega, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times\Omega$ to X).
- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times\Omega, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times\Omega$ to X) with the supremum norm.
- $PC(\mathbb{R}, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R} \to X$ such that $f(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$, $f(t_i^+)$, $f(t_i^-)$ exist, and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC(\mathbb{R}\times\Omega, X)$: the space formed by all piecewise continuous functions $f : \mathbb{R}\times\Omega \to X$ such that for any $x \in \Omega$, $f(\cdot, x) \in PC(\mathbb{R}, X)$, and for any $t \in \mathbb{R}$, $f(t, \cdot)$ is continuous at $x \in \Omega$.
- $L(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.
- $l^\infty(\mathbb{Z}, X) = \{x : \mathbb{Z} \to X : \|x\| = \sup_{n\in\mathbb{Z}} \|x(n)\| < \infty\}$.

2.1. Fixed point theorem and compactness criterion

First, we recall the definition of strong continuous evolution family and Krasnoselskii's fixed point theorem, which will be used later.

Definition 2.1 *[13] A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space X is called a strong continuous evolution family if*

(*i*) *$U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$.*

(*ii*) *The map $(t,s) \to U(t,s)x$ is continuous for all $x \in X, t \geq s$ and $t, s \in \mathbb{R}$.*

Theorem 2.1 *([27] Krasnoselskii's fixed point theorem) Let $\mathcal{M}$ be a closed convex nonempty subset of a Banach space X. Suppose that A and B map $\mathcal{M}$ into X such that* (*i*) $Au + Bv \in \mathcal{M}$ $(\forall u, v \in \mathcal{M})$,

(*ii*) A *is compact and continuous,*

(*iii*) B *is a contraction mapping.*

Then there exists $v \in \mathcal{M}$ *such that* $Av + Bv = v$.

Next, we recall a useful compactness criterion on $PC(\mathbb{R}, X)$.

Let $h : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}$ and $h(t) \to \infty$ as $|t| \to \infty$. Define

$$PC_h^0(\mathbb{R}, X) := \left\{ f \in PC(\mathbb{R}, X) : \lim_{|t|\to\infty} \frac{\|f(t)\|}{h(t)} = 0 \right\}$$

endowed with the norm $\|f\|_h = \sup_{t\in\mathbb{R}} \dfrac{\|f(t)\|}{h(t)}$, it is a Banach space.

Lemma 2.1 *[19] A set* $B \subseteq PC_h^0(\mathbb{R}, X)$ *is relatively compact if and only if it verifies the following conditions:*

(1) $\lim_{|t|\to\infty} \dfrac{\|f(t)\|}{h(t)} = 0$ *uniformly for* $f \in B$.

(2) $B(t) = \{f(t) : f \in B\}$ *is relatively compact in* X *for every* $t \in \mathbb{R}$.

(3) *The set* B *is equicontinuous on each interval* (t_i, t_{i+1}) $(i \in \mathbb{Z})$.

2.2. Piecewise asymptotic almost periodicity

Definition 2.2 *[14] A function* $f \in C(\mathbb{R}, X)$ *is said to be almost periodic if for each* $\varepsilon > 0$ *there exists an* $l(\varepsilon) > 0$, *such that every interval* J *of length* $l(\varepsilon)$ *contains a number* τ *with the property that* $\|f(t+\tau) - f(t)\| < \varepsilon$ *for all* $t \in \mathbb{R}$. *Denote by* $AP(\mathbb{R}, X)$ *the set of such functions.*

Definition 2.3 *[26] A sequence* $\{x_n\}$ *is called almost periodic if for any* $\varepsilon > 0$ *there exists a relatively dense set of its* ε*-periods, i.e. there exists a natural number* $l = l(\varepsilon)$, *such that for* $k \in \mathbb{Z}$, *there is at least one number* p *in* $[k, k+l]$, *for which inequality* $\|x_{n+p} - x_n\| < \varepsilon$ *holds for all* $n \in \mathbb{N}$. *Denote by* $AP(\mathbb{Z}, X)$ *the set of such sequences.*

Define

$$AAP_0(\mathbb{Z}, X) = \left\{ x_n \in l^\infty(\mathbb{Z}, X) : \lim_{n\to\infty} \|x_n\| = 0 \right\}.$$

Definition 2.4 *[25] A sequence* $\{x_n\}_{n\in\mathbb{Z}} \in l^\infty(\mathbb{Z}, X)$ *is called asymptotically almost periodic if* $x_n = x_n^1 + x_n^2$, *where* $x_n^1 \in AP(\mathbb{Z}, X)$, $x_n^2 \in AAP_0(\mathbb{Z}, X)$. *Denote by* $AAP(\mathbb{Z}, X)$ *the set of such sequences.*

For $\{t_i\}_{i\in\mathbb{Z}} \in T$, $\{t_i^j\}$ is defined by

$$\left\{ t_i^j = t_{i+j} - t_i \right\}, i \in \mathbb{Z}, j \in \mathbb{Z}.$$

It is easy to verify that the numbers t_i^j satisfy

$$t_{i+k}^j - t_i^j = t_{i+j}^k - t_i^k, \quad t_i^j - t_i^k = t_{i+k}^{j-k} \quad \text{for} \ \ i,j,k \in \mathbb{Z}.$$

Definition 2.5 *[4] A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise almost periodic if the following conditions are fulfilled:*

(1) $\left\{t_i^j = t_{i+j} - t_i\right\}, i,j \in \mathbb{Z}$ *are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods common for all of the sequences $\{t_i^j\}$.*

(2) *For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.*

(3) *For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then*

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ that satisfy the condition $|t - t_i| > \varepsilon, i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, X)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $\mathcal{U}PC(\mathbb{R}, X)$ be the space of all functions $f \in PC(\mathbb{R}, X)$ such that f satisfies the condition (2) in Definition 2.5.

