On Armendariz-like properties in amalgamated algebras along ideals

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Abstract: Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this paper, we investigate the transfer of Armendariz-like properties to the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$) introduced and studied by D'Anna, Finocchiaro, and Fontana in 2009. Our aim is to provide necessary and sufficient conditions for $A \bowtie^f J$ to be an Armendariz ring, nil-Armendariz ring, and weak Armendariz ring.

Key words: Amalgamated algebra along an ideal, reduced ring, Armendariz ring, nil-Armendariz ring, weak Armendariz, semicommutative ring

1. Introduction

All rings considered are associative with identity elements and all modules are unital. Given a ring R, $nil(R)$ denotes the set of all nilpotent elements of R and the polynomial ring over R is denoted by $R[x]$. For a polynomial $f(x) \in R[x]$, the content of $f(x)$, denoted by $c(f)$, is the ideal of R generated by all coefficients of $f(x)$. In [13], Rege and Chhawchharia introduced the notion of the Armendariz ring as an associative ring R with identity such that for all polynomials $f(x) = \sum_{i=0}^{i=m} a_i x^i$ and $g(x) = \sum_{j=0}^{j=n} b_j x^j$ in $R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for every i, j. The name was chosen because Armendariz had shown that a reduced ring (i.e. a ring without nonzero nilpotent elements) satisfies this property [3]. Later, in 1998, Anderson and Camillo continued this investigation by studying Armendariz rings and Gauss rings (recall that a ring R is said to be a Gauss ring if for all polynomials $f(x)$ and $g(x)$ in $R[x]$, $c(fg) = c(f)c(g)$). Among others, they proved that a commutative ring R is Gaussian if and only if each homomorphic image of R is an Armendariz ring [1]. Since then, various generalizations of Armendariz rings such as the skew Armendariz ring, weak Armendariz ring, central Armendariz ring, and nil-Armendariz ring appeared in the literature.

In 2006, Liu and Zhao [11] introduced the notion of a weak Armendariz ring as a ring R such that whenever two polynomials $f(x) = \sum_{i=0}^{i=m} a_i x^i$ and $g(x) = \sum_{j=0}^{j=n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in nil(R)$ for every i, j. Among others, they proved that a ring R is a weak Armendariz ring if and only if for every positive integer n, the n-by-n upper triangular matrix ring $T_n(R)$ is a weak Armendariz ring. Moreover,

if R is a semicommutative ring (i.e. a ring such that whenever $ab = 0$, $aRb = 0$), then the polynomial ring $R[x]$ and the ring $R[x]/(x^n)$ are weak Armendariz rings. Here, it is worth noting that a weaker version of the Armendariz ring notion, also called a "weak Armendariz ring" (or 1-Armendariz ring), is due to Lee and Wong [10] in the sense that whenever two linear polynomials $f(x) = a_0 + a_1 x$ and $g(x) = b_0 + b_1 x$ satisfy $fg = 0$, then $a_i b_j = 0$ for every $i, j = 0, 1$.

In 2008, observing that in all examples found in the literature of Armendariz and weak Armendariz rings, the set of nilpotent elements forms an ideal, Antoine proved that this is not true in general and he provided an example of an Armendariz ring R for which $nil(R)$ is not an ideal [2, Example 4.8]. However, if $nil(R)$ is an ideal of R, then R is a weak Armendariz ring, and in fact R satisfies a stronger condition. This allowed him to introduce the notion of nil-Armendariz ring as a ring R such that whenever two polynomials $f(x) = \sum_{i=0}^{i=m} a_i x^i$ and $g(x) = \sum_{j=0}^{i=n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $a_i b_j \in nil(R)$ for every i, j. He proved that if R is a nil-Armendariz ring, then $nil(R)$ is a subring without the unit of R. He also studied the conditions under which the polynomial ring over a nil-Armendariz ring is a nil-Armendariz ring.

The following diagram of implication summarizes the relation between the above notions: reduced ring $\implies$ Armendariz ring $\implies$ nil-Armendariz ring $\implies$ weak Armendariz ring. The reverses of the first and second implications are not, in general, true and examples can be found in [9, Proposition 2.1] and [2, Example 4.9]. However, we do not know so far any example of a weak Armendariz ring that is not a nil-Armendariz ring. This question was left open in [2].

Let A and B be two rings with unity, let J be an ideal of B, and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\},$$

called *the amalgamation of A and B along J with respect to f*. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [5, 7, 8]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations)(cf. [12, page 2]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [6, Examples 2.5 and 2.6] and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [4]) are strictly related to it [6, Example 2.7 and Remark 2.8]. In [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [8], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, chain properties, and Krull dimension.

This paper aims at studying the transfer of the notions of "Armendariz ring", "nil-Armendariz ring", and "weak Armendariz ring" to the amalgamation of algebras along ideals. It contains, in addition to the introduction, three sections and each section deals respectively with one of the mentioned notions. The main results (Theorem 2.2, Theorem 3.1, and Theorem 4.1) can be summarized as follows:

Theorem 1.1 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism, and J be a proper ideal of B.*

1. *If $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring, then A is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*

2. *If A and $f(A) + J$ are Armendariz (resp. nil-Armendariz, resp. weak Armendariz) rings, then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*

3. *Assume that $J \cap S \neq \varnothing$ where S is the set of regular central elements of B. Then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring if and only if A and $f(A) + J$ are Armendariz (resp. nil-Armendariz, resp. weak Armendariz) rings.*

4. *Assume that $J \cap nil(B) = (0)$ (resp. $J \subseteq nil(B)$). Then $A \bowtie^f J$ is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring if and only if A is an Armendariz (resp. a nil-Armendariz, resp. a weak Armendariz) ring.*

5. *Assume that $f^{-1}(J) \cap nil(A) = (0)$ (resp. $f^{-1}(J) \subseteq nil(A)$). If $f(A) + J$ is an Armendariz (resp. a weak Armendariz) ring, then $A \bowtie^f J$ is an Armendariz (resp. a weak Armendariz) ring, and the equivalence holds for nil-Armendariz property.*

6. *Assume that f is injective.*
   (i) $f(A) \cap J = \{0\}$. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A) + J$ is a weak Armendariz ring.
   (ii) $J \subseteq nil(B)$. If $f(A) + J$ is a weak Armendariz, then $A \bowtie^f J$ is a weak Armendariz ring.

