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Dissipative operator and its Cayley transform

Ekin UĞURLU*, **Kenan TAŞ**
Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara, Turkey

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Abstract: In this paper, we investigate the spectral properties of the maximal dissipative extension of the minimal symmetric differential operator generated by a second order differential expression and dissipative and eigenparameter dependent boundary conditions. For this purpose we use the characteristic function of the maximal dissipative operator and inverse operator. This investigation is done by the characteristic function of the Cayley transform of the maximal dissipative operator, which is a completely nonunitary contraction belonging to the class C_0. Using Solomyak's method we also introduce the self-adjoint dilation of the maximal dissipative operator and incoming/outgoing eigenfunctions of the dilation. Moreover, we investigate other properties of the Cayley transform of the maximal dissipative operator.

Key words: Cayley transform, completely nonunitary contraction, unitary colligation, characteristic function, CMV matrix

1. Introduction

If an operator T_1 acting on a Hilbert space H_1 is equivalent to another operator T_2 acting on another Hilbert space H_2 in a certain sense, then one can say that T_2 is a model of T_1. There exist models up to unitary equivalence, similarity equivalence, quasi-similarity, pseudo-similarity, and other equivalences [27]. A useful model was given by Sz.-Nagy and Foiaş [24, 25]. Sz.-Nagy and Foiaş constructed the model operator for the contractive operators acting on Hilbert spaces. This construction is based on the dilation. An operator $\mathcal{U}$ acting on a Hilbert space $\mathcal{H}$ is called a dilation of an operator T acting on a Hilbert space H such that $H \subset \mathcal{H}$ if

$$T^n f = P_H \mathcal{U}^n f, \; f \in H, \; n \geq 0,$$

where P_H is the orthogonal projection of $\mathcal{H}$ onto H, and $\mathcal{H}$ is called the dilation space. If $\mathcal{U}$ is unitary on $\mathcal{H}$, then $\mathcal{U}$ is called unitary dilation of T. Moreover, if the minimal subspace of $\mathcal{H}$ containing H and being invariant with respect to $\mathcal{U}$ and $\mathcal{U}^*$ coincides with $\mathcal{H}$, then $\mathcal{U}$ is called minimal. In 1965, Sarason gives the geometric structure of the dilation space [32]. In fact, Sarason showed that an operator $\mathcal{U}$ acting on $\mathcal{H}$ is a dilation of its compression $P_H \mathcal{U} \mid H$ if and only if $\mathcal{H}$ decomposes in the following way

$$\mathcal{H} = G_* \oplus H \oplus G,$$

where $\mathcal{U}G \subset G$ and $\mathcal{U}^* G_* \subset G_*$. Moreover, if an operator T has a unitary dilation then T is a contraction, i.e. $\|T\| \leq 1$. The geometric structure of the dilation space allows one to give a more useful description of the

*Correspondence: ekinugurlu@cankaya.edu.tr
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minimal unitary dilation of a contraction. Namely, if $\mathcal{U}$ is a minimal unitary dilation on $\mathcal{H} = G_* \oplus H \oplus G$ of contraction T acting on H, then

$$\mathcal{U} = \begin{bmatrix} P_{G_*}\mathcal{U} \mid G_* & 0 & 0 \\ D_{T^*}V_*^* & T & 0 \\ -VT^*V_*^* & VD_T & \mathcal{U} \mid G \end{bmatrix},$$

where $D_{T^*} = (I - TT^*)^{1/2}$, $D_T = (I - T^*T)^{1/2}$, V is a partial isometry with initial space $\mathfrak{D}_T = \overline{D_T H}$ and final space $E = G \ominus \mathcal{U}G$ and V_* is a partial isometry with initial space $\mathfrak{D}_{T^*} = \overline{D_{T^*}H}$ and final space $E_* = G_* \ominus \mathcal{U}^* G_*$. Note that an operator $V : H \to K$ is called partial isometry if $H = H_i \oplus H_0$, where $V : H_i \to H$ is an isometry and $V \mid H_0 = 0$. H_i is called the initial space of V and its range $VH_i = VH$ the final space of V.

Another description of the dilation $\mathcal{U}$ of contraction T can be given with the following unitary mappings

$$\mathcal{U} \mid span\left(\mathcal{U}^n \left(G \ominus \mathcal{U}G\right) : n \in \mathbb{Z}\right) = \mathcal{U} \mid \bigoplus_{n\in\mathbb{Z}} \mathcal{U}^n \left(G \ominus \mathcal{U}G\right),$$
$$\mathcal{U} \mid span\left(\mathcal{U}^n \left(G_* \ominus \mathcal{U}^* G_*\right) : n \in \mathbb{Z}\right) = \mathcal{U} \mid \bigoplus_{n\in\mathbb{Z}} \mathcal{U}^n \left(G_* \ominus \mathcal{U}^* G_*\right).$$

Let us consider the spaces E and E_* such that

$$\dim E = \dim\left(G \ominus \mathcal{U}G\right), \ \ \dim E_* = \dim\left(G_* \ominus \mathcal{U}^* G_*\right).$$

Then the unitary mappings given above are the multiplication operators $g(z) \to zg(z)$ on $L^2(E)$ and $L^2(E_*)$, respectively. Let

$$\upsilon : E \to G \ominus \mathcal{U}G, \ \upsilon_* : E_* \to G_* \ominus \mathcal{U}^* G$$

be the unitary mappings and

$$\prod : \begin{array}{lcl} L^2(E) \oplus L^2(E_*) & \to & \mathcal{H}, \\ (f, g) & \to & \pi f + \pi_* g, \end{array}$$

where

$$\pi : \begin{array}{lcl} L^2(E) & \to & \bigoplus_{n\in\mathbb{Z}} \mathcal{U}^n(G \ominus \mathcal{U}G), \\ \sum_n z^n e_n & \to & \sum_n \mathcal{U}^n \upsilon e_n, \end{array}$$

and

$$\pi_* : \begin{array}{lcl} L^2(E_*) & \to & \bigoplus_{n\in\mathbb{Z}} \mathcal{U}^n(G_* \ominus \mathcal{U}^* G_*), \\ \sum_n z^n e_n^* & \to & \sum_n \mathcal{U}^{n+1} \upsilon_* e_n^*. \end{array}$$

The operators π and π_* are called functional embeddings. The function

$$\pi_*^* \pi = \Theta : E \to E_*$$

is called the characteristic function of the contraction T. An explicit formula for the characteristic function is given as

$$\Theta_T(z)h = V_*(-T + zD_{T^*}(I - zT^*)^{-1}D_T)V^*h,\ h \in E,$$

where V and V_* are the unitary identifications such as

$$V : \mathfrak{D}_T \leftrightarrow E,\ V_* : \mathfrak{D}_{T^*} \leftrightarrow E_*.$$

If one chooses $E = \mathfrak{D}_T$ and $E_* = \mathfrak{D}_{T^*}$, then Θ_T is reduced to

$$\Theta_T = -T + zD_{T^*}(I - zT^*)^{-1}D_T,$$

which is the well-known characteristic function given by Sz.-Nagy and Foiaş.

An operator B with domain $D(B)$ acting on a Hilbert space H is called dissipative if

$$\mathrm{Im}(By, y) \geq 0,\ y \in D(B)$$

and accumulative if

$$\mathrm{Im}(By, y) \leq 0,\ y \in D(B).$$

There is a connection between dissipative operators and contractions. Indeed, the Cayley transform of a dissipative operator defines a contraction. Solomyak used this connection and free parameters to obtain the characteristic function of the maximal dissipative operator [35]. For this purpose, Solomyak used the boundary spaces of the maximal dissipative operator. Indeed, let B be a maximal dissipative operator, G_B be its Hermitian part, and $\mathcal{P}$ be the natural projection defined on the quotient space such as $\mathcal{P} : D(B) \to D(B)/G_B$. The completion $F(B)$ of the quotient space $D(B)/G_B$ is called the boundary space. Similarly, $F_*(B)$ is defined as $F_*(B) = F(-B^*)$. Moreover, $\mathcal{P}_*$ is defined from $D(B^*)$ onto $D(B^*)/G_B$. Then Solomyak introduced the connection of the characteristic functions between the maximal dissipative operator B and its Cayley transform T as

$$S_B(\lambda) = \Theta_T\left(\frac{\lambda - i}{\lambda + i}\right),\ \mathrm{Im}\lambda > 0$$

with the rule

$$S_B(\lambda) = \mathcal{P}_*(B^* - \lambda I)^{-1}(B - \lambda I)\mathcal{P}^{-1}.$$

Moreover, with the help of free parameters Solomyak constructed a self-adjoint dilation of the maximal dissipative operator. Using the characteristic function he described directly the generalized eigenfunctions of the self-adjoint dilation.