Lemma 2.2 *[26] If the sequences $\{t_i^j\}$ are equipotentially almost periodic, then for each $j > 0$ there exists a positive integer N such that on each interval of length j there are no more than N elements of the sequence $\{t_i\}$, i.e.*

$$i(t,s) \leq N(t-s) + N,$$

where $i(t,s)$ is the number of the points $\{t_i\}$ in the interval $[s,t]$.

Definition 2.6 *[26] $f \in PC(\mathbb{R} \times \Omega, X)$ is said to be piecewise almost periodic in t uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that $\|f(t+\tau, x) - f(t,x)\| \leq \varepsilon$ for all $x \in K, \tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times \Omega, X)$ the set of all such functions.*

Define

$$PC_T^0(\mathbb{R}, X) = \left\{f \in PC(\mathbb{R}, X) : \lim_{t \to \infty} \|f(t)\| = 0\right\},$$

$$PC_T^0(\mathbb{R} \times \Omega, X) = \Big\{f \in PC(\mathbb{R} \times \Omega, X) : \lim_{t \to \infty} \|f(t,x)\| dt = 0 \ \text{ uniformly with respect to} \ x \in K, \ \ \text{where } K \text{ is an arbitrary compact subset of } \Omega\Big\}.$$

Definition 2.7 *A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise asymptotically almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R}, X)$ and $\varphi \in PC_T^0(\mathbb{R}, X)$. Denote by $AAP_T(\mathbb{R}, X)$ the set of all such functions.*

Similarly as the proof of [11, Lemma 2.5], one has

Lemma 2.3 *Let* $\{f_n\}_{n\in\mathbb{N}} \subset PC_T^0(\mathbb{R}, X)$ *be a sequence of functions. If* f_n *converges uniformly to* f*, then* $f \in PC_T^0(\mathbb{R}, X)$.

Definition 2.8 *Let* $AAP_T(\mathbb{R}\times\Omega, X)$ *consist of all functions* $f \in PC(\mathbb{R}\times\Omega, X)$ *such that* $f = g + \varphi$*, where* $g \in AP_T(\mathbb{R}\times\Omega, X)$ *and* $\varphi \in AAP_T^0(\mathbb{R}\times\Omega, X)$.

Similarly as the proof of [19, Theorem 3.1], the composition theorems hold for piecewise asymptotically almost periodic function.

Theorem 2.2 *Suppose* $f \in AAP_T(\mathbb{R}\times\Omega, X)$*. Assume that the following conditions hold:*

(i) $\{f(t,u) : t \in \mathbb{R}, u \in K\}$ *is bounded for every bounded subset* $K \subseteq \Omega$.

(ii) $f(t,\cdot)$ *is uniformly continuous in each bounded subset of* Ω *uniformly in* $t \in \mathbb{R}$.

If $\varphi \in AAP_T(\mathbb{R}, X)$ *such that* $R(\varphi) \subset \Omega$*, then* $f(\cdot, \varphi(\cdot)) \in AAP_T(\mathbb{R}, X)$.

Corollary 2.1 *Let* $f \in AAP_T(\mathbb{R}\times\Omega, X)$*,* $\varphi \in AAP_T(\mathbb{R}, X)$*, and* $R(\varphi) \subset \Omega$*. Assume that there exists a constant* $L_f > 0$ *such that*

$$\|f(t,u) - f(t,v)\| \le L_f \|u - v\|, \quad t \in \mathbb{R}, \quad u, v \in \Omega,$$

then $f(\cdot, \varphi(\cdot)) \in AAP_T(\mathbb{R}, X)$.

3. Neutral Volterra integro-differential equations with impulsive effects

In this section, we investigate the existence and uniqueness of a piecewise asymptotically almost periodic mild solution of (1.1).

First, we make the following assumptions:

(H_1) There exist constants $\lambda_0 \ge 0, \theta \in (\frac{\pi}{2}, \pi), L, \widetilde{M} \ge 0$, and $\beta, \gamma \in (0,1)$ with $\beta + \gamma > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{\widetilde{M}}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0) R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \le L|t - s|^\beta |\lambda|^{-\gamma}$$

for $t, s \in \mathbb{R}, \Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\lambda| \le \theta\}$.

(H_2) $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}, L(X))$.

(H_3) The evolution family $(U(t,s))_{t\ge s}$ generated by $A(t)$ is exponentially stable, i.e. there exist constants $M > 0$, $\omega > 0$ such that $\|U(t,s)\| \le Me^{-\omega(t-s)}$, $t \ge s$, $t, s \in \mathbb{R}$.

(H_4) $f \in AAP_T(\mathbb{R}\times\Omega, X)$ and there exists a constant $L_f > 0$ such that

$$\|f(t,u) - \|f(t,v)\| \le L_f\|u-v\|, \quad t\in\mathbb{R}, \quad u,v\in\Omega.$$

(H_5) $g \in AAP_T(\mathbb{R}\times\Omega, X)$ and $g(t,\cdot)$ is uniformly continuous in each bounded subset of Ω uniformly in $t\in\mathbb{R}$.

(H_6) $h \in AAP_T(\mathbb{R}\times\Omega, X)$ and $h(t,\cdot)$ is uniformly continuous in each bounded subset of Ω uniformly in $t\in\mathbb{R}$.

(H_7) $\gamma_i \in AAP(\mathbb{Z}, X)$, $\delta_i \in AAP(\mathbb{Z}, X)$ and $\sup\limits_{i\in\mathbb{Z}}\|\gamma_i\| \le \varpi$, $\sup\limits_{i\in\mathbb{Z}}\|\delta_i\| \le \kappa$, $i\in\mathbb{Z}$.

(H_8) $k\in C(\mathbb{R}^+,\mathbb{R})$ and $|k(t)| \le C_k e^{-\eta t}$ for some positive constants C_k, η.