7. *Assume that J is semicommutative. If A is a weak Armendariz ring, then so is $A \bowtie^f J$.*

8. *Assume that $f^{-1}(J)$ is semicommutative. If $f(A) + J$ is a weak Armendariz ring, then so is $A \bowtie^f J$.*

It is worth mentioning that the proofs of some assertions of the above theorem are very similar, and for the convenience of the reader, we separate the three notions in three sections and we omit the similar proofs to avoid repetitions as much as possible.

Definition 1.2 (1) *A ring R is called a reduced ring if it has no nonzero nilpotent elements.*
(2) *A ring R is called a semicommutative ring if for every $a, b \in R$, $ab = 0$ implies that $aRb = 0$ ([14, Lemma 1.2]).*
(3) *A ring R is called an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1 x + ... + a_n x^n$, $g(x) = b_0 + b_1 x + ... + b_m x^m$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j [13, Definition 1.1].*

(4) *A ring R is called a nil-Armendariz ring if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_i x^i$ and*

$g(x) = \sum_{j=0}^{j=m} b_j x^j$ *in $R[x]$ satisfies $f(x)g(x) \in nil(R)[x]$, then $a_i b_j \in nil(R)$ for each i, j [2, Definition 2.3].*

(5) *A ring R is called a weak Armendariz ring if whenever the product of two polynomials $f(x) = \sum\limits_{i=0}^{i=n} a_i x^i$ and*

$g(x) = \sum\limits_{j=0}^{j=m} b_j x^j$ *in $R[x]$ satisfies $f(x)g(x) = 0$, then $a_i b_j \in nil(R)$ for each i, j [11, Definition 2.1].*

2. Armendariz property in amalgamated algebra along an ideal

We start this section by the following proposition, which characterizes when the amalgamated algebra $A \bowtie^f J$ is a reduced ring.

Proposition 2.1 *([6, Proposition 5.4]) Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism, and J be a proper ideal of B. The following conditions are equivalent:*

1. *$A \bowtie^f J$ is a reduced ring.*

2. *A is a reduced ring and $nil(B) \cap J = (0)$*

In particular, if A and B are reduced, then $A \bowtie^f J$ is reduced; conversely, if J is a radical ideal of B and $A \bowtie^f J$ is reduced, then B (and A) is reduced.

Our next theorem states necessary and sufficient conditions under which the amalgamated algebra $A \bowtie^f J$ is an Armendariz ring. We notice that statements (1) and (2) are immediate consequences of the fact that Armendariz-like conditions pass trivially to subrings and finite products. For the convenience of the reader, we give simple proofs.

Theorem 2.2 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism, and J be a proper ideal of B.*

1. *If $A \bowtie^f J$ is an Armendariz ring, then so is A.*

2. *If A and $f(A) + J$ are Armendariz rings, then so is $A \bowtie^f J$.*

3. *Assume that $J \cap S \neq \varnothing$ where S the set of regular central elements of B. Then $A \bowtie^f J$ is an Armendariz ring if and only if $f(A) + J$ and A are Armendariz rings.*

4. *Assume that $J \cap nil(B) = (0)$. Then $A \bowtie^f J$ is an Armendariz ring if and only if A is an Armendariz ring.*

5. *Assume that $f^{-1}(J) \cap nil(A) = (0)$. If $f(A) + J$ is an Armendariz ring, then $A \bowtie^f J$ is an Armendariz ring.*

Proof   (1) Assume that $A \bowtie^f J$ is Armendariz and let $f_A(x) = \sum\limits_{i=0}^{i=n} a_i x^j$ and $g_A(x) = \sum\limits_{j=0}^{j=m} b_j x^j$ be two

polynomials in $A[x]$ such that $f_A(x)g_A(x) = 0$. Then for every $k \in \{0, ..., n + m\}$, $\sum\limits_{i+j=k} a_i b_j = 0$. Set

$$F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i))x^i \text{ and } G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j))x^j. \text{ Then}$$

$$
\begin{aligned}
F(x)G(x) &= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}(a_ib_j, f(a_ib_j)))x^k \\
&= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}a_ib_j, \sum_{i+j=k}f(a_ib_j))x^k \\
&= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}a_ib_j, f(\sum_{i+j=k}a_ib_j))x^k.
\end{aligned}
$$

Hence, $F(x)G(x) = 0$ and so $(a_ib_j, f(a_ib_j)) = 0$ since $A \bowtie^f J$ is Armendariz. Thus, $a_ib_j = 0$ and consequently A is Armendariz.

(2) Assume that A and $f(A) + J$ are Armendariz and let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$, $g_B(x) = \sum_{j=0}^{j=m}(f(b_j) + k_j)x^j$, $f_A(x) = \sum_{i=0}^{i=n}a_ix^j$, and $g_A(x) = \sum_{j=0}^{j=m}b_jx^j$. Then $F(x)G(x) = 0$ implies that $f_A(x)g_A(x) = 0$ and $f_B(x)g_B(x) = 0$, which in turn implies that $(f(a_i) + j_i)(f(b_j) + k_j) = 0$ and $a_ib_j = 0$ for every i, j since $f(A) + J$ and A are Armendariz rings. Therefore, $A \bowtie^f J$ is Armendariz.