In this paper, we investigate the spectral properties of the maximal dissipative extension generated by a second order differential expression in the limit-circle case and two boundary conditions in which the domain of the minimal symmetric operator contains a spectral parameter in the boundary conditions. To indicate that the extension is maximal dissipative we use Gorbachuks' theorem on extension, which requires the equal deficiency indices [19]. Although the second order differential expression is in the limit-circle case, the minimal symmetric operator has the deficiency indices $(1, 1)$. This connection has been studied by Maozhu et al. in [22]. On the other hand, it is known that any symmetric operator with deficiency indices (n, n) has a boundary value space with dimension n. Therefore, this relation allows us to construct a maximal dissipative extension of the minimal symmetric operator.

The investigation of the spectral properties of the maximal dissipative operator is based on the characteristic function and the inverse operator of the maximal dissipative operator. Using the connection between the characteristic functions of the maximal dissipative operator and its Cayley transform we obtain the characteristic function of the Cayley transform. We also prove that the Cayley transform is a completely nonunitary (c.n.u.) contraction belonging to the class C_0, which consists of those c.n.u. contractions T for which there exists a nonzero function $u \in H^\infty$ (H^p denotes the Hardy class) such that $u(T) = 0$. It is well known that u has a canonical factorization into the product of an inner function u_i and outer function u_e. The equation $u(T) = 0$ implies that $u_i(T) = 0$ and therefore one may ask whether for a c.n.u. contraction T belonging to the class C_0 there exists an inner function u with $u(T) = 0$ such that every other function $v \in H^\infty$ with $v(T) = 0$ is a multiple of u. Such a function m_T is called a minimal function of T and this function is determined up to a constant factor of modulus one. Sz.-Nagy and Foiaş proved that for every contraction $T \in C_0$ there exists a minimal function m_T. With the help of m_T, some spectral properties of $T \in C_0$ can be obtained. For example, the spectrum of the contraction $T \in C_0$ and the zeros of the minimal function m_T in the open disc $\boldsymbol{D}$ and of the complement, in the unit circle $\boldsymbol{C}$, of the union of the arcs of $\boldsymbol{C}$ on which m_T is analytic, coincide with each other. Moreover, the points of the spectrum in the interior of the unit circle $\boldsymbol{C}$ are eigenvalues of T. As a characteristic value of T, λ has finite index, equal to its multiplicity as a zero of m_T. Completeness of the root vectors of T associated with the points of the spectrum of T in $\boldsymbol{D}$ can be proved as showing that the minimal function m_T is a Blaschke product.

We should note that this method is new and differs from Pavlov's method [1–6, 30, 31, 36]. In fact, Pavlov's method is based on the fact that there is a connection between a continuous semigroup of contractions $\{Z(t)\}_{t\geq 0}$ and its cogenerator Z. Therefore every model of Z generates a model of $\{Z(t)\}_{t\geq 0}$. In this paper, we only use the Cayley transform of the maximal dissipative operator.

For an arbitrary bounded operator it is important to find the least subspace that is a generating subspace for the bounded operator. The dimension of such a subspace is called the spectral multiplicity or multiplicity of the bounded operator. The characteristic function may help one to find the multiplicity of the contraction. Therefore we find the multiplicity of the contraction of the maximal dissipative operator.

It is known that unitary colligation theory is more general than the characteristic function theory of contractions given by Sz.-Nagy and Foiaş. Since the Cayley transform of our maximal dissipative operator has finite defect indices, embedding the contraction to its unitary colligation we introduce some results with the help of the results reported by Arlinskiĭ et al. [10].

Jacobi matrices are useful to understand the characterization of self-adjoint, nonself-adjoint, and unitary operators acting on separable Hilbert spaces. Indeed, multiplication operators on the Hilbert spaces $L^2(\mathbb{R})$ or $L^2(\boldsymbol{C})$ associated with the probability measure $\boldsymbol{m}$ on the real line $\mathbb{R}$ or on the unit circle $\boldsymbol{C}$, respectively, are unitary equivalent to the self-adjoint or unitary operators with a simple spectrum acting on some Hilbert spaces [8]. Tri-diagonal Jacobi matrix representation of self-adjoint operators with simple spectrum was introduced by Stone [9]. The nonself-adjoint version of Stone's theorem has been introduced by Arlinskiĭ and Tsekanovskiĭ [11]. Moreover, the canonical matrix representation of unitary operators with simple spectrum has been introduced by Cantero et al. [15] with the help of five-diagonal unitary matrices called CMV matrices. Arlinskiĭ et al. [10] obtained a connection between truncated CMV matrix and Sz.-Nagy-Foiaş characteristic function. Therefore, we introduce a truncated CMV matrix associated with the Cayley transform.

This paper is organized as follows. In section 2, we construct the maximal dissipative extension of the minimal symmetric differential operator. In section 3, passing to the Cayley transform of the dissipative operator we obtain a contraction. Moreover, we find the characteristic functions of the maximal dissipative operator and its Cayley transform and using the properties of the Cayley transform and its characteristic function we introduce some results about the spectral analysis of both the maximal dissipative operator and its Cayley transform. In section 4, we obtain the inverse operator of the dissipative operator, which is an integral operator with finite-rank imaginary component. Then we introduce the complete spectral analysis of the dissipative operator. In section 5, we construct the self-adjoint dilation and its incoming/outgoing eigenfunctions directly. In section 6, we introduce some results on the Cayley transform of the dissipative operator.

Finally we should note that the notations $\boldsymbol{C}$ and $\boldsymbol{D}$ will be used to describe the unit circle $\boldsymbol{C} = \{\mu : |\mu| = 1\}$ and unit disc $\boldsymbol{D} = \{\mu : |\mu| < 1\}$.

2. Maximal dissipative extension of the minimal symmetric operator

In this paper we consider the following second order differential equation:

$$\ell(y) = \lambda y,\ x \in \mathbb{I}, \tag{2.1}$$

where

$$\ell(y) = \frac{1}{w(x)}\left[-(p(x)y')' + q(x)y\right],$$

$\mathbb{I} = [a, b),\ -\infty < a < b \leq \infty$, and λ is a complex parameter. We assume that a is the regular point and b is the singular point for the equation (2.1), p, q, w are real-valued Lebesgue measurable functions on $\mathbb{I}$, p^{-1}, q and w are locally integrable functions on $\mathbb{I}$ and $w > 0$ for almost all $x \in \mathbb{I}$.

Let $L^2_w(\mathbb{I})$ denote the Hilbert space consisting of all functions y satisfying

$$\int_{\mathbb{I}} |y|^2\, w dx < \infty$$

with the inner product

$$(y, \chi) = \int_{\mathbb{I}} y\overline{\chi} w dx.$$

We denote by M a subset of $L^2_w(\mathbb{I})$ that consists of those functions $y \in L^2_w(\mathbb{I})$ such that y and py' are locally absolutely continuous functions on $\mathbb{I}$ and $\ell(y) \in L^2_w(\mathbb{I})$. The operator L, defined by $Ly = \ell(y),\ y \in M$, is called the maximal operator with domain M. Let $L^* = L_0$ with domain M_0. The set M_0 consists of those functions $y \in M$ satisfying $y(a) = (py')(a) = 0$ and $[y, \chi](b) = 0$ for all $\chi \in M$.

The deficiency indices of the minimal operator L_0 are defined as the numbers (m, n) such that

$$m = \dim\left(\left(L_0 - \overline{\lambda} I\right) D(L_0)\right)^{\perp},\ n = \dim\left(\left(L_0 - \lambda I\right) D(L_0)\right)^{\perp},$$

where $D(L_0)$ is the domain of L_0 and λ is the complex number. It is well known that L_0 is self-adjoint if and only if $m = n = 0$ and L_0 has self-adjoint extensions if and only if $m = n$. $(1, 1)$ is known as Weyl's limit-point case and $(2, 2)$ is known as Weyl's limit-circle case for a second order operator.

In this paper, we assume that w, p and q satisfy Weyl's limit-circle case conditions at singular point b. In other words, we assume that the deficiency indices of L_0 are $(2,2)$ [12, 17, 20, 26, 38].

For two arbitrary functions $y, \chi \in M$ the following Green's formula holds

$$\int_{\mathbb{I}} \{\ell(y)\chi - y\ell(\chi)\} \, w dx = [y, \chi](b) - [y, \chi](a),$$

where $[y, \chi](x) := y(x)\chi^{[1]}(x) - y^{[1]}(x)\chi(x)$ and $y^{[1]} = py'$. Green's formula implies that for arbitrary two functions $y, \chi \in M$, the values $[y, \chi](b)$ and $[y, \overline{\chi}](b)$ exist and are finite. The secondary one follows from the fact that p, q and w are real valued functions on $\mathbb{I}$.