(H_9) For any $L>0$, $C_{1L} = \sup\limits_{t\in\mathbb{R},\|u\|\le L}\|g(t,u)\| < \infty$, $C_{2L} = \sup\limits_{t\in\mathbb{R},\|u\|\le L}\|h(t,u)\| < \infty$. Moreover, there exists a constant $L_0 > 0$ such that

$$L_f L_0 + \sup_{t\in\mathbb{R}}\|f(t,0)\| + \frac{M(C_k\eta^{-1}C_{1L_0} + C_{2L_0})}{\omega} + \frac{M(\varpi L_0 + \kappa)}{1-e^{-\omega\alpha}} \le L_0.$$

(H_{10}) For fixed $t,s\in\mathbb{R}, t\ge s$, the operator $U(t,s): X\to X$ is compact.

Remark 3.1 *(H_1) is usually called "Acquistapace–Terreni" conditions, which was first introduced in [3] and widely used to study nonautonomous differential equations in [2, 3, 12, 13]. If (H_1) holds, there exists a unique evolution family $(U(t,s))_{t\ge s}$ on X, which governs the homogeneous version of (1.1) [2].*

Before starting our main results, we recall the definition of the mild solution to (1.1).

Definition 3.1 *[10] A function $u:\mathbb{R}\to X$ is called a mild solution of (1.1) if for any $t\in\mathbb{R}$, $t>\sigma$, $\sigma\ne t_i$, $i\in\mathbb{Z}$,*

$$\begin{aligned} u(t) = {} & U(t,\sigma)(u(\sigma)+f(\sigma,u(\sigma))) - f(t,u(t)) + \int_\sigma^t U(t,s)((Ku)(s)+h(s,u(s)))ds \\ & + \sum_{\sigma<t_i<t} U(t,t_i)(\gamma_i u(t_i)+\delta_i), \end{aligned} \tag{3.1}$$

where

$$(Ku)(t) = \int_{-\infty}^t k(t-s)g(s,u(s))ds.$$

Note that if (H_3) holds, then (3.1) can be replaced by

$$u(t) = -f(t,u(t)) + \int_{-\infty}^t U(t,s)((Ku)(s)+h(s,u(s)))ds + \sum_{t_i<t} U(t,t_i)(\gamma_i u(t_i)+\delta_i).$$

Lemma 3.1 *[22] Assume that (H_1)–(H_3) hold; then for each $\varepsilon>0$ and $h>0$, there is a relatively dense set $\Omega_{\varepsilon,h}$ such that*

$$\|U(t+\tau,s+\tau) - U(t,s)\| \le \varepsilon e^{-\frac{\omega}{2}(t-s)}, \qquad t-s>h, t,s\in\mathbb{R}, \tau\in\Omega_{\varepsilon,h}.$$

This property can be abbreviated by writing $U\in AP(L(X))$.

Lemma 3.2 *[4] Assume that $f \in AP_T(\mathbb{R}, X)$, $U \in AP(L(X))$, the sequence $\{x_i\}_{i\in\mathbb{Z}} \in AP(\mathbb{Z}, X)$, and $\{t_i^j\}$, $j \in \mathbb{Z}$ are equipotentially almost periodic. Then for each $\varepsilon > 0$, there exist relatively dense sets Ω_ε of $\mathbb{R}$ and Q_ε of $\mathbb{Z}$ such that*

(i) $\|f(t+\tau) - f(t)\| < \varepsilon$ *for all* $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$, *and* $i \in \mathbb{Z}$.

(ii) $\|U(t+\tau, s+\tau) - U(t,s)\| \le \varepsilon e^{-\frac{\omega}{2}(t-s)}$ *for all* $t, s \in \mathbb{R}$, $|t-s| > 0$, $|s - t_i| > \varepsilon$, $|t - t_i| > \varepsilon$, $\tau \in \Omega_\varepsilon$, *and* $i \in \mathbb{Z}$.

(iii) $\|x_{i+q} - x_i\| < \varepsilon$ *for all* $q \in Q_\varepsilon$ *and* $i \in \mathbb{Z}$.

(iv) $|t_i^q - \tau| < \varepsilon$ *for all* $q \in Q_\varepsilon$, $\tau \in \Omega_\varepsilon$, *and* $i \in \mathbb{Z}$.

Lemma 3.3 *Assume that (H_1)–(H_3), (H_5), (H_8), (H_9) hold, if $u \in AAP_T(\mathbb{R}, X)$, then*

$$(Ku)(t) = \int_{-\infty}^{t} k(t-s)g(s, u(s))ds \in AAP_T(\mathbb{R}, X).$$

Proof For $u \in AAP_T(\mathbb{R}, X)$, it is not difficult to see that $\phi(\cdot) = g(\cdot, u(\cdot)) \in AAP_T(\mathbb{R}, X)$ by Theorem 2.2. Let $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP_T(\mathbb{R}, X)$, $\phi_2 \in PC_T^0(\mathbb{R}, X)$; then

$$\begin{aligned}(Ku)(t) &= \int_{-\infty}^{t} k(t-s)\phi(s)ds = \int_{-\infty}^{t} k(t-s)\phi_1(s)ds + \int_{-\infty}^{t} k(t-s)\phi_2(s)ds \\ &:= \Psi_1(t) + \Psi_2(t).\end{aligned}$$

(i) $\Psi_1 \in AP_T(\mathbb{R}, X)$.