(3) Let S be the set of regular central elements of B. Assume that $J \cap S \neq \varnothing$ and $A \bowtie^f J$ is Armendariz. Let $f_A(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$ and $g_A(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$ be two polynomials in $(f(A)+J)[x]$ such that $f_A(x)g_A(x) = 0$ and let e be a regular element of J. Set $F(x) = \sum_{i=0}^{i=n}(0, e(f(a_i) + j_i))x^i$ and $G(x) = \sum_{j=0}^{j=m}(0, e(f(b_j) + k_j))x^j$. Clearly

$$
\begin{aligned}
F(x)G(x) &= \sum_{k=0}^{k=n+m}(\sum_{i+j=k}(0, e^2(f(a_i) + j_i)(f(b_j) + k_j))x^k \\
&= \sum_{k=0}^{k=n+m}(0, e^2\sum_{i+j=k}(f(a_i) + j_i)(f(b_j) + k_j))x^k = 0.
\end{aligned}
$$

Thus, $(0, e(f(a_i) + j_i))(0, e(f(b_j) + k_j)) = 0$ since $A \bowtie^f J$ is Armendariz, which implies that $e^2(f(a_i) + j_i)(f(b_j) + k_j) = 0$ for every i, j. Hence, $(f(a_i) + j_i)(f(b_j) + k_j) = 0$, and this shows that $f(A) + J$ is Armendariz.

(4) Assume that $J \cap nil(B) = (0)$ and A is Armendariz. Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i) + j_i)x^i$ and $G(x) =$

$\sum_{t=0}^{t=m}(b_t, f(b_t)+k_t)x^t$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$,

$g_B(x) = \sum_{t=0}^{t=m}(f(b_t) + k_t)x^t$, $f_A(x) = \sum_{i=0}^{i=n}a_ix^j$, and $g_A(x) = \sum_{t=0}^{t=m}b_tx^t$. Then $F(x)G(x) = 0$ implies that

$f_A g_A = 0$ (and $f_B g_B = 0$), which in turn implies that $a_i b_t = 0$ since A is an Armendariz ring. Thus, $(f(a_i) + j_i)(f(b_t) + k_t) \in J$ for every i, t. Next, we show that $(f(a_i) + j_i)(f(b_t) + k_t) = 0$ for every i, t. For this, we proceed by induction on the degree n of $F(x)$. If $n = 0$, it is clear. Suppose that $n \geq 1$ and the induction hypothesis.

Claim: $(f(a_0) + j_0)(f(b_t) + k_t) = 0$ for every $0 \leq t \leq m$. Indeed, suppose that $\exists t \in \{0, ..., m\}$ such that $(f(a_0)+j_0)(f(b_t) + k_t) \neq 0$ and let l be the smallest integer in $\{0, ..., m\}$ such that $(f(a_0)+j_0)(f(b_l)+k_l) \neq 0$. Then for $t \in \{0, ..., l - 1\}$, $(f(a_0) + j_0)(f(b_t) + k_t) = 0$ and so $((f(b_t) + k_t)J(f(a_0) + j_0))^2 = 0$. Thus, $(f(b_t)+k_t)J(f(a_0)+j_0) = 0$ since $J \cap nil(B) = (0)$. Hence, $(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)((f(a_0)+j_0)(f(b_l)+k_l))^2 = (f(a_{l-t})+j_{l-t})(f(b_t)+k_t)(f(a_0)+j_0)(f(b_l)+k_l)(f(a_0)+j_0)(f(b_l)+k_l) \in (f(a_{l-t})+j_{l-t})((f(b_t)+k_t)J(f(a_0)+j_0))(f(b_l)+k_l) = 0$, but since the coefficient of the term x^l in $f_B g_B = 0$ is zero, we obtain
$0 = (f(a_0)+j_0)(f(b_l)+k_l)+(f(a_1)+j_1)(f(b_{l-1})+k_{l-1})+...+(f(a_l)+j_l)(f(b_0)+k_0) = (f(a_0)+j_0)(f(b_l)+k_l)+$

$\sum_{t=1}^{t=l-1}(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)$. Multiplying $((f(a_0)+j_0)(f(b_l)+k_l))^2$ to the preceding equation on the right

side, we obtain: $((f(a_0)+j_0)(f(b_l)+k_l))^3 + \sum_{t=1}^{t=l-1}(f(a_{l-t})+j_{l-t})(f(b_t)+k_t)((f(a_0)+j_0)(f(b_l)+k_l))^2 = 0$. Hence,

$((f(a_0)+j_0)(f(b_l)+k_l))^3 = 0$ and so $(f(a_0)+j_0)(f(b_l)+k_l) \in J \cap nil(B) = 0$. Thus, $(f(a_0)+j_0)(f(b_l)+k_l) = 0$, which is a contradiction. Consequently, $(f(a_0) + j_0)(f(b_t) + k_t) = 0$ for every $t \in \{0, ..., m\}$. Now, set $F_1(x) = (f(a_1) + j_1) + (f(a_2) + j_2)x + ... + (f(a_n) + j_n)x^{n-1}$. Then $F(x) = (a_0, f(a_0) + j_0) + xF_1(x)$, and by the claim, $(a_0, f(a_0) + j_0)G(x) = 0$. Thus, $F_1(x)G(x) = 0$, and by the induction hypothesis, $(f(a_i) + j_i)(f(b_t) + k_t) = 0$ for every $1 \leq i \leq n$ and $0 \leq t \leq m$. Therefore, $(f(a_i) + j_i)(f(b_t) + k_t) = 0$ for every $0 \leq i \leq n$ and $0 \leq t \leq m$ and hence $(a_i, f(a_i) + j_i)(b_t, f(b_t) + k_t) = 0$ for every $0 \leq i \leq n$ and $0 \leq t \leq m$. It follows that $A \bowtie^f J$ is Armendariz.