Let u and v be the real solutions of the equation $\ell(y) = 0, \ x \in \mathbb{I}$, satisfying the conditions

$$u(a) = \alpha_2, \ u^{[1]}(a) = \alpha_1, \ v(a) = \gamma_2, \ v^{[1]}(a) = \gamma_1, \tag{2.2}$$

such that $[u, v](a) = 1$. Moreover, Green's formula and (2.2) imply that $[u, v] = 1$ for all $x \in \mathbb{I}$. Therefore for two arbitrary functions $y, \chi \in M$ one has

$$[y, \chi] = [y, u][\chi, v] - [y, v][\chi, u], \ x \in \mathbb{I}. \tag{2.3}$$

Since the limit-circle case holds for ℓ, u and v belong to $L_w^2(\mathbb{I})$ and M. Therefore for arbitrary $y \in M$, the values $[y, u](b)$ and $[y, v](b)$ exist and are finite.

For $y \in M$, we consider the following boundary conditions:

$$\begin{array}{c} \alpha_1 y(a) - \alpha_2 y^{[1]}(a) = \lambda \left(\beta_1 y(a) - \beta_2 y^{[1]}(a)\right), \\ [y, v](b) + h[y, u](b) = 0, \end{array} \tag{2.4}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers such that $\delta := \beta_1\alpha_2 - \beta_2\alpha_1 > 0$ and h is a complex number such that $h = h_1 + ih_2$ with $h_2 > 0$.

Let $H = L_w^2(\mathbb{I}) \oplus \mathbb{C}$ be the Hilbert space with the inner product

$$\langle \mathcal{Y}, \mathcal{Z} \rangle_H = (y, z) + \frac{1}{\delta} y_1 \overline{z}_1,$$

where

$$\mathcal{Y} = \begin{pmatrix} y(x) \\ y_1 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} z(x) \\ z_1 \end{pmatrix} \in H.$$

We consider the set $D(\mathcal{L})$ consisting of all functions $\mathcal{Y} = \binom{y}{y_1}$ such that $y \in M$ satisfying $y_1 = \beta_1 y(a) - \beta_2 y^{[1]}(a)$. Let $\mathcal{L}$ be the operator defined on $D(\mathcal{L})$ with the rule

$$\mathcal{L}\mathcal{Y} = \begin{pmatrix} \ell(y) \\ \alpha_1 y(a) - \alpha_2 y^{[1]}(a) \end{pmatrix}.$$

Let $D(\mathcal{L}_0)$ be the set consisting of all functions $\mathcal{Y} \in D(\mathcal{L})$ such that $[y, z](b) = 0$ for all

$$\mathcal{Z} = \begin{pmatrix} z(x) \\ z_1 \end{pmatrix} \in H.$$

The operator $\mathcal{L}_0$ is defined as the restriction of the operator $\mathcal{L}$ to the set $D(\mathcal{L}_0)$. It is known that $\mathcal{L}_0$ is closed, densely defined, symmetric, and $\mathcal{L}_0^* = \mathcal{L}$ in H [22]. Moreover, since L_0 has the deficiency indices $(2,2)$, the deficiency indices of $\mathcal{L}_0$ are $(1,1)$ [22].

Recall that [19] a triple $(\mathfrak{H}, \Gamma_1, \Gamma_2)$, where $\mathfrak{H}$ is a Hilbert space and Γ_1, Γ_2 are linear mappings of $D(A^*)$ into $\mathfrak{H}$, is called a boundary value space of the operator A if,

(i) for any $f, g \in D(A^*)$,

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_2 g)_{\mathfrak{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathfrak{H}},$$

(ii) for any $F_1, F_2 \in \mathfrak{H}$ there exists a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1, \Gamma_2 f = F_2$.

Theorem 2.1. [19] *For any symmetric operator with deficiency indices (n,n) $(n \leq \infty)$ there exists a boundary value space $(\mathfrak{H}, \Gamma_1, \Gamma_2)$ with $\dim \mathfrak{H} = n$.*

Now consider the following linear mappings:

$$\Gamma_1 \mathcal{Y} = [y, u](b), \ \Gamma_2 \mathcal{Y} = [y, v](b).$$

Then we have the following theorem.

Theorem 2.2. *$(\mathbb{C}, \Gamma_1, \Gamma_2)$ is a space of boundary values of $\mathcal{L}_0$.*

Proof Let $\mathcal{Y} \in D(\mathcal{L})$. Then $y \in M$ and for any complex numbers c_1 and c_2, the values $[y,u](b) = c_1$ and $[y,v](b) = c_2$ exist [1, 2, 26]. Moreover, for $\mathcal{Y}, \mathcal{Z} \in D(\mathcal{L})$ one gets

$$\langle \mathcal{L}_0^* \mathcal{Y}, \mathcal{Z} \rangle_H - \langle \mathcal{Y}, \mathcal{L}_0^* \mathcal{Z} \rangle_H = [y, \overline{z}](b). \tag{2.5}$$

On the other hand, we have

$$(\Gamma_1 \mathcal{Y}, \Gamma_2 \mathcal{Z})_{\mathbb{C}} - (\Gamma_2 \mathcal{Y}, \Gamma_1 \mathcal{Z})_{\mathbb{C}} = [y, \overline{z}](b). \tag{2.6}$$

Therefore (2.5) and (2.6) complete the proof. □

The following theorem, given by Gorbachuks, describes all maximal dissipative maximal accumulative and maximal self-adjoint extension of a given minimal symmetric operator.

Theorem 2.3. [19] *If K is a contraction on $\mathfrak{H}$, then the restriction of the operator A^* to the set of vectors $f \in D(A^*)$ satisfying the condition*

$$(K - I)\Gamma_1 f + i\,(K + I)\,\Gamma_2 f = 0, \tag{2.7}$$

or

$$(K - I)\Gamma_1 f - i\,(K + I)\,\Gamma_2 f = 0, \tag{2.8}$$

is a maximal dissipative, respectively, a maximal accumulative extension of A. Conversely, any maximal dissipative (maximal accumulative) extension of A is the restriction of A^* to the set of vectors $f \in D(A^*)$ satisfying (2.7) ((2.8)), where a contraction K is uniquely determined by an extension. The maximal symmetric extensions of an operator A on *$\mathfrak{H}$ are described by the conditions (2.7) ((2.8)), where K is a isometric operator. These conditions define a self-adjoint extension if K is unitary. In the last case (2.7), (2.8) are equivalent to*

$$(\cos C)\Gamma_2 f - (\sin C)\Gamma_1 f = 0,$$

where C is a self-adjoint operator on $\mathfrak{H}$. The general form of dissipative (accumulative) extension of A is given by the condition

$$K(\Gamma_1 f + i\Gamma_2 f) = \Gamma_1 f - i\Gamma_2 f,\ \Gamma_1 f + i\Gamma_2 f \in D(K), \tag{2.9}$$

respectively,

$$K(\Gamma_1 f - i\Gamma_2 f) = \Gamma_1 f + i\Gamma_2 f,\ \Gamma_1 f - i\Gamma_2 f \in D(K), \tag{2.10}$$

where K is a linear operator satisfying $\|Kf\| \leq \|f\|$ $(f \in D(K))$, while symmetric extensions are described by the formulas (2.9) and (2.10), where K is an isometric operator.

Therefore Theorems 2.2 and 2.3 give the following.

Theorem 2.4. *All maximal dissipative extensions of the operator $\mathcal{L}_0$ are given by the boundary condition*

$$[y, v](b) + h[y, u](b) = 0,\ y \in M, \tag{2.11}$$

where h is a complex number as $h = h_1 + ih_2$ with $h_2 > 0$.

Let $D(\mathcal{L}_h)$ be the set consisting of all functions $\mathcal{Y} \in D(\mathcal{L}_h)$ satisfying (2.11). We define the operator $\mathcal{L}_h$ on $D(\mathcal{L}_h)$ with the rule

$$\mathcal{L}_h\mathcal{Y} = \mathcal{L}\mathcal{Y},\ \mathcal{Y} \in D(\mathcal{L}_h).$$

Therefore $\mathcal{L}_h$ is the maximal dissipative extension of $\mathcal{L}_0$ and the equation

$$\mathcal{L}_h\mathcal{Y} = \lambda\mathcal{Y}$$

coincides with the problem (2.1), (2.4).