It is not difficult to see that $\Psi_1 \in \mathcal{U}PC(\mathbb{R}, X)$. Since $\phi_1 \in AP_T(\mathbb{R}, X)$, for any $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that

$$\|\phi_1(t+\tau) - \phi_1(t)\| < \varepsilon \quad \text{for} \ \ \tau \in \Omega_\varepsilon, t \in \mathbb{R}, |t - t_i| > \varepsilon, i \in \mathbb{Z}.$$

Thus, by (H_8), for $t \in \mathbb{R}, |t - t_i| > \varepsilon, i \in \mathbb{Z}$, one has

$$\begin{aligned}\|\Psi_1(t+\tau) - \Psi_1(t)\| &= \left\| \int_{-\infty}^{t+\tau} k(t+\tau-s)\phi_1(s)ds - \int_{-\infty}^{t} k(t-s)\phi_1(s)ds \right\| \\ &= \left\| \int_{-\infty}^{t} k(t-s)(\phi_1(s+\tau) - \phi_1(s))ds \right\| \\ &\le \int_{-\infty}^{t} C_k e^{-\eta(t-s)} \|\phi_1(s+\tau) - \phi_1(s)\| ds \\ &< \frac{C_k}{\eta}\varepsilon,\end{aligned}$$

which implies that $\Psi_1 \in AP_T(\mathbb{R}, X)$.

(ii) $\Psi_2 \in PC_T^0(\mathbb{R}, X)$.

Since $\phi_2 \in PC_T^0(\mathbb{R}, X)$, for each $\varepsilon > 0$, there exists $T_0 > 0$ such that $\|\phi_2(s)\| \le \varepsilon$ for all $s > T_0$; then for all $t > 2T_0$, one has

$$\begin{aligned}\|\Psi_2(t)\| &\le \int_{-\infty}^{t} \|k(t-s)\phi_2(s)\| ds \\ &\le \int_{-\infty}^{t} C_k e^{-\eta(t-s)} \|\phi_2(s)\| ds \\ &= \int_{-\infty}^{t/2} C_k e^{-\eta(t-s)} \|\phi_2(s)\| ds + \int_{t/2}^{t} C_k e^{-\eta(t-s)} \|\phi_2(s)\| ds \\ &\le C_k \|\phi_2\| \int_{t/2}^{\infty} e^{-\eta s} ds + \varepsilon C_k \int_{0}^{\infty} e^{-\eta s} ds,\end{aligned}$$

and, therefore, $\lim_{t\to\infty} \|\Psi_2(t)\| = 0$, that is $\Psi_2 \in PC_T^0(\mathbb{R}, X)$. This completes the proof. □

Theorem 3.1 *Assume that* (H_1)–(H_{10}) *hold; then* (1.1) *has a mild solution* $u \in AAP_T(\mathbb{R}, X)$.

Proof Let $\Gamma : AAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X) \to PC(\mathbb{R}, X)$ be the operator defined by

$$\begin{aligned}(\Gamma u)(t) &= -f(t, u(t)) + \int_{-\infty}^{t} U(t,s)((Ku)(s) + h(s, u(s)))ds + \sum_{t_i<t} U(t, t_i)(\gamma_i u(t_i) + \delta_i) \\ &:= (\Gamma_1 u)(t) + (\Gamma_2 u)(t),\end{aligned} \tag{3.2}$$

where

$$(\Gamma_1 u)(t) = -f(t, u(t)), \quad (\Gamma_2 u)(t) = \int_{-\infty}^{t} U(t,s)((Ku)(s) + h(s, u(s)))ds + \sum_{t_i<t} U(t, t_i)(\gamma_i u(t_i) + \delta_i).$$

Let $\mathcal{M} = \{u \in AAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X) : \|u\| \le L_0\}$. We next show that Γ has a fixed point in $\mathcal{M}$ and divide the proof into several steps.

(i) For $u, v \in \mathcal{M}$, we have $\Gamma_1 u, \Gamma_2 v \in \mathcal{U}PC(\mathbb{R}, X)$.

For $u, v \in \mathcal{M}$, it is not difficult to see that $\Gamma_1 u \in \mathcal{U}PC(\mathbb{R}, X)$. Next, we will show that $\Gamma_2 v \in \mathcal{U}PC(\mathbb{R}, X)$. Let $t', t'' \in (t_i, t_{i+1})$, $i \in \mathbb{Z}$, $t'' < t'$, $v \in AAP_T(\mathbb{R}, X) \cap \mathcal{U}PC(\mathbb{R}, X)$, and one has

$$\begin{aligned}&(\Gamma_2 v)(t') - (\Gamma_2 v)(t'') \\ &= \int_{-\infty}^{t'} U(t',s)((Kv)(s) + h(s, v(s)))ds + \sum_{t_i<t'} U(t', t_i)(\gamma_i v(t_i) + \delta_i) \\ &\quad - \int_{-\infty}^{t''} U(t'',s)((Kv)(s) + h(s, v(s)))ds - \sum_{t_i<t''} U(t'', t_i)(\gamma_i v(t_i) + \delta_i) \\ &= \int_{-\infty}^{t''} [U(t',s) - U(t'',s)]((Kv)(s) + h(s, v(s)))ds + \int_{t''}^{t'} U(t',s)((Kv)(s) + h(s, v(s)))ds \\ &\quad + \sum_{t_i<t''} [U(t', t_i) - U(t'', t_i)](\gamma_i v(t_i) + \delta_i).\end{aligned} \tag{3.3}$$

Moreover,

$$\begin{aligned}
&\int_{-\infty}^{t''}[U(t',s)-U(t'',s)]((Kv)(s)+h(s,v(s)))ds \\
&=\int_0^{\infty}[U(t',t''-s)-U(t'',t''-s)]((Kv)(t''-s)+h(t''-s,v(t''-s)))ds \\
&=\int_0^{\infty}[U(t',t'')U(t'',t''-s)-U(t'',t''-s)]((Kv)(t''-s)+h(t''-s,v(t''-s)))ds \\
&=\int_0^{\infty}[U(t',t'')-I]U(t'',t''-s)((Kv)(t''-s)+h(t''-s,v(t''-s)))ds.
\end{aligned}$$