(5) Assume that $f^{-1}(J) \cap nil(A) = (0)$ and $f(A)+J$ is an Armendariz ring. Our argument is similar to the

one in (4). Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j)+k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$

such that $F(x)G(x) = 0$. Set $f_B(x) = \sum_{i=0}^{i=n}(f(a_i) + j_i)x^i$, $g_B(x) = \sum_{j=0}^{j=m}(f(b_j) + k_j)x^j$, $f_A(x) = \sum_{i=0}^{i=n}a_ix^j$, and

$g_A(x) = \sum_{j=0}^{j=m}b_jx^j$. Since $F(x)G(x) = 0$, $f_A g_A = 0$ and $f_B g_B = 0$. Thus, $(f(a_i) + j_i)(f(b_j) + k_j) = 0$

for every i, j since $f(A) + J$ is an Armendariz ring, and hence $a_i b_j \in f^{-1}(J)$. To show that $a_i b_j = 0$ for every i, j, we proceed by induction on the degree n of $F(x)$. If $n = 0$, this is trivial. Suppose that $n \geq 1$

and the induction hypothesis. First we show that $a_0 b_j = 0$ for every $0 \le j \le m$. Indeed, suppose that $\exists j \in \{0; ...; m\}$ such that $a_0 b_j \ne 0$. Let k be the smallest positive integer in $\{0, ..., m\}$ such that $a_0 b_k \ne 0$. Then for $j \in \{0, ..., k-1\}$, $a_0 b_j = 0$ and so $(b_j f^{-1}(J) a_0)^2 = 0$. Then $b_j f^{-1}(J) a_0 \subseteq f^{-1}(J) \cap nil(A) = (0)$ and so $b_j f^{-1}(J) a_0 = 0$. Hence, $(a_{k-j} b_j)(a_0 b_k)^2 = a_{k-j} b_j a_0 b_k a_0 b_k \in a_{k-j}(b_j f^{-1}(J) a_0) b_k = 0$. The coefficient of the term x^k in $f_A(x) g_A(x) = 0$ is $0 = a_0 b_k + a_1 b_{k-1} + ... + a_k b_0 = a_0 b_k + \sum_{j=1}^{j=k-1} a_{k-j} b_j$. Multiplying

$(a_0 b_k)^2$ to the preceding equation on the right side, we obtain $(a_0 b_k)^3 + \sum_{j=1}^{j=k-1} (a_{k-j} b_j)(a_0 b_k)^2 = 0$. Hence,

$(a_0 b_k)^3 = 0$ and so $(a_0 b_k) \in f^{-1}(J) \cap nil(A) = (0)$, which is a contradiction. Consequently, $a_0 b_j = 0$, for every $j \in \{0, ..., m\}$. Finally, as in (4), set $F_1(x) = (f(a_1) + j_1) + (f(a_2) + j_2)x + ... + (f(a_n) + j_n)x^{n-1}$. Then $F(x) = (a_0, f(a_0) + j_0) + x F_1(x)$, and by the claim, $(a_0, f(a_0) + j_0)G(x) = 0$. Thus, $F_1(x)G(x) = 0$, and by the induction hypothesis $a_i b_j = 0$ for every $1 \le i \le n$ and $0 \le j \le m$. Therefore, $a_i b_j = 0$ for every i, j and hence $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j) = 0$ for every i, j. It follows that $A \bowtie^f J$ is an Armendariz ring. $\qquad \square$

3. Nil-Armendariz property in amalgamated algebra along an ideal

Theorem 3.1 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism, and J be a proper ideal of B; then:*

1. *If $A \bowtie^f J$ is a nil-Armendariz ring, then so is A.*

2. *If A and $f(A) + J$ are nil-Armendariz rings, then so is $A \bowtie^f J$.*

3. *Assume that $J \cap S \ne \varnothing$ where S is the set of regular central elements of B. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A) + J$ and A are nil-Armendariz rings.*

4. *Assume that $J \subseteq nil(B)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if A is a nil-Armendariz ring.*

5. *Assume that $f^{-1}(J) \subseteq nil(A)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A) + J$ is a nil-Armendariz ring.*

6. *Assume that f is injective.*
   (i) $f(A) \cap J = 0$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A) + J$ is a nil-Armendariz ring.
   (ii) $J \subseteq nil(B)$. Then $A \bowtie^f J$ is a nil-Armendariz ring if and only if $f(A) + J$ is a nil-Armendariz ring.

Proof The proofs of assertions (1), (2), and (3) are similar to (1), (2), and (3) in Theorem 2.2.

(4) Suppose that A is a nil-Armendariz ring. Then $\dfrac{A \bowtie^f J}{0 \times J} \simeq A$ is a nil-Armendariz ring. Let $F(x) = \sum_{i=0}^{i=n} (a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m} (b_j, f(b_j) + k_j)x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) = $

$$\sum_{k=0}^{k=n+m} (\sum_{i+j=k} (a_i b_j, (f(a_i) + j_i)(f(b_j) + k_j))) x^k \in nil((A \bowtie^f J)[x]). \quad \text{Set} \quad \overline{F(x)} = \sum_{i=0}^{i=n} \overline{(a_i, f(a_i) + j_i)} x^i \quad \text{and}$$

$$\overline{G(x)} = \sum_{j=0}^{j=m} \overline{(b_j, f(b_j) + k_j)} x^j \quad \text{in} \quad \frac{A \bowtie^f J}{0 \times J}[x]. \quad \text{Then} \quad F(x)G(x) \in nil(A \bowtie^f J)[x] \quad \text{implies that} \quad \overline{F(x)}\,\overline{G(x)} \in$$