Definition 2.5. *A nonself-adjoint operator L acting on a Hilbert space $\mathcal{H}$ is called simple if there is no invariant subspace of $\mathcal{H}$ on which L has a self-adjoint part there.*

Theorem 2.6. *$\mathcal{L}_h$ is simple on H.*

Proof Let H_s be a subspace of H on which $\mathcal{L}_h$ has a selfadjoint part there. For $\mathcal{Y} \in D(\mathcal{L}_h) \cap H_s$ one gets

$$0 = \langle \mathcal{L}_h\mathcal{Y}, \mathcal{Y}\rangle_{H_s} - \langle \mathcal{Y}, \mathcal{L}_h\mathcal{Y}\rangle_{H_s} = 2ih_2 \left|[y, u](b)\right|^2.$$

Therefore $[y, u](b) = 0$ and hence $[y, v](b) = 0$. This implies that $y \equiv 0$ on $[a, b)$ and $y_1 = 0$. Consequently $\mathcal{Y} \equiv 0$. This completes the proof. □

Let us consider the solution $\varphi(x, \lambda)$ of the equation (2.1) satisfying the conditions

$$\varphi(a, \lambda) = \alpha_2 - \lambda\beta_2,\ \varphi^{[1]}(a, \lambda) = \alpha_1 - \lambda\beta_1.$$

Then the zeros of the function

$$\Delta_h(\lambda) = [\varphi(x, \lambda), v(x)](b) + h[\varphi(x, \lambda), u(x)](b)$$

coincide with the eigenvalues of $\mathcal{L}_h$. It can be obtained that Δ_h is an entire function of λ of order ≤ 1 of growth and of minimal type. Therefore the eigenvalues of $\mathcal{L}_h$ are purely discrete and possible limit points of these eigenvalues can only occur at infinity. However, more detailed analysis will be obtained with the help of the characteristic function and inverse operator $\mathcal{L}_h$.

3. Characteristic function

The following lemma gives a nice connection between maximal dissipative operators and related contractions [24, 25, 35].

Lemma 3.1. *(i) Assume the operator L_0 is dissipative. Then the operator $T_0 = K(L_0) = (L_0 - iI)(L_0 + iI)^{-1}$ is a contraction from $(L_0 + iI)D(L_0)$ onto $(L_0 - iI)D(L_0)$ and $L_0 = i(I + T_0)(I - T_0)^{-1}$. For each contraction T_0 such that $1 \notin \sigma_p(T_0)$ (the point spectrum of the operator), operator $L_0 = K^{-1}(T_0)$, $D(L_0) = (I - T_0)D(T_0)$, is dissipative.*

(ii) Each dissipative operator L_0 has a maximal dissipative extension L. A maximal dissipative operator is closed.

(iii) A maximal dissipative operator is maximal dissipative if and only if $T = K(L)$ is a contraction such that $D(T) = H$ and $1 \notin \sigma_p(T)$.

(iv) If L is a maximal dissipative operator, $L = K^{-1}(T)$, then $-L^*$ is also maximal dissipative, $L^* = -K^{-1}(T^*)$.

(v) If L is a maximal dissipative operator, then $\sigma(T) \subset \overline{\mathbb{C}}_+$, $\left\|(L - \lambda I)^{-1}\right\| \leq |\mathrm{Im}\lambda|^{-1}$, $\lambda \in \mathbb{C}_-$.

For a maximal dissipative operator B with domain $D(B)$, the subspace

$$G_B = \{y \in D(B) \cap D(B^*) : By = B^*y\}$$

is called the Hermitian part of the domain of B.

Let $\mathcal{P}$ be the natural projection defined as

$$\mathcal{P} : D(B) \to D(B)/G_B,$$

where $D(B)/G_B$ is the quotient space. On the quotient space the following inner product is defined

$$\langle \mathcal{P}y, \mathcal{P}\chi \rangle = \frac{i}{2}\left((y, B\chi) - (By, \chi)\right), \ y, \chi \in D(B).$$

We denote by $F(B)$ the completion of the quotient space $D(B)/G_B$ with respect to the corresponding norm. Similarly, we define $F_*(B)$ as $F_*(B) = F(-B^*)$. Here the projection $\mathcal{P}_*$ is defined as

$$\mathcal{P}_* : D(B^*) \to D(B^*)/G_B.$$

Therefore we have

$$\|\mathcal{P}y\|_F^2 = \mathrm{Im}(By, y), \ \ \|\mathcal{P}_*z\|_{F_*}^2 = -\mathrm{Im}(B^*z, z). \tag{3.1}$$

$F(B)$ and $F_*(B)$ are Hilbert spaces and are called boundary spaces of the operator B.

We have from (3.1) that

$$\|\mathcal{P}\mathcal{Y}\|_F^2 = h_2 \left|[y, u](b)\right|^2, \|\mathcal{P}_*\mathcal{Z}\|_F^2 = h_2 \left|[z, u](b)\right|^2.$$

Note that if one has all dissipative extensions of a symmetric operator B, then G_B is dense in the Hilbert space. If $y \in D(B) \cap D(B^*)$, then $By = B^*y$, i.e. $D(B) \cap D(B^*) = G_B$.

Let $\mathcal{C}_h$ be the Cayley transform of the dissipative operator $\mathcal{L}_h$ as $\mathcal{C}_h = (\mathcal{L}_h - i\mathbf{1})(\mathcal{L}_h + i\mathbf{1})^{-1}$ from $(\mathcal{L}_h + i\mathbf{1})D(\mathcal{L}_h)$ onto $(\mathcal{L}_h - i\mathbf{1})D(\mathcal{L}_h)$, where $\mathbf{1}$ is the identity operator in the direct sum Hilbert space H.

Since $\mathcal{L}_h$ is maximal dissipative, the domain of $\mathcal{C}_h$ is the Hilbert space H. Let $D_{\mathcal{C}_h} = (\mathbf{1} - \mathcal{C}_h^* \mathcal{C}_h)^{1/2}$ and $D_{\mathcal{C}_h^*} = (\mathbf{1} - \mathcal{C}_h \mathcal{C}_h^*)^{1/2}$ be the *defect operators* of $\mathcal{C}_h$ acting on H, $\mathfrak{D}_{\mathcal{C}_h} = \overline{D_{\mathcal{C}_h} H}$ and $\mathfrak{D}_{\mathcal{C}_h^*} = \overline{D_{\mathcal{C}_h^*} H}$ be the *defect spaces*, and $\mathfrak{d}_{\mathcal{C}_h} = \dim \mathfrak{D}_{\mathcal{C}_h}$ and $\mathfrak{d}_{\mathcal{C}_h^*} = \dim \mathfrak{D}_{\mathcal{C}_h^*}$ be the *defect indices* of $\mathcal{C}_h$.

Definition 3.2. *A contraction C on a Hilbert space H is called c.n.u. if for no nonzero reducing subspace $\mathcal{C}$ for C is $C \mid \mathcal{C}$ is a unitary operator.*

For $(\mathcal{L}_h + i\mathbf{1})\mathcal{F} = \mathcal{Y}$, the inequality

$$\|(\mathcal{L}_h - i\mathbf{1})\mathcal{Y}\|_H^2 < \|(\mathcal{L}_h + i\mathbf{1})\mathcal{Y}\|_H^2 \tag{3.2}$$

holds if and only if

$$2\mathrm{Im}\,\langle \mathcal{L}_h \mathcal{Y}, \mathcal{Y}\rangle_H > 0.$$

Particularly, (3.2) shows

$$\|\mathcal{C}_h\|_H < 1. \tag{3.3}$$

Note that 1 cannot belong to the point spectrum of $\mathcal{C}_h$ [24, 25, 35]. On the other hand,

$$\mathcal{L}_h = i(\mathbf{1} + \mathcal{C}_h)(\mathbf{1} - \mathcal{C}_h)^{-1}.$$

However, this does not imply that $\mathcal{L}_h$ is a bounded operator (see [25], p. 171).

From (3.3) one gets the following.

Theorem 3.3. *$\mathcal{C}_h$ is a c.n.u. contraction in H.*

It is known that there exist isometric isomorphisms between $F(\mathcal{L}_h)$ $(F_*(\mathcal{L}_h))$ and $\mathfrak{D}_{\mathcal{C}_h}$ $(\mathfrak{D}_{\mathcal{C}_h^*})$. Indeed, the mappings $\rho : F(\mathcal{L}_h) \to \mathfrak{D}_{\mathcal{C}_h}$ and $\rho_* : F_*(\mathcal{L}_h) \to \mathfrak{D}_{\mathcal{C}_h^*}$ [35] such that

$$\rho \mathcal{P}(\mathbf{1} - \mathcal{C}_h) = D_{\mathcal{C}_h}, \ \rho_* \mathcal{P}_*(\mathbf{1} - \mathcal{C}_h^*) = D_{\mathcal{C}_h^*}$$

define isometric isomorphism.