Note that for any $\varepsilon>0$, there exists $0\le\delta<\dfrac{\varepsilon}{3M(C_k\eta^{-1}+\|\Psi\|_\infty)}$ such that if t',t'' belong to the same continuity and $0<t'-t''<\delta$, then

$$\|U(t',t'')-I\|\le\min\left\{\frac{\omega\varepsilon}{3M(C_k\eta^{-1}+\|\Psi\|_\infty)},\frac{(1-e^{-\omega\alpha})\varepsilon}{3MN(\varpi\|v\|_\infty+\kappa)}\right\},$$

where $\|\Psi\|_\infty=\sup_{t\in\mathbb{R}}\|h(t,v(t))\|$, N is the constant in the Lemma 2.2. Hence

$$\begin{aligned}
&\left\|\int_{-\infty}^{t''}[U(t',s)-U(t'',s)]((Kv)(s)+h(s,v(s)))ds\right\| \\
&\le\int_0^{\infty}\|U(t',t'')-I\|\|U(t'',t''-s)\|\|(Kv)(t''-s)+h(t''-s,v(t''-s))\|ds \\
&\le\int_0^{\infty}\frac{\omega\varepsilon}{3M(C_k\eta^{-1}+\|\Psi\|_\infty)}Me^{-\omega s}(C_k\eta^{-1}+\|\Psi\|_\infty)ds \\
&<\frac{\varepsilon}{3},
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
\left\|\int_{t''}^{t'}U(t',s)((Kv)(s)+h(s,v(s)))ds\right\| &\le\int_{t''}^{t'}\|U(t',s)\|\|((Kv)(s)+h(s,v(s)))\|ds \\
&<\delta M(C_k\eta^{-1}+\|\Psi\|_\infty)<\frac{\varepsilon}{3}.
\end{aligned}\tag{3.5}$$

Similarly,

$$\begin{aligned}
\left\|\sum_{t_i<t''}[U(t',t_i)-U(t'',t_i)](\gamma_i v(t_i)+\delta_i)\right\| &\le\left\|\sum_{t_i<t''}[U(t',t'')-I]U(t'',t_i)(\gamma_i v(t_i)+\delta_i)\right\| \\
&\le\sum_{t_i<t''}\|U(t',t'')-I\|\|U(t'',t_i)\|\|\gamma_i v(t_i)+\delta_i\| \\
&\le\sum_{t_i<t''}\frac{(1-e^{-\omega\alpha})\varepsilon}{3MN(\varpi\|v\|_\infty+\kappa)}Me^{-\omega(t''-t_i)}(\varpi\|v\|_\infty+\kappa) \\
&<\frac{\varepsilon}{3}.
\end{aligned}\tag{3.6}$$

Hence, by (3.3) − (3.6), if t', t'' belong to the same continuity and $0 < t' - t'' < \delta$, then

$$\|(\Gamma_2 v)(t') - (\Gamma_2 v)(t'')\| < \varepsilon,$$

which implies that $\Gamma_2 v \in \mathcal{U}PC(\mathbb{R}, X)$.

(ii) For $u, v \in \mathcal{M}$, we have $\Gamma_1 u, \Gamma_2 v \in AAP_T(\mathbb{R}, X)$.

For $u, v \in \mathcal{M}$, by Theorem 2.2, one has $\Gamma_1 u \in AAP_T(\mathbb{R}, X)$. Next, we will show that $\Gamma_2 v \in AAP_T(\mathbb{R}, X)$. Similarly as the proof of Lemma 3.3, one has

$$\int_{-\infty}^{t} U(t,s)((Kv)(s) + h(s, v(s)))ds \in AAP_T(\mathbb{R}, X).$$

It remains to show that

$$\sum_{t_i < t} U(t, t_i)(\gamma_i v(t_i) + \delta_i) \in AAP_T(\mathbb{R}, X).$$

It is not difficult to see that $\gamma_i v(t_i) + \delta_i \in AAP(\mathbb{Z}, X)$; then let $\gamma_i v(t_i) + \delta_i = \beta_i + \sigma_i$, where $\beta_i \in AP(\mathbb{Z}, X)$ and $\sigma_i \in AAP_0(\mathbb{Z}, X)$, and so

$$\sum_{t_i < t} U(t, t_i)(\gamma_i v(t_i) + \delta_i) = \sum_{t_i < t} U(t, t_i)\beta_i + \sum_{t_i < t} U(t, t_i)\sigma_i := \Pi_1(t) + \Pi_2(t).$$

For any $\varepsilon > 0$, by Lemma 3.2, there exist relative dense sets of real numbers Ω_ε and integers Q_ε, such that for $t_i < t \le t_{i+1}$, $\tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon$, $|t - t_i| > \varepsilon$, $|t - t_{i+1}| > \varepsilon$, $j \in \mathbb{Z}$, one has

$$t + \tau > t_i + \varepsilon + \tau > t_{i+q},$$

and

$$t_{i+q+1} > t_{i+1} + \tau - \varepsilon > t + \tau,$$

that is $t_{i+q} < t + \tau < t_{i+q+1}$; then, by Lemma 2.2, one has

$$
\begin{aligned}
\|\Pi_1(t+\tau)-\Pi_1(t)\| &= \left\| \sum_{t_i<t+\tau} U(t+\tau,t_i)\beta_i - \sum_{t_i<t} U(t,t_i)\beta_i \right\| \\
&\leq \left\| \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_{i+q} - \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_i \right\| \\
&\quad + \left\| \sum_{t_i<t} U(t+\tau,t_{i+q})\beta_i - \sum_{t_i<t} U(t,t_i)\beta_i \right\| \\
&\leq \sum_{t_i<t} \|U(t+\tau,t_{i+q})\| \|\beta_{i+q}-\beta_i\| \\
&\quad + \sum_{t_i<t} \|U(t+\tau,t_{i+q})-U(t,t_i)\| \|\beta_i\| \\
&\leq \sum_{t_i<t} Me^{-\omega(t-t_i)}\varepsilon + \sum_{t_i<t} \varepsilon M_{\beta_i} e^{-\frac{\omega}{2}(t-t_i)} \\
&\leq \sum_{j=0}^{+\infty} \sum_{j<t-t_i\leq j+1} Me^{-\omega(t-t_i)}\varepsilon + \sum_{j=0}^{+\infty} \sum_{j<t-t_i\leq j+1} \varepsilon M_{\beta_i} e^{-\frac{\omega}{2}(t-t_i)} \\
&\leq \frac{NM\varepsilon}{1-e^{-\omega\alpha}} + \frac{NM_{\beta_i}\varepsilon}{1-e^{-\frac{\omega}{2}\alpha}},
\end{aligned}
$$

where $M_{\beta_i} = \sup_{i\in\mathbb{Z}} \|\beta_i\|$. Thus $\Pi_1 \in AP_T(\mathbb{R}, X)$.