$nil \frac{A \bowtie^f J}{0 \times J}[x]. \quad \text{Consequently,} \quad \overline{(a_i, f(a_i) + j_i)}\,\overline{(b_j, f(b_j) + k_j)} \in nil \frac{A \bowtie^f J}{0 \times J} \quad \text{since} \quad \frac{A \bowtie^f J}{0 \times J} \quad \text{is nil-Armendariz.}$

Hence, $(a_i b_j, (f(a_i) + j_i)(f(b_j) + k_j))^{p_{ij}} \in 0 \times J$ for some integer p_{ij}. Therefore, $((f(a_i) + j_i)(f(b_j) + k_j))^{p_{ij}} \in J \subseteq nil(B)$. Hence, $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j) \in nil(A \bowtie^f J)$ and this shows that $A \bowtie^f J$ is nil-Armendariz.

(5) Assume that $f^{-1}(J) \subseteq nil(A)$ and suppose that $A \bowtie^f J$ is nil-Armendariz. Let $f_A(x) = \sum_{i=0}^{i=n} (f(a_i) + j_i) x^i$

and $g_A(x) = \sum_{j=0}^{j=m} (f(b_j) + k_j) x^j$ such that $f_A(x)g_A(x) \in nil(f(A) + J)[x]$. Let $F(x) = \sum_{i=0}^{i=n} (a_i, f(a_i) + j_i) x^i$ and

$G(x) = \sum_{j=0}^{j=m} (b_j, f(b_j) + k_j) x^j$. Since $f_A(x)g_A(x) = \sum_{k=0}^{k=n+m} (\sum_{i+j=k} (f(a_i) + j_i)(f(b_j) + k_j)) x^k \in nil(f(A) + J)[x]$,

$\sum_{i+j=k} (f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ for every $k \in \{0, ..., n + m\}$. Thus, $\sum_{i+j=k} (f(a_i b_j) + t_{ij}) \in$

$nil(f(A) + J)$ with $t_{ij} \in J$. Hence, for every $k \in \{0, ..., n + m\}$, $f(\sum_{i+j=k} a_i b_j) + \sum_{i+j=k} t_{ij}$ is nilpotent,

so $(f(\sum_{i+j=k} a_i b_j))^{n_{ij}} \in J$ for some positive integer n_{ij}, and therefore $(\sum_{i+j=k} a_i b_j)^{n_{ij}} \in f^{-1}(J) \subseteq nil(A)$,

which in turn implies that $\sum_{i+j=k} a_i b_j \in nil(A)$. Consequently, $F(x)G(x) \in nil(A \bowtie^f J)[x]$ and hence

$(\sum_{i+j=k} a_i b_j, \sum_{i+j=k} (f(a_i) + j_i)(f(b_j) + k_j)) \in nil(A \bowtie^f J)$. Since $A \bowtie^f J$ is a nil-Armendariz ring, $(a_i b_j, (f(a_i) + j_i)(f(b_j) + k_j)) = (a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j)$ is nilpotent and so $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$. Hence, $f(A) + J$ is nil-Armendariz, as desired.

The converse is similar to (4) by using the fact that $\dfrac{A \bowtie^f J}{f^{-1}(J) \times 0} \simeq f(A) + J$.

(6) Assume that f is injective.

(i) $f(A) \cap J = 0$. In this case $A \bowtie^f J \simeq f(A) + J$ and the conclusion follows.

(ii) Assume that $J \subseteq nil(B)$ and suppose that $f(A) + J$ is nil-Armendariz. Let $F(x) = \sum_{i=0}^{i=n} (a_i, f(a_i) + j_i) x^i$

and $G(x) = \sum_{j=0}^{j=m} (b_j, f(b_j) + k_j) x^j$ be two polynomials in $(A \bowtie^f J)[x]$ such that $F(x)G(x) \in nil(A \bowtie^f J)[x]$.

Set $f_B(x) = \sum_{i=0}^{i=n} (f(a_i) + j_i) x^i$ and $g_B(x) = \sum_{j=0}^{j=m} (f(b_j) + k_j) x^j$. Then $F(x)G(x) \in nil(A \bowtie^f J)[x]$ implies that

$f_B(x)g_B(x) \in nil(f(A) + J)[x]$. Hence, $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ since $f(A) + J$ is nil-Armendariz.

Now we show that $a_i b_j$ is nilpotent. Indeed, since $(f(a_i)+j_i)(f(b_j)+k_j) = (f(a_i b_j)+t_{ij}) \in nil(f(A)+J), t_{ij} \in J$, $(f(a_i b_j) + t_{ij})^{n_{ij}} = 0$ for some positive integer n_{ij}. Therefore, $(f(a_i b_j))^{n_{ij}} = f((a_i b_j)^{n_{ij}}) \in J \subseteq nil(B)$ and so $(f((a_i b_j)^{n_{ij}}))^{m_{ij}} = 0$ for some positive integer m_{ij}. Hence, $f(((a_i b_j)^{n_{ij}})^{m_{ij}}) = 0$ and therefore $(a_i b_j)^{n_{ij} m_{ij}} = 0$ since f is injective. Consequently, $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j)$ is nilpotent and this shows that $A \bowtie^f J$ is nil-Armendariz.