A c.n.u. contraction T is defined to within a unitary equivalence of the characteristic function

$$\Theta_T : E \to E_*,$$

where E and E_* are auxiliary Hilbert spaces, isomorphic to $\mathfrak{D}_{\mathcal{C}_h}$ and $\mathfrak{D}_{\mathcal{C}_h^*}$, respectively. Fixing isometric isomorphisms $\Omega : E \to D_{\mathcal{C}_h}$, $\Omega_* : E_* \to D_{\mathcal{C}_h^*}$, one has

$$\Theta_{\mathcal{C}_h}(\mu) = \Omega_*^* \left(-\mathcal{C}_h + \mu D_{\mathcal{C}_h^*}(\mathbf{1} - \mu \mathcal{C}_h^*) D_{\mathcal{C}_h}\right) \Omega$$

and

$$\Theta_{\mathcal{C}_h}(\mu) \Omega^* D_{\mathcal{C}_h} = \Omega_*^* D_{\mathcal{C}_h^*} (\mathbf{1} - \mu \mathcal{C}_h^*)^{-1} (\mu \mathbf{1} - \mathcal{C}_h),$$

where $\mu \in \boldsymbol{D}$.

For a simple maximal dissipative operator B and its Cayley transform T, the characteristic function of B is defined as

$$S_B(\lambda) = \Theta_T\left(\frac{\lambda - i}{\lambda + i}\right), \ \mathrm{Im}\lambda > 0. \tag{3.4}$$

The characteristic function $S_B : F(B) \to F_*(B)$ on the sense set $D(B)/G_B$ is defined by

$$S_B(\lambda) = \mathcal{P}_*(B^* - \lambda I)^{-1}(B - \lambda I)\mathcal{P}^{-1}. \tag{3.5}$$

Therefore we have the following theorem.

Theorem 3.4. *The characteristic function of* $\mathcal{L}_h$ *is as follows*

$$S_{\mathcal{L}_h}(\lambda) = \frac{\Delta_h(\lambda)}{\Delta_{\overline{h}}(\lambda)}.$$

Proof Since $\mathcal{L}_h$ is a dissipative extension of the symmetric operator $\mathcal{L}_0$, we have $G_{\mathcal{L}_h} = D(\mathcal{L}_h) \cap D(\mathcal{L}_h^*)$. Moreover, $F(\mathcal{L}_h) = D(\mathcal{L}_h)/G_{\mathcal{L}_h}$ and $F_*(\mathcal{L}_h) = D(\mathcal{L}_h^*)/G_{\mathcal{L}_h}$.

Let $\mathcal{P}$ and $\mathcal{P}_*$ be the natural projections such that $\mathcal{P} : D(\mathcal{L}_h) \to F(\mathcal{L}_h)$ and $\mathcal{P}_* : D(\mathcal{L}_h^*) \to F_*(\mathcal{L}_h)$.

We set $E = E_* = \mathbb{C}$ and we define the isometric isomorphisms Ψ and Ψ_* such as

$$\begin{array}{rccc} \Psi : & E & \to & F(\mathcal{L}_h), \\ & a & \to & \Psi(a) = \mathcal{P}\mathcal{Y}, \end{array} \tag{3.6}$$

where $\mathcal{Y} \in D(\mathcal{L}_h)$ with

$$[y, u](b) = \frac{a}{\sqrt{h_2}}$$

and

$$\begin{array}{rccc} \Psi_* : & E_* & \to & F_*(\mathcal{L}_h), \\ & a & \to & \Psi_*(a) = \mathcal{P}_*\mathcal{Z}, \end{array} \tag{3.7}$$

where $\mathcal{Z} \in D(\mathcal{L}_h^*)$ with

$$[z, u](b) = \frac{a}{\sqrt{h_2}}.$$

Using (3.5) we obtain the characteristic function $S_{\mathcal{L}_h}$ as

$$S_{\mathcal{L}_h}(\lambda) = \Psi_*^*\mathcal{P}_*(\mathcal{L}_h^* - \lambda\mathbf{1})^{-1}(\mathcal{L}_h - \lambda\mathbf{1})\mathcal{P}^{-1}\Psi. \tag{3.8}$$

Therefore (3.6) and (3.8) give

$$S_{\mathcal{L}_h}(\lambda)a = \Psi_*^*\mathcal{P}_*\mathcal{Z},$$

where

$$\mathcal{Z} = (\mathcal{L}_h^* - \lambda\mathbf{1})^{-1}(\mathcal{L}_h - \lambda\mathbf{1})\mathcal{Y}, \ \mathcal{Y} \in D(\mathcal{L}_h) \tag{3.9}$$

with $[y, u](b) = a/\sqrt{h_2}$. Moreover, (3.7) implies that

$$\Psi_*^*\mathcal{P}_*\mathcal{Z} = [z, u](b)\sqrt{h_2}.$$

Therefore using (3.8) one can write

$$S_{\mathcal{L}_h}(\lambda)a = \frac{[z,u](b)}{[y,u](b)}a. \tag{3.10}$$

(3.9) implies that

$$(\mathcal{L}_h^* - \lambda \mathbf{1})\mathcal{Z} = (\mathcal{L}_h - \lambda \mathbf{1})\mathcal{Y},$$

where $\mathcal{Z} \in D(\mathcal{L}_h^*)$ and $\mathcal{Y} \in D(\mathcal{L}_h)$. Consequently, one should find a solution of the equation

$$-(p\varkappa')' + q\varkappa = \lambda\varkappa,$$

from the space $L_w^2(\mathbb{I})$ subject to the condition

$$\alpha_1\varkappa(a) - \alpha_2\varkappa^{[1]}(a) = \lambda\left(\beta_1\varkappa(a) - \beta_2\varkappa^{[1]}(a)\right),$$

where $\varkappa = z - y$ such that $\widehat{\varkappa} = \binom{\varkappa(x,\lambda)}{\varkappa_1}$ and $\varkappa_1 = \beta_1\varkappa(a,\lambda) - \beta_2\varkappa^{[1]}(a,\lambda)$.

Let

$$\widehat{\varphi} = \begin{pmatrix} \varphi(x,\lambda) \\ \varphi_1 \end{pmatrix},$$

where $\varphi_1 = \beta_1\varphi(a,\lambda) - \beta_2\varphi^{[1]}(a,\lambda)$. Then one can infer that $\widehat{\varkappa} = c\widehat{\varphi}$, where c is a constant.

From the equation

$$[\varphi,v](b) = \frac{[\varphi,v](b)}{[\varphi,u](b)}[\varphi,u](b)$$

we obtain

$$[z,u](b)\left([\varphi,v](b) + \overline{h}[\varphi,u](b)\right) = [y,u](b)\left([\varphi,v](b) + h[\varphi,u](b)\right). \tag{3.11}$$

Substituting (3.11) into (3.10) we complete the proof. □

Now using (3.4) we obtain the following theorem.

Theorem 3.5. *The characteristic function of* $\mathcal{C}_h$ *is as follows:*

$$\Theta_{\mathcal{C}_h}(\mu) = \frac{\Delta_h(\lambda)}{\Delta_{\overline{h}}(\lambda)}, \ \mu = \frac{\lambda - i}{\lambda + i}, \ \mathrm{Im}\lambda > 0.$$

Since there is a connection between the characteristic functions of $\mathcal{C}_h$ and $\mathcal{C}_h^*$ with the rule

$$\Theta_{\mathcal{C}_h^*}(\mu) = \Theta_{\mathcal{C}_h}^*(\overline{\mu}), \ \mu \in \boldsymbol{D},$$

we obtain the following corollary.

Corollary 3.6. *The characteristic function of* $\mathcal{C}_h^*$ *is*

$$\Theta_{\mathcal{C}_h^*}(\mu) = \frac{\Delta_{\overline{h}}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_h\left(-i\frac{1+\mu}{1-\mu}\right)}, \ \mu = \frac{\lambda - i}{\lambda + i}, \ \mathrm{Im}\lambda > 0.$$

Remark 3.7. *Since $\mathcal{C}_h$ is a c.n.u. contraction,* 1 *cannot belong to the point spectrum of $\mathcal{C}_h$. On the other hand, the spectrum of $\mathcal{C}_h$ coincides with those μ belonging to the disc $\boldsymbol{D}$ for which the operator $\Theta_{\mathcal{C}_h}(\mu)$ is not boundedly invertible, together with those $\mu \in \boldsymbol{C}$ not lying on any of the open arcs of $\boldsymbol{C}$ on which $\Theta_{\mathcal{C}_h}(\mu)$ is a unitary operator valued analytic function of μ and point spectrum of $\mathcal{C}_h$ coincides with those $\mu \in \boldsymbol{D}$ for which $\Theta_{\mathcal{C}_h}(\mu)$ is not invertible at all. Since the zeros of $\Delta_h(\lambda)$, $\mathrm{Im}\lambda > 0$, are eigenvalues of $\mathcal{L}_h$, $\lambda = i(1+\mu)/(1-\mu)$ for $\lambda = is$, $\lim_{s\to\infty}(is) =: \lambda_\infty$ cannot be a zero of $\Delta_h(\lambda)$ or equivalently an eigenvalue of $\mathcal{L}_h$.*

Definition 3.8. [24, 25, 28, 29] *The classes $C_{0.}$ and $C_{.0}$ of contractions are defined as*

$$\begin{aligned} T \in C_{0.} &\text{ if } T^n f \to 0 \text{ for all } f, \\ T \in C_{.0} &\text{ if } T^{*n} f \to 0 \text{ for all } f. \end{aligned}$$

Asymptotic classifications of $C_{0.}$ and $C_{.0}$ are given as

$$\begin{aligned} C_{0.} &= \{T : \|T\| \le 1, \lim_n \|T^n f\| = 0 \text{ for every } f\}, \\ C_{.0} &= \{T : \|T\| \le 1, \lim_n \|T^{*n} f\| = 0 \text{ for every } f\}. \end{aligned}$$

C_{00} is defined as $C_{00} = C_{0.} \cap C_{.0}$.