Next, we show that $\Pi_2 \in PC_T^0(\mathbb{R}, X)$. For a given $i \in \mathbb{Z}$, define the function $\rho(t)$ by

$$
\rho(t) = U(t,t_i)\sigma_i, \qquad t_i < t \leq t_{i+1},
$$

then

$$
\lim_{t\to\infty} \|\rho(t)\| = \lim_{t\to\infty} \|U(t,t_i)\sigma_i\| \leq \lim_{t\to\infty} Me^{-\omega(t-t_i)}\|\sigma_i\| = 0,
$$

and then $\rho \in PC_T^0(\mathbb{R}, X)$. Define $\rho_k : \mathbb{R} \to X$ by

$$
\rho_k(t) = U(t,t_{i-k})\sigma_{i-k}, \qquad t_i < t \leq t_{i+1}, \quad k \in \mathbb{N},
$$

Hence $\rho_k \in PC_T^0(\mathbb{R}, X)$. Moreover,

$$
\|\rho_k(t)\| = \|U(t,t_{i-k})\sigma_{i-k}\| \leq M \sup_{i\in\mathbb{Z}} \|\sigma_i\| e^{-\omega(t-t_{i-k})} \leq M \sup_{i\in\mathbb{Z}} \|\sigma_i\| e^{-\omega(t-t_i)} e^{-\omega\alpha k}.
$$

Therefore, the series $\sum_{k=0}^{\infty} \rho_k$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.3, one has

$$
\Pi_2(t) = \sum_{t_i<t} U(t,t_i)\sigma_i = \sum_{k=0}^{\infty} \rho_k \in PC_T^0(\mathbb{R}, X).
$$

Thus $\Gamma_2 v \in AAP_T(\mathbb{R}, X)$.

(iii) For all $u, v \in \mathcal{M}$, we claim that $\Gamma_1 u + \Gamma_2 v \in \mathcal{M}$.

For $u, v \in \mathcal{M}$, one has

$$\begin{aligned}\|\Gamma_1 u\| &\leq \|f(t,u(t)) - f(t,0)\| + \|f(t,0)\| \\ &\leq L_f\|u\| + \sup_{t\in\mathbb{R}}\|f(t,0)\| \\ &\leq L_f L_0 + \sup_{t\in\mathbb{R}}\|f(t,0)\|,\end{aligned}$$

and

$$\begin{aligned}\|(\Gamma_2 v)(t)\| &\leq \int_{-\infty}^{t}\|U(t,s)\|\|(Kv)(s) + h(s,v(s))\|ds + \sum_{t_i<t}\|U(t,t_i)\|\|\gamma_i v(t_i) + \delta_i\| \\ &\leq \int_{-\infty}^{t} Me^{-\omega(t-s)}\|(Kv)(s) + h(s,v(s))\|ds + \sum_{t_i<t} Me^{-\omega(t-t_i)}\|\gamma_i v(t_i) + \delta_i\| \\ &\leq (C_k\eta^{-1}C_{1L_0} + C_{2L_0})\int_{-\infty}^{t} Me^{-\omega(t-s)}ds + (\varpi L_0 + \kappa)\sum_{t_i<t} Me^{-\omega(t-t_i)} \\ &\leq \frac{M(C_k\eta^{-1}C_{1L_0} + C_{2L_0})}{\omega} + \frac{M(\varpi L_0 + \kappa)}{1 - e^{-\omega\alpha}},\end{aligned}$$

and then $\|\Gamma_1 u + \Gamma_2 v\| \leq L_0$ by (H_9). Hence, by (i) and (ii), we claim that $\Gamma_1 u + \Gamma_2 v \in \mathcal{M}$.

(iv) Γ_1 is a contraction mapping.

For $u, v \in \mathcal{M}$, one has

$$\|\Gamma_1 u - \Gamma_1 v\| = \|f(t,u) - f(t,v)\| \leq L_f\|u - v\|,$$

and it follows that Γ_1 is a contraction mapping by (H_9).

(v) Γ_2 is continuous.

Let $\{u_n\} \subset \mathcal{M}$, $u_n \to u$ as $n \to \infty$; then there exists a bounded subset $\widetilde{\Omega} \subseteq \Omega$ such that $R(u) \subseteq \widetilde{\Omega}$, $R(u_n) \subseteq \widetilde{\Omega}$, $n \in \mathbb{N}$. By (H_5)–(H_7), for any $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that $u, v \in \widetilde{\Omega}$ and $\|u - v\| < \delta$ implies that

$$\begin{aligned}&\|g(t,u) - g(t,v)\| < \varepsilon \qquad \text{for all } \ t \in \mathbb{R}, \\ &\|h(t,u) - h(t,v)\| < \varepsilon \qquad \text{for all } \ t \in \mathbb{R}.\end{aligned}$$

For the above $\delta > 0$, there exists n_0 such that $\|u_n(t) - u(t)\| < \delta$ for all $n > n_0$, $t \in \mathbb{R}$; then, for $n > n_0$, one has

$$\begin{aligned}&\|g(t,u_n(t)) - g(t,u(t))\| < \varepsilon, \qquad \text{for all } \ t \in \mathbb{R}, \\ &\|h(t,u_n(t)) - h(t,u(t))\| < \varepsilon, \qquad \text{for all } \ t \in \mathbb{R}.\end{aligned}$$