Conversely, suppose that $A \bowtie^f J$ is nil-Armendariz and let $f_A(x) = \sum_{i=0}^{i=n}(f(a_i) + j_i)x^i$ and $g_A(x) = \sum_{j=0}^{j=m}(f(b_j) + k_j)x^j$ be two polynomials in $(f(A) + J)[x]$ such that $f_A(x)g_A(x) \in nil(f(A) + J)[x]$. Set $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j) + k_j)x^j$. Then

$$
\begin{aligned}
F(x)G(x) &= \sum_{k=0}^{k=n+m}\left(\sum_{i+j=k}(a_i b_j, (f(a_i) + j_i)(f(b_j) + k_j))\right)x^k \\
&= \sum_{k=0}^{k=n+m}\left(\sum_{i+j=k}a_i b_j, \sum_{i+j=k}(f(a_i) + j_i)(f(b_j) + k_j)\right)x^k,
\end{aligned}
$$

but $f_A(x)g_A(x) \in nil(f(A) + J)[x]$ implies that $(\sum_{i+j=k}(f(a_i) + j_i)(f(b_j) + k_j))^{n_{ij}} = 0$ for some positive integer n_{ij}. Thus, $(\sum_{i+j=k}(f(a_i b_j) + t_{ij}))^{n_{ij}} = 0$ for some positive integer n_{ij} and so $(f(\sum_{i+j=k}a_i b_j) + \sum_{i+j=k}t_{ij})^{n_{ij}} = 0$. Hence, $(f(\sum_{i+j=k}a_i b_j)^{n_{ij}} \in J \subseteq nil(B)$ and therefore $f((\sum_{i+j=k}a_i b_j)^{m_{ij}}) = 0$ for some positive integer m_{ij}. Since f is injective, $(\sum_{i+j=k}a_i b_j)^{m_{ij}} = 0$ and hence $F(x)G(x) \in nil(A \bowtie^f J)[x]$, which in turn implies that $(a_i b_j, (f(a_i) + j_i)(f(b_j) + k_j)) \in nil(A \bowtie^f J)$. Therefore, $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ and this shows that $f(A) + J$ is nil-Armendariz. □

4. Weak Armendariz property in amalgamated algebra along an ideal

Theorem 4.1 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism, and J be a proper ideal of B; then:*

1. *If $A \bowtie^f J$ is a weak Armendariz ring, then so is A.*

2. *If A and $f(A) + J$ are weak Armendariz rings, then so is $A \bowtie^f J$.*

3. *Assume that $J \cap S \neq \varnothing$ where S is the set of regular central elements of B. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A) + J$ and A are weak Armendariz rings.*

4. *Assume that $J \subseteq nil(B)$. Then A is weak Armendariz ring if and only if $A \bowtie^f J$ is a weak Armendariz ring.*

5. Assume that $f^{-1}(J) \subseteq nil(A)$. If $f(A)+J$ is a weak Armendariz ring, then $A \bowtie^f J$ is a weak Armendariz ring.

6. Assume that f is injective.

   (i) $f(A) \cap J = 0$. Then $A \bowtie^f J$ is a weak Armendariz ring if and only if $f(A)+J$ is a weak Armendariz ring.

   (ii) $J \subseteq nil(B)$. If $f(A)+J$ is a weak Armendariz ring, then $A \bowtie^f J$ is a weak Armendariz ring.

7. Assume that J is semicommutative. If A is a weak Armendariz ring, then so is $A \bowtie^f J$.

8. Assume that $f^{-1}(J)$ is semicommutative. If $f(A)+J$ is a weak Armendariz ring, then so is $A \bowtie^f J$.

Proof   The assertions (1), (2), and (3) are similar to (1), (2), and (3) in Theorem 2.2, and the assertions (4), (5), and (6) are similar to (4), (5), and (6) in Theorem 3.1.

(7) Assume that J is semicommutative and A is weak Armendariz. Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j) + k_j)x^j$ be two polynomials in $A \bowtie^f J[x]$ such that $F(x)G(x) = 0$ and set $f_A(x) = \sum_{i=0}^{i=n} a_i x^j$, $g_A(x) = \sum_{j=0}^{j=m} b_j x^j$, $f_B(x) = \sum_{i=0}^{i=n}(f(a_i)+j_i)x^i$, and $g_B(x) = \sum_{j=0}^{j=m}(f(b_j)+k_j)x^j$. Then $F(x)G(x) = 0$ implies that

$$f_A(x)g_A(x) = \sum_{l=0}^{l=n+m} (\sum_{i+j=l} a_i b_j)x^l = 0 \text{ and } f_B(x)g_B(x) = \sum_{l=0}^{l=n+m} (\sum_{i+j=l}(f(a_i) + j_i)(f(b_j) + k_j))x^l = 0. \text{ Hence,}$$

$\sum_{i+j=l} a_i b_j = 0$ for all $l = 0, 1..., n+m$ and $\sum_{i+j=l}(f(a_i) + j_i)(f(b_j) + k_j) = 0$ for all $l = 0, 1..., n+m$. Thus, $a_i b_j \in nil(A)$ since A is weak Armendariz, and so $(a_i b_j)^{n_{ij}} = 0$ for some positive integer n_{ij}. To show that $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$, we proceed by induction on $i + j$.

   If $i + j = 0$, we have $(f(a_0) + j_0)(f(b_0) + k_0) = 0 \in nil(f(A) + J)$.