Theorem 3.9. $\mathcal{C}_h \in C_{00}$.

Proof The inequality

$$\|\mathcal{C}_h^n \mathcal{F}\|_H \le \|\mathcal{C}_h\|_H^n \|\mathcal{F}\|_H$$

and (3.3) give that $\mathcal{C}_h \in C_{0.}$. Since $\|\mathcal{C}_h\|_H = \|\mathcal{C}_h^*\|_H$ one arrives at $\mathcal{C}_h \in C_{.0}$. This completes the proof. □

Remark 3.10. *Since $\mathcal{C}_h \in C_{.0}$, $\Theta_{\mathcal{C}_h}(\mu)$ is an inner function.*

Lemma 3.11. $\mathfrak{d}_{\mathcal{C}_h} = \mathfrak{d}_{\mathcal{C}_h^*} = 1$.

Proof Let $(\mathcal{L}_h + i\mathbf{1})\mathcal{Y} = \mathcal{F}$, where $\mathcal{Y} \in D(\mathcal{L}_h)$, $\mathcal{F} \in H$. Then using the idea of [28] we obtain

$$D_{\mathcal{C}_h}^2 \mathcal{F} = (\mathcal{L}_h + i\mathbf{1})\,\mathcal{Y} - (\mathcal{L}_h^* + i\mathbf{1})\,\mathcal{Z}, \tag{3.12}$$

where $\mathcal{Z} \in D(\mathcal{L}_h^*)$ and

$$\mathcal{Z} = (\mathcal{L}_h^* - i\mathbf{1})^{-1}\,(\mathcal{L}_h - i\mathbf{1})\,\mathcal{Y}. \tag{3.13}$$

From (3.13) one infers that

$$(\mathcal{L}_h^* - i\mathbf{1})\,\mathcal{Z} = (\mathcal{L}_h - i\mathbf{1})\,\mathcal{Y}.$$

Then we have

$$D_{\mathcal{C}_h}^2 \mathcal{F} = (\mathcal{L}_h + i\mathbf{1})\,(\mathcal{Y} - \mathcal{Z}) = 2ic\widehat{\varphi}(i,\lambda).$$

Therefore $D_{\mathcal{C}_h}$ is spanned by $\widehat{\varphi}(i,\lambda)$. Namely, the equation

$$-(p\varkappa')' + q\varkappa = 2ic\varkappa,$$

has two linearly independent solutions belonging to $L_w^2(\mathbb{I})$. However, only one of them satisfies the condition

$$\alpha_1\varkappa(a) - \alpha_2\varkappa^{[1]}(a) = 2ic\left(\beta_1\varkappa(a) - \beta_2\varkappa^{[1]}(a)\right).$$

This solution can be regarded as a multiple of $\varphi(i,\lambda)$. Consequently, $\mathfrak{d}_{\mathcal{C}_h}=1$.

Now let $(\mathcal{L}_h^*-i\mathbf{1})\mathcal{Y}=\mathcal{F}$, where $\mathcal{Y}\in D(\mathcal{L}_h^*)$, $\mathcal{F}\in H$. Then

$$D^2_{\mathcal{C}_h^*}\mathcal{F}=(\mathcal{L}_h^*-i\mathbf{1})\,\mathcal{Y}-(\mathcal{L}_h-i\mathbf{1})\,\mathcal{Z},$$

where $\mathcal{Z}\in D(\mathcal{L}_h)$ and

$$\mathcal{Z}=(\mathcal{L}_h+i\mathbf{1})^{-1}\,(\mathcal{L}_h^*+i\mathbf{1})\,\mathcal{Y}.$$

Consequently a similar argument shows that $D_{\mathcal{C}_h^*}$ is spanned by $\widehat{\varphi}(-i,\lambda)$, i.e., $\mathfrak{d}_{\mathcal{C}_h^*}=1$. Therefore the proof is completed. □

Theorems 3.9 and 3.11 give the following theorem.

Theorem 3.12. *The c.n.u. contraction $\mathcal{C}_h$ belongs to the class C_0. Moreover, the characteristic function $\Theta_{\mathcal{C}_h}(\mu)$ of $\mathcal{C}_h$ coincides with the minimal function $m_{\mathcal{C}_h}(\mu)$ of $\mathcal{C}_h$.*

Theorem 3.13. *$\Theta_{\mathcal{C}_h}$ is a Blaschke product in the disc $\boldsymbol{D}$.*

Proof According to Remark 3.10, $\Theta_{\mathcal{C}_h}$ can be represented as

$$\Theta_{\mathcal{C}_h}(\lambda)=\mathbb{B}(\lambda)\exp(i\lambda b),\ b>0,\ \mathrm{Im}\lambda>0,$$

where $\mathbb{B}(\lambda)$ is a Blaschke product in the upper half-plane. Hence we get

$$|\Theta_{\mathcal{C}_h}(\lambda)|\le\exp(-b\mathrm{Im}\lambda). \tag{3.14}$$

For $\lambda_s=is$ from (3.14) we get that $\Delta_h(\lambda)\to0$ as $s\to\infty$. Therefore $\lambda_\infty=\lim_{s\to\infty}is$ is a zero of $\Delta_h(\lambda)$ or equivalently is an eigenvalue of $\mathcal{L}_h$. However, according to Remark 3.7 this is not possible. Therefore there cannot be a singular factor in the factorization of $\Theta_{\mathcal{C}_h}(\lambda)$, $\mathrm{Im}\lambda>0$. Letting $\mu=(\lambda-i)/(\lambda+i)$ the proof is completed. □

According to the well-known theorem given by Sz.-Nagy and Foiaş we can introduce the following theorem.

Theorem 3.14. *Root functions of $\mathcal{C}_h$ associated with the points of the spectrum of $\mathcal{C}_h$ in $\boldsymbol{D}$ span the Hilbert space H.*

Definition 3.15. *Let all root functions of the operator A span the Hilbert space H. Such an operator is called a complete operator. If every $A-$ invariant subspace is generated by root vectors of A belonging to the subspace then it is said A admits spectral synthesis.*

It is well known that any complete operator belonging to the class C_0 admits spectral synthesis [28, 29]. Therefore the following theorem is obtained.

Theorem 3.16. *$\mathcal{C}_h$ admits spectral synthesis.*

There is a connection between the completeness of the root functions of a linear operator and its Cayley transform [13] (p. 42). Therefore we obtain the following.

Theorem 3.17. *Root functions of $\mathcal{L}_h$ associated with the point spectrum of $\mathcal{L}_h$ in the open upper half-plane $\mathrm{Im}\lambda>0$ span the Hilbert space H.*

More detailed analysis of the spectrum of $\mathcal{L}_h$ will be obtained with the help of the inverse operator of $\mathcal{L}_h$ in the next section.

4. Bounded integral operator with finite-rank imaginary component

In this section we find the inverse operator of $\mathcal{L}_h$. For this purpose let us consider the equality

$$\mathcal{L}_h \mathcal{Y} = \mathcal{F}, \tag{4.1}$$

where

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \end{pmatrix} \in D(\mathcal{L}_h),\ \mathcal{F} = \begin{pmatrix} f \\ f_1 \end{pmatrix} \in H.$$

Equation (4.1) is equivalent to the nonhomogeneous equation

$$\ell(y) = f(x),\ x \in \mathbb{I}, \tag{4.2}$$

subject to the conditions

$$\begin{array}{l} \alpha_1 y(a) - \alpha_2 y^{[1]}(a) = f_1, \\ [y,v](b) + h[y,u](b) = 0. \end{array} \tag{4.3}$$

Let us consider the solutions u and $\tau = v + hu$. Then the method of variation of parameters gives the solution y of (4.2), (4.3) as the form

$$y(x) = \int_{\mathbb{I}} G(x,t) f(t) w(t) dt - f_1 \tau(x). \tag{4.4}$$

where

$$G(x,t) = \left\{ \begin{array}{l} -\tau(x)u(t),\ a \le t \le x \\ -\tau(t)u(x),\ x \le t \le b \end{array} \right. . \tag{4.5}$$

On the other hand, from (4.5) one obtains

$$G_1(x) = \beta_1 G(x,a) - \beta_2 G^{[1]}(x,a) = -\delta \tau(x). \tag{4.6}$$

Therefore substituting (4.6) in (4.4) we get that

$$y(x) = \int_{\mathbb{I}} G(x,t) f(t) w(t) dt + \frac{G_1(x) f_1}{\delta}$$

or

$$y(x) = \left\langle \mathcal{G}(x,t), \overline{\mathcal{F}}(t) \right\rangle_H ,$$

where

$$\mathcal{G}(x,t) = \begin{pmatrix} G(x,t) \\ G_1(x) \end{pmatrix}.$$

If we define the operator $\mathcal{K}$ as

$$\mathcal{K}\mathcal{F} = \left\langle \mathcal{G}(x,t), \overline{\mathcal{F}}(t) \right\rangle_H , \tag{4.7}$$

then $\mathcal{K}$ is the inverse of $\mathcal{L}_h$. Since the completeness of the root functions of $\mathcal{K}$ and $\mathcal{L}_h$ coincide, we obtain the following theorem.