Hence

$$\begin{aligned}\|(\Gamma_2 u_n)(t)-(\Gamma_2 u)(t)\| &\le \int_{-\infty}^{t}\|U(t,s)\|\|((Ku_n)(s)+h(s,u_n(s)))-((Ku)(s)+h(s,u(s)))\|ds\\ &\quad+\sum_{t_i<t}\|U(t,t_i)\|\|\gamma_i u_n(t_i)-\gamma_i u(t_i)\|\\ &\le \int_{-\infty}^{t} Me^{-\omega(t-s)}\|((Ku_n)(s)+h(s,u_n(s)))-((Ku)(s)+h(s,u(s)))\|ds\\ &\quad+\sum_{t_i<t} Me^{-\omega(t-t_i)}\|\gamma_i u_n(t_i)-\gamma_i u(t_i)\|\\ &\le \int_{-\infty}^{t} Me^{-\omega(t-s)}(C_k\eta^{-1}+1)\varepsilon ds+\sum_{t_i<t} Me^{-\omega(t-t_i)}\varpi\varepsilon\\ &\le \left(\frac{M(C_k\eta^{-1}+1)}{\omega}+\frac{MN\varpi}{1-e^{-\omega\alpha}}\right)\varepsilon,\end{aligned}$$

which implies that Γ_2 is continuous.

(vi) $B(t)=\{(\Gamma_2 u)(t): u\in\mathcal{M}\}$ is a relatively compact subset of X in each $t\in\mathbb{R}$.

For each $t\in\mathbb{R}$, $0<\varepsilon<1$, $u\in\mathcal{M}$, define

$$\begin{aligned}(\Gamma_2^{\varepsilon}u)(t) &:= \int_{-\infty}^{t-\varepsilon} U(t,s)((Ku)(s)+h(s,u(s)))ds+\sum_{t_i<t-\varepsilon} U(t,t_i)(\gamma_i v(t_i)+\delta_i)\\ &= U(t,t-\varepsilon)\left[\int_{-\infty}^{t-\varepsilon} U(t-\varepsilon,s)((Ku)(s)+h(s,u(s)))ds+\sum_{t_i<t-\varepsilon} U(t-\varepsilon,t_i)(\gamma_i v(t_i)+\delta_i)\right]\\ &= U(t,t-\varepsilon)(\Gamma_2 u)(t-\varepsilon).\end{aligned}$$

Since $\{(\Gamma_2 u)(t-\varepsilon): u\in\mathcal{M}\}$ is bounded in X and $U(t,t-\varepsilon)$ is compact by (H_{10}), $\{(\Gamma_2^{\varepsilon}u)(t): u\in\mathcal{M}\}$ is a relatively compact subset of X. Moreover,

$$\begin{aligned}\|(\Gamma_2 u)(t)-(\Gamma_2^{\varepsilon}u)(t)\| &= \left\|\int_{t-\varepsilon}^{t} U(t,s)((Ku)(s)+h(s,u(s)))ds+\sum_{t-\varepsilon<t_i<t} U(t,t_i)(\gamma_i v(t_i)+\delta_i)\right\|\\ &\le \int_{t-\varepsilon}^{t}\|U(t,s)\|\|(Ku)(s)+h(s,u(s))\|ds+\sum_{t-\varepsilon<t_i<t}\|U(t,t_i)\|\|\gamma_i v(t_i)+\delta_i\|\\ &\le \int_{t-\varepsilon}^{t} Me^{-\omega(t-s)}\|(Ku)(s)+h(s,u(s))\|ds+\sum_{t-\varepsilon<t_i<t} Me^{-\omega(t-t_i)}\|\gamma_i v(t_i)+\delta_i\|\\ &\le \frac{\varepsilon M(C_k\eta^{-1}C_{1L_0}+C_{2L_0})}{\omega}+\frac{\varepsilon M(\varpi L_0+\kappa)}{\alpha}.\end{aligned}$$

Thus $\{(\Gamma_2 u)(t): u\in\mathcal{M}\}$ is a relatively compact subset of X in each $t\in\mathbb{R}$.

By (i), $\{\Gamma_2 u : u\in\mathcal{M}\}$ is equipotentially continuous at each interval (t_i,t_{i+1}) $(i\in\mathbb{Z})$. Since $\{\Gamma_2 u : u\in\mathcal{M}\}\subset PC_h^0(\mathbb{R},X)$, then $\{\Gamma_2 u : u\in\mathcal{M}\}$ is a relatively compact set by Lemma 2.1, and then

Γ_2 is a compact operator. Since $\mathcal{M}$ is a closed convex set, by Krasnoselskii's fixed point theorem (Theorem 2.1), Γ has a fixed point u in $\mathcal{M}$, which is the piecewise asymptotically almost periodic mild solution of (1.1). $\square$

The following existence result is based on the Banach contraction mapping principle.

Theorem 3.2 *Assume that* (H_1)–(H_4), (H_7), (H_8) *hold and satisfy the following conditions:*

(H_5') $g \in AAP_T(\mathbb{R} \times \Omega, X)$ *and there exists a constant* $L_g > 0$ *such that*

$$\|g(t,u) - \|g(t,v)\| \leq L_g\|u - v\|, \qquad t \in \mathbb{R}, \quad u, v \in \Omega.$$

(H_6') $h \in AAP_T(\mathbb{R} \times \Omega, X)$ *and there exists a constant* $L_h > 0$ *such that*

$$\|h(t,u) - \|h(t,v)\| \leq L_h\|u - v\|, \qquad t \in \mathbb{R}, \quad u, v \in \Omega.$$

Then (1.1) *has a unique mild solution* $u \in AAP_T(\mathbb{R}, X)$ *if* $\frac{M(C_k\eta^{-1}L_g + L_h)}{\omega} + \frac{MN\varpi}{1-e^{-\omega\alpha}} + L_f < 1$.