   Let l be a positive integer such that $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ when $i + j < l$. We will show that $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ when $i + j = l$. We have $((f(a_0) + j_0)(f(b_l) + k_l))^{n_{0l}=p} \in J$ since $(a_0 b_l)^p = 0$. By the induction hypothesis, $(f(a_0) + j_0)(f(b_{l-1}) + k_{l-1}) \in nil(f(A) + J)$. Let t be a positive integer such that $((f(a_0) + j_0)(f(b_{l-1}) + k_{l-1}))^t = 0$. Then $((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0))^{t+1} = 0$, and hence $(((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}(f(a_1) + j_1)) \times ((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0))^{t+1}((f(b_{l-1}) + k_{l-1})((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}) = 0$. Since $(((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}(f(a_1) + j_1))(f(b_{l-1}) + k_{l-1})(f(a_0) + j_0) \in J$, $((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0))^t((f(b_{l-1}) + k_{l-1})((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}) \in J$, $(f(b_l) + k_l)((f(a_0) + j_0)(f(b_l) + k_l))^p)(f(a_1) + j_1) \in J$ and J is semicommutative. It follows that $(((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}(f(a_1) + j_1))((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0)) \times (f(b_l) + k_l)((f(a_0) + j_0)(f(b_l) + k_l))^p)(f(a_1) + j_1) \times ((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0))^t((f(b_{l-1}) + k_{l-1})((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}) = 0$. Hence, $[((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}]^2(f(a_1) + j_1) \times ((f(b_{l-1}) + k_{l-1})(f(a_0) + j_0))^t((f(b_{l-1}) + k_{l-1})((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}) = 0$. Continuing this procedure, we obtain $[((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}]^{t+3} = 0$.

Thus, $((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))((f(a_0) + j_0)(f(b_l) + k_l))^{p+1} \in nil(J)$.

Similarly, we can show that $((f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_0)+j_0)(f(b_l)+k_l))^{p+1} \in nil(J)$ for $i = 2, ..., l$.

Since J is semicommutative, $nil(J)$ is an ideal and consequently $\sum_{i=1}^{i=l}(f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_0)+$
$j_0)(f(b_l) + k_l))^{p+1} \in nil(J)$. Multiplying the equation $\sum_{i+j=l} (f(a_i) + j_i)(f(b_j) + k_j) = 0$ on the right side by

$((f(a_0) + j_0)(f(b_l) + k_l))^{p+1}$, we obtain $((f(a_0) + j_0)(f(b_l) + k_l))^{p+2} = -\sum_{i=1}^{i=l}(f(a_i) + j_i)(f(b_j) + k_j)((f(a_0) +$
$j_0)(f(b_l) + k_l))^{p+1} \in nil(J)$.

Thus, $(f(a_0) + j_0)(f(b_l) + k_l) \in nil(f(A) + J)$.

Let $q = n_{1,l-1}$. Then $((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^q \in J$. By analogy with the above proof, we have

$$\sum_{i=2}^{i=l}(f(a_i) + j_i)(f(b_{l-i}) + k_{l-i})((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1} \in nil(J).$$

Suppose that $((f(a_0) + j_0)(f(b_l) + k_l))^s = 0$. Then

$$((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1}((f(a_0) + j_0)(f(b_l) + k_l))^s((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1} = 0.$$

Since $((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1} \in J$ and J is semicommutative,

$$((f(a_0) + j_0)(f(b_l) + k_l)((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1})^{s+1} = 0.$$

Therefore,

$$((f(a_0) + j_0)(f(b_l) + k_l)((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1} \in nil(J).$$

Multiplying the equation $\sum_{i+j=l} (f(a_i)+j_i)(f(b_j)+k_j)) = 0$ on the right side by $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}$,

we obtain $((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+2} = -\sum_{i=2}^{i=l}((f(a_i)+j_i)(f(b_{l-i})+k_{l-i}))((f(a_1)+j_1)(f(b_{l-1})+k_{l-1}))^{q+1}-$
$((f(a_0) + j_0)(f(b_l) + k_l))((f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}))^{q+1} \in nil(J)$. Therefore, $(f(a_1) + j_1)(f(b_{l-1}) + k_{l-1}) \in nil(f(A) + J)$.

A similar argument shows that

$$(f(a_2) + j_2)(f(b_{l-2}) + k_{l-2}) \in nil(f(A) + J) \dots (f(a_l) + j_l)(f(b_0) + k_0) \in nil(f(A) + J).$$

Consequently, $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ when $i + j = l$, and therefore $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ for every i, j. Hence, $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j) \in nil(A \bowtie^f J)$, and this shows that $A \bowtie^f J$ is weak Armendariz.

(8) Assume that $f^{-1}(J)$ is semicommutative and $f(A)+J$ is weak Armendariz. Let $F(x) = \sum_{i=0}^{i=n}(a_i, f(a_i)+$
$j_i)x^i$ and $G(x) = \sum_{j=0}^{j=m}(b_j, f(b_j) + k_j)x^j$ be two polynomials in $A \bowtie^f J[x]$ such that $F(x)G(x) = 0$ and set

$$f_A(x) = \sum_{i=0}^{i=n} a_i x^j, \ \ g_A(x) = \sum_{j=0}^{j=m} b_j x^j, \ \ f_B(x) = \sum_{i=0}^{i=n} (f(a_i) + j_i) x^i, \ \text{and} \ \ g_B(x) = \sum_{j=0}^{j=m} (f(b_j) + k_j) x^j. \ \text{Then}$$

$F(x)G(x) = 0$ implies that $f_A(x)g_A(x) = \sum_{l=0}^{l=n+m} (\sum_{i+j=l} a_i b_j) x^l = 0$ and $f_B(x)g_B(x) = \sum_{l=0}^{l=n+m} (\sum_{i+j=l} (f(a_i) +$

$j_i)(f(b_j) + k_j)) x^l = 0$. Hence, $\sum_{i+j=l} a_i b_j = 0$ for all $l = 0, 1..., n+m$ and $\sum_{i+j=l} (f(a_i) + j_i)(f(b_j) + k_j) = 0$ for

all $l = 0, 1..., n+m$. Therefore, $(f(a_i) + j_i)(f(b_j) + k_j) \in nil(f(A) + J)$ since $f(A) + J$ is weak Armendariz.
Since $(f(a_i) + j_i)(f(b_j) + k_j) = (f(a_i b_j) + t_{ij}) \in nil(f(A) + J)$, where $t_{ij} \in J$, $(f(a_i b_j) + t_{ij})^{n_{ij}} = 0$, for some
positive integer n_{ij}, therefore $(f(a_i b_j))^{n_{ij}} = f((a_i b_j)^{n_{ij}}) \in J$, and hence $(a_i b_j)^{n_{ij}} \in f^{-1}(J)$. Now we show
that $a_i b_j \in nil(A)$ by induction on $i + j$.