Theorem 4.1. *The system of all root functions of* $\mathcal{K}$ *is complete in* $\mathcal{H}$.

Since $h = h_1 + ih_2$, one can infer from (4.5)–(4.7) that $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{K}_1 + i\mathcal{K}_2$, where

$$\mathcal{K}_1\mathcal{F} = \left\langle \mathcal{G}_1(x,t), \overline{\mathcal{F}}_1(t) \right\rangle_H$$

with

$$\mathcal{G}_1(x,t) = \begin{pmatrix} G_{(1)}(x,t) \\ G_{(1)1}(x) \end{pmatrix},\ G_{(1)}(x,t) = \begin{cases} -(v(x) + h_1 u(x))u(t), & a \le t \le x \\ -(v(t) + h_1 u(t))u(x), & x \le t \le b \end{cases},$$

and

$$\mathcal{K}_2\mathcal{F} = \left\langle \mathcal{G}_2(x,t), \overline{\mathcal{F}}_1(t) \right\rangle_H$$

with

$$\mathcal{G}_2(x,t) = \begin{pmatrix} G_{(2)}(x,t) \\ G_{(2)1}(x) \end{pmatrix},\ G_{(2)}(x,t) = h_2 u(x)u(t).$$

The following theorem is important to understand the nature of the imaginary part of a densely defined operator.

Theorem 4.2. [37] *Assume that a densely defined operator* B *is invertible and has a dense range. If* $\mathfrak{E}$ *and* $\mathfrak{F}$ *are linear components of*

$$\{y \in D(B) \cap D(B^*) : By = B^*y\}$$

in $D(B)$ and $D(B^*)$, respectively, then the range of the imaginary component $\operatorname{Im}(B^{-1})$ of the inverse B^{-1} is contained in $\mathfrak{E} \oplus \mathfrak{F}$.

Now consider the operator $\mathcal{K} = \mathcal{K}_1 + i\mathcal{K}_2$. Since $\mathcal{L}_h\mathcal{Y} = \mathcal{L}_h^*\mathcal{Y}$ for all minimal domain functions $\mathcal{Y}$ and $D(\mathcal{L}_h)$ and $D(\mathcal{L}_h^*)$ are only one-dimensional extensions of the minimal domain, $\mathcal{K}_2$ is a finite-rank (nuclear) operator. Therefore $\mathcal{K}_2$ is a compact operator.

Because a complete dissipative operator with a nuclear imaginary component admits spectral synthesis [23], we have the following.

Theorem 4.3. $\mathcal{K}$ *admits spectral synthesis.*

It is known that the nonreal spectrum of an operator with a compact imaginary part consists of eigenvalues of finite algebraic multiplicities (dimensions of the corresponding root subspace) and the limit points of the nonreal spectrum belong to the spectrum of the real part of the operator [10]. Therefore, together with the results given in [21], we obtain the following theorem.

Theorem 4.4. (i) *Eigenvalues of* $\mathcal{K}$ *are countable,*

(ii) *zero is the only possible limit point of the eigenvalues,*

(iii) *zero must belong to the spectrum of* $\mathcal{K}$, *but may not be an eigenvalue of* $\mathcal{K}$,

(iv) the nonreal spectrum of $\mathcal{K}$ consists of eigenvalues of finite algebraic multiplicities and limit points of the nonreal spectrum belong to the spectrum of the real part $\mathcal{K}_1$.

Note that $\mathcal{K}_1$ is the inverse of the real part $\mathrm{Re}\mathcal{L}_h$ of $\mathcal{L}_h$, which is generated by ℓ and the conditions (2.4) with $[y,v](b)+h_1[y,u](b)=0,\ h_1=\mathrm{Re}h$.

Let λ_j and $\mathrm{Re}\lambda_j$ denote the eigenvalues of $\mathcal{L}_h$ and $\mathrm{Re}\mathcal{L}_h$, respectively. Then $1/\lambda_j$ and $1/\mathrm{Re}\lambda_j$ are the eigenvalues of $\mathcal{K}$ and $\mathcal{K}_1$, respectively. Therefore, we immediately obtain the following corollary.

Corollary 4.5. *(i) Eigenvalues of $\mathcal{L}_h$ are countable,*

(ii) infinity is the only possible limit point of the eigenvalues of $\mathcal{L}_h$,

(iii) infinity must belong to the spectrum of $\mathcal{L}_h$, but may not be an eigenvalue of $\mathcal{L}_h$,

(iv) infinity (on the real axis) belongs to the spectrum of $\mathrm{Re}\mathcal{L}_h$.

5. Dilation of the maximal dissipative operator $\mathcal{L}_h$

5.1. Self-adjoint dilation

Let T be the Cayley transform of the maximal dissipative operator B and let $\mathcal{U}$ be the minimal unitary dilation of T acting in the direct sum Hilbert space $\mathcal{H}=G_*\oplus H\oplus G$, where G and G_* are $\mathcal{U}$ and $\mathcal{U}^*$ invariant subspaces, respectively. Setting E and E_* as isomorphic spaces with $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$, respectively, one can select $\mathcal{H}$ as follows:

$$\mathcal{H}=H^2_-(\boldsymbol{D},E_*)\oplus H\oplus H^2_+(\boldsymbol{D},E).$$

Moreover, a more useful representation of the space $\mathcal{H}$ can be obtained with the help of the maximal dissipative operator B.

A self-adjoint operator $\mathcal{B}$ acting on the Hilbert space $\mathcal{H}$ is called a self-adjoint dilation of the maximal dissipative operator B acting on the Hilbert space H if one of the following statements hold:

(i) $(B-\lambda I)^{-1}=P_H(\mathcal{B}-\lambda I)^{-1}\mid H,\ \lambda\in\mathbb{C}_-$,

(ii) $(B+iI)^{-n}=P_H(\mathcal{B}+iI)^{-n}\mid H,\ n\geq 0$,

(iii) $\exp(iBt)=P_H\exp(i\mathcal{B}t)\mid H,\ t>0$,

(iv) $\mathcal{U}=(\mathcal{B}-iI)(\mathcal{B}+iI)^{-1}$ is a unitary dilation of $T=(B-iI)(B+iI)^{-1}$.

The following theorem gives the characterization of the minimal self-adjoint dilation on the space $\mathcal{H}=L^2(\mathbb{R}_-,E_*)\oplus H\oplus L^2(\mathbb{R}_+,E)$, where $\mathbb{R}_-:=(-\infty,0]$ and $\mathbb{R}_+:=[0,\infty)$.

Theorem 5.1.1. [35] *Let B be a maximal dissipative operator in the Hilbert space H and let T be its Cayley transform. Then its minimal self-adjoint dilation $\mathcal{B}$ in the space $\mathcal{H}=L^2(\mathbb{R}_-,E_*)\oplus H\oplus L^2(\mathbb{R}_+,E)$ has the form*

$$\mathcal{B}\left\langle \upsilon_-,f,\upsilon_+\right\rangle=\left\langle i\upsilon_-',i\left\{2(I-T)^{-1}\left[f-\frac{i}{2}D_{T^*}\Omega_*\upsilon_-(0)\right]-f\right\},i\upsilon_+'\right\rangle,$$

on the domain $D(\mathcal{B})$, which consists of those functions $\langle\upsilon_-,f,\upsilon_+\rangle$ such that $\upsilon_-\in W_2^1(\mathbb{R}_-,E_*)$, $\upsilon_+\in W_2^1(\mathbb{R}_+,E)$,

$$f-\frac{i}{\sqrt{2}}D_{T^*}\Omega_*\upsilon_-(0)\in(I-T)H=D(B),$$

$$i\sqrt{2}D_T(I-T)^{-1}\left(f-\frac{i}{\sqrt{2}}D_{T^*}\Omega_*\upsilon_-(0)\right)=T^*\Omega_*\upsilon_-(0)+\Omega\upsilon_+(0),$$

where $\Omega : E \to \mathfrak{D}_T,\ \Omega_* : E_* \to \mathfrak{D}_{T^*}$ *are the free parameters.*

If $\mathfrak{d}_T, \mathfrak{d}_{T^*} < \infty$ then it is convenient to consider the boundary spaces $F(B)$ and $F_*(B)$ instead of $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$. Therefore the following theorem can be introduced.