Proof Define the operator Γ as in (3.2). Similarly as the proof of Theorem 3.1, for $u \in AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$, one has $\Gamma u \in AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$. Hence $\Gamma(AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)) \subset AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$. It suffices now to show that Γ has a fixed point in $AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$. For $u, v \in AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$, one has

$$\begin{aligned}
\|(\Gamma u)(t) - (\Gamma v)(t)\| &\leq \int_{-\infty}^{t} \|U(t,s)\|\|((Ku)(s) + h(s,u(s))) - ((Kv)(s) + h(s,v(s)))\|ds \\
&\quad + \sum_{t_i < t} \|U(t,t_i)\|\|\gamma_i u(t_i) - \gamma_i v(t_i)\| + \|f(t,u(t)) - f(t,v(t))\| \\
&\leq \int_{-\infty}^{t} Me^{-\omega(t-s)}\|((Ku)(s) + h(s,u(s))) - ((Kv)(s) + h(s,v(s)))\|ds \\
&\quad + \sum_{t_i < t} Me^{-\omega(t-t_i)}\|\gamma_i u(t_i) - \gamma_i v(t_i)\| + \|f(t,u(t)) - f(t,v(t))\| \\
&\leq \left(\int_{-\infty}^{t} Me^{-\omega(t-s)}(C_k\eta^{-1}L_g + L_h)ds + \sum_{t_i < t} \varpi Me^{-\omega(t-t_i)} + L_f\right)\|u - v\| \\
&\leq \left(\frac{M(C_k\eta^{-1}L_g + L_h)}{\omega} + \frac{MN\varpi}{1 - e^{-\omega\alpha}} + L_f\right)\|u - v\|.
\end{aligned}$$

Since $\frac{M(C_k\eta^{-1}L_g + L_h)}{\omega} + \frac{MN\varpi}{1-e^{-\omega\alpha}} + L_f < 1$, Γ is a contraction. By the Banach contraction mapping principle, Γ has a unique fixed point in $AAP_T(\mathbb{R}, X) \cap \mathcal{UPC}(\mathbb{R}, X)$, which is the unique piecewise asymptotically almost periodic mild solution to (1.1). $\square$

4. Example

Consider the impulsive partial differential equations with Dirichlet conditions

$$\begin{cases} \dfrac{\partial}{\partial t}(u(t,x)+f(t,x,u(t,x))) = \dfrac{\partial^2}{\partial x^2}(u(t,x)+f(t,x,u(t,x))) - 2(u(t,x)+f(t,x,u(t,x))) \\ + \left(\sin t + \sin\sqrt{2}t\right)(u(t,x)+f(t,x,u(t,x))) + \displaystyle\int_{-\infty}^{t} k(t-s)g(s,x,u(s,x))ds + h(t,x,u(t,x)), \\ \Delta u(t_i,x) = \beta_i u(t_i,x), i \in \mathbb{Z}, \quad x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad t \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $f,g,h \in AAP_T(\mathbb{R}\times[0,\pi]\times L^2[0,\pi], L^2[0,\pi])$, $t_i = i+\frac{1}{4}|\sin i + \sin\sqrt{2}i|$, $\beta_i \in AAP(\mathbb{Z},\mathbb{R})$, and $\sup\limits_{i\in\mathbb{Z}}|\beta_i| \le \varpi$. Note that $\{t_i^j\}$, $i\in\mathbb{Z}$, $j\in\mathbb{Z}$ are equipotentially almost periodic and $\alpha = \inf\limits_{i\in\mathbb{Z}}(t_{i+1}-t_i) > 0$; one can see [19, 26] for more details.

Take $X = L^2[0,\pi]$ is equipped with its natural topology and define

$$\mathcal{D}(A) = \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0\},$$
$$Au = u'' - 2u, \qquad \text{for all} \quad u \in \mathcal{D}(A).$$

Let $\varphi_n(t) = \sqrt{\dfrac{2}{\pi}}\sin(nt)$ for all $n \in \mathbb{N}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge0}$ on $L^2[0,\pi]$ with $\|T(t)\| \le e^{-3t}$ for $t \ge 0$. Moreover,

$$T(t)\varphi = \sum_{n=1}^{\infty} e^{-(n^2+2)t}\langle\varphi,\varphi_n\rangle\varphi_n,$$

for each $\varphi \in L^2[0,\pi]$.

Define a family of linear operators $A(t)$ by

$$\mathcal{D}(A(t)) = \mathcal{D}(A),$$
$$A(t)\varphi(x) = \left(A + \sin t + \sin\sqrt{2}t\right)\varphi(x), \qquad \forall\, x \in [0,\pi], \quad \varphi \in \mathcal{D}(A).$$

Then the system

$$u'(t) = A(t)u(t), \qquad t \ge s,$$
$$u(s) = \varphi \in L^2[0,\pi],$$

has an associated evolution family $(U(t,s))_{t\ge s}$ on $L^2[0,\pi]$, which can be explicitly expressed by

$$U(t,s)\varphi = T(t-s)e^{\int_s^t(\sin\tau+\sin\sqrt{2}\tau)d\tau}\varphi.$$

Moreover,

$$\|U(t,s)\| \le e^{-(t-s)} \qquad \text{for every} \quad t \ge s.$$

Note that $\sin t + \sin\sqrt{2}t \in AP(\mathbb{R},\mathbb{R})$ and it is not difficult to verify that (H_1)–(H_3), (H_7) hold with $M = 1, \omega = 1$. One can see [11] for more details.

Now the following theorem is an immediate consequence of Theorem 3.2.

Theorem 4.1 *Under the assumptions* (H_4), (H_5'), (H_6'), (H_8), (4.1) *admits a unique mild solution* $u \in AAP_T(\mathbb{R}, L^2[0,\pi])$ *if* $C_k\eta^{-1}L_g + L_h + \frac{N\varpi}{1-e^{-\alpha}} + L_f < 1$.

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