If $i + j = 0$, we have $a_0 b_0 = 0 \in nil(A)$ and so we are done.

Let l be a positive integer such that $a_i b_j \in nil(A)$ when $i + j < l$. As in (7), we will show that
$a_i b_j \in nil(A)$ when $i + j = l$.

We have $(a_0 b_l)^{n_{0l}=p} \in f^{-1}(J)$, and by the induction hypothesis, $(a_0 b_{l-1}) \in nil(A)$. Let t be a positive integer such that $(a_0 b_{l-1})^t = 0$. Then $(b_{l-1} a_0)^{t+1} = 0$ and hence $((a_1 b_{l-1})(a_0 b_l)^{p+1} a_1)(b_{l-1} a_0)^{t+1}(b_{l-1}(a_0 b_l)^{p+1}) = 0$. Since

$$(a_1 b_{l-1})(a_0 b_l)^{p+1} a_1)(b_{l-1} a_0) \in f^{-1}(J)$$

$$(b_{l-1} a_0)^t (b_{l-1}(a_0 b_l)^{p+1}) \in f^{-1}(J)$$

$$(b_l(a_0 b_l)^p a_1 \in f^{-1}(J)$$

and $f^{-1}(J)$ is semicommutative, we obtain

$$((a_1 b_{l-1})(a_0 b_l)^{p+1} a_1)(b_{l-1} a_0)(b_l(a_0 b_l)^p a_1)(b_{l-1} a_0)^t (b_{l-1}(a_0 b_l)^{p+1}) = 0.$$

Hence, $[(a_1 b_{l-1})(a_0 b_l)^{p+1}]^2 a_1 (b_{l-1} a_0)^t (b_{l-1}(a_0 b_l)^{p+1}) = 0$.

Iterating this process, we obtain:

$$[(a_1 b_{l-1})(a_0 b_l)^{p+1}]^{t+3} = 0.$$

Thus, $(a_1 b_{l-1})(a_0 b_l)^{p+1} \in nil f^{-1}(J)$, and similarly we have $(a_i b_{l-i})(a_0 b_l)^{p+1} \in nil f^{-1}(J)$ for $i = 2, ..., l$. Since
$f^{-1}(J)$ is semicommutative, $nil f^{-1}(J)$ is an ideal and consequently

$$\sum_{i=1}^{i=l} (a_i b_{l-i})(a_0 b_l)^{p+1} \in nil f^{-1}(J).$$

Multiplying the equation $\sum_{i+j=l} a_i b_j = 0$ on the right side by $(a_0 b_l)^{p+1}$, we get:

$$(a_0 b_l)^{p+2} = -\sum_{i=1}^{i=l} (a_i b_{l-i})(a_0 b_l)^{p+1} \in nil f^{-1}(J).$$

Thus, $a_0 b_l \in nil(A)$. Now let $q = n_{1,l-1}$. Then $(a_1 b_{l-1})^q \in f^{-1}(J)$. As in the above proof, we have

$$\sum_{i=2}^{i=l} (a_i b_{l-i})(a_1 b_{l-1})^{q+1} \in nil f^{-1}(J).$$

Suppose that $(a_0 b_l)^s = 0$. Then

$$(a_1 b_{l-1})^{q+1} (a_0 b_l)^s (a_1 b_{l-1})^{q+1} = 0.$$

Since $(a_1 b_{l-1})^{q+1} \in f^{-1}(J)$ and $f^{-1}(J)$ is semicommutative,

$$((a_0 b_l)(a_1 b_{l-1})^{q+1})^{s+1} = 0.$$

Therefore,

$$(a_0 b_l)(a_1 b_{l-1})^{q+1} \in nil(f^{-1}(J)).$$

If we multiply the equation $\displaystyle\sum_{i+j=l} a_i b_j = 0$ on the right side by $(a_1 b_{l-1})^{q+1}$, we obtain

$$(a_1 b_{l-1})^{q+2} = -\sum_{i=2}^{i=l} (a_i b_{l-i})(a_1 b_{l-1})^{q+1} - (a_0 b_l)(a_1 b_{l-1})^{q+1} \in nil f^{-1}(J).$$

Therefore, $a_1 b_{l-1} \in nil(A)$. Similarly, we have $a_2 b_{l-2} \in nil(A), ..., a_l b_0 \in nil(A)$ and consequently $a_i b_j \in nil(A))$ when $i + j = l$. Therefore, $a_i b_j \in nil(A)$ for every i, j and hence $(a_i, f(a_i) + j_i)(b_j, f(b_j) + k_j) \in nil A \bowtie^f J$. This shows that $A \bowtie^f J$ is weak Armendariz and completes the proof. □

Remark As we mentioned in the introduction, we do not know any example to date of a weak Armendariz ring that is not a nil-Armendariz ring. This question was left open in [2]. We were not able to answer the question of whether $A \bowtie^f J$ is a nil-Armendariz ring if and only it is a weak Armendariz ring. A negative answer will provide a counterexample of a weak Armendariz ring that is not nil-Armendariz. However, a positive answer shows that amalgamation of algebras along ideals, as a source of examples and counterexamples, cannot provide such an example if it exists.

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