Theorem 5.1.2. [35] *Let* B *be a maximal dissipative operator in the Hilbert space* H *with finite defects. Assume that there are given isometric isomorphisms* $\Psi : E \to F(B),\ \Psi_* : E_* \to F_*(B)$. *The minimal selfadjoint dilation* $\mathcal{B}$ *in the space* $\mathcal{H} = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$ *has the form*

$$\mathcal{B}\langle \upsilon_-, f, \upsilon_+\rangle = \left\langle i\upsilon_-', B\left(f-\frac{i}{\sqrt{2}}\left[\Psi_*\upsilon_-(0)\right]\right)+\frac{i}{\sqrt{2}}B^*\left[\Psi_*\upsilon_-(0)\right], i\upsilon_+'\right\rangle,$$

where $[.]$ *denotes some representative of the quotient class mod* G_B, *on the domain* $D(\mathcal{B})$, *which consists of those functions* $\langle \upsilon_-, f, \upsilon_+\rangle$ *such that*

$$f-\frac{i}{\sqrt{2}}\left[\Psi_*\upsilon_-(0)\right]\in D(B),$$

$$f-\frac{i}{\sqrt{2}}\left[\Psi_*\upsilon_-(0)\right]+\frac{i}{\sqrt{2}}\left[\Psi\upsilon_+(0)\right]\in G_B.$$

In the case that G_B is dense in H, then $G_B = D(B) \cap D(B^*)$ and one has the following corollary.

Theorem 5.1.3. [35] *Let* B *be a maximal dissipative operator with finite defects such that* G_B *is dense in* H. *Then the self-adjoint dilation has the form*

$$B\langle \upsilon_-, f, \upsilon_+\rangle = \left\langle i\upsilon_-', \widetilde{B}f, i\upsilon_+'\right\rangle, \quad \widetilde{B} = (B \mid G_B)^* = \left\{\begin{array}{c} B \text{ on } D(B) \\ B^* \text{ on } D(B^*) \end{array}\right. ,$$

on the set $D(\mathcal{B})$ *consisting of all functions* $\langle \upsilon_-, f, \upsilon_+\rangle$ *such that* $\upsilon_- \in W_2^1(\mathbb{R}_-, E_*),\ \upsilon_+ \in W_2^1(\mathbb{R}_+, E)$,

$$f-\frac{i}{\sqrt{2}}\left[\Psi_*\upsilon_-(0)\right]\in D(B),\ f+\frac{i}{\sqrt{2}}\left[\Psi\upsilon_+(0)\right]\in D(B^*).$$

Hence we are ready to introduce the self-adjoint dilation $\mathfrak{L}$ of the maximal dissipative operator $\mathcal{L}_h$.

Theorem 5.1.4. *The self-adjoint dilation* $\mathfrak{L}$ *of the maximal dissipative operator* $\mathcal{L}_h$ *acts on the direct sum Hilbert space* $L^2(\mathbb{R}_-) \oplus H \oplus L^2(\mathbb{R}_+)$, *where* $H = L_w^2(\mathbb{I}) \oplus \mathbb{C}$, *and has the following form:*

$$\mathfrak{L}\langle \upsilon_-, \mathcal{F}, \upsilon_+\rangle = \left\langle i\upsilon_-', \mathcal{L}_h\mathcal{F}, i\upsilon_+'\right\rangle,$$

where $\upsilon_- \in W_2^1(\mathbb{R}_-),\ \upsilon_+ \in W_2^1(\mathbb{R}_+)$ *such that*

$$[f,v](b)+h[f,u](b)=-\sqrt{2h_2}\upsilon_-(0), \quad [f,v](b)+\overline{h}[f,u](b)=-\sqrt{2h_2}\upsilon_+(0).$$

Proof Let $\gamma_{-} := \mathcal{F} - \frac{i}{\sqrt{2}}[\Psi_{*}\upsilon_{-}(0)] \in D(\mathcal{L}_h)$ and $\gamma_{+} := \mathcal{F} + \frac{i}{\sqrt{2}}[\Psi\upsilon_{+}(0)] \in D(\mathcal{L}_h^{*})$. Consider the function $\mathcal{Y} = \binom{y}{y_1} \in D(\mathcal{L}_h)$ such that $[y,u](b) = \upsilon_{+}(0)(h_2)^{-1/2}$ and $\mathcal{Y} = \binom{z}{z_1} \in D(\mathcal{L}_h^{*})$ such that $[z,u](b) = \upsilon_{-}(0)(h_2)^{-1/2}$. Therefore for $\gamma_{-} \in D(\mathcal{L}_h)$ one has

$$[f,v](b) - \frac{i}{\sqrt{2}}[z,v](b) = -h[f,u](b) + \frac{i}{\sqrt{2}}h[z,u](b)$$

and

$$[f,v](b) + h[f,u](b) = -\sqrt{2h_2}\upsilon_{-}(0).$$

Similarly for $\gamma_{+} \in D(\mathcal{L}_h^{*})$ one has

$$[f,v](b) + \frac{i}{\sqrt{2}}[y,v](b) = -\overline{h}[f,u](b) - \frac{i}{\sqrt{2}}\overline{h}[z,u](b)$$

and

$$[f,v](b) + \overline{h}[f,u](b) = -\sqrt{2h_2}\upsilon_{+}(0).$$

This completes the proof. □

5.2. Functional embeddings

Let $\mathcal{H} = G_{*} \oplus H \oplus G$, $\mathcal{B}$ is the minimal selfadjoint dilation on $\mathcal{H}$ of the maximal dissipative operator B acting on H, and the following are satisfied:

(i) $\exp(i\mathcal{B}t)G \subset G,\ t > 0$;

(ii) $\exp(i\mathcal{B}t)G_{*} \subset G_{*},\ t < 0$.

Consider the following isometries:

$$\pi^{\mathbb{R}} : L^2(\mathbb{R}, E) \to \mathcal{H},\ \ \dim E = \dim F(B),$$
$$\pi_{*}^{\mathbb{R}} : L^2(\mathbb{R}, E_{*}) \to \mathcal{H},\ \ \dim E_{*} = \dim F_{*}(B).$$

$\pi^{\mathbb{R}}$ and $\pi_{*}^{\mathbb{R}}$ are called functional embeddings. Under the condition $(\mathcal{B}+iI)^{-1}\pi^{\mathbb{R}} = \pi^{\mathbb{R}}(Z+iI)^{-1}$, $(\mathcal{B}+iI)^{-1}\pi_{*}^{\mathbb{R}} = \pi_{*}^{\mathbb{R}}(Z+iI)^{-1}$, $\pi^{\mathbb{R}}H^2(\mathbb{C}_{+}, E) = G$, $\pi_{*}^{\mathbb{R}}H^2(\mathbb{C}_{-}, E_{*}) = G_{*}$, $\pi^{\mathbb{R}}$ and $\pi_{*}^{\mathbb{R}}$ are uniquely determined to within multiplications by unitary constants in E and E_{*}.

The operator

$$S = (\pi_{*}^{\mathbb{R}})^{*}\pi^{\mathbb{R}}$$

acts from $L^2(\mathbb{R}, E)$ into $L^2(\mathbb{R}, E_{*})$, maps $H^2(\mathbb{C}_{+}, E)$ into $H^2(\mathbb{C}_{+}, E_{*})$, and commutes with the multiplication $(z+i)^{-1}$. Therefore S is multiplication by a function [35].

$S_B(\lambda) : E \to E_{*}$ is called the characteristic function of B. Therefore

$$S(\lambda) = S_B(\lambda) = \Theta_T\left(\frac{\lambda - i}{\lambda + i}\right),$$

where Θ_T is the characteristic function of the Cayley transform of B.

Generalized eigenfunctions of the dilation $\mathcal{B}$ can be described by the characteristic function of the maximal dissipative operator B. In fact, incoming eigenfunctions are of the form

$$\left\langle S(\lambda)\exp(-i\lambda\xi)d, \frac{i}{\sqrt{2}}\left((B^{*}-\overline{\lambda}I)^{-1}(B-\overline{\lambda}I)-I\right)\mathcal{P}^{-1}\Psi d, \exp(-i\lambda\zeta)d\right\rangle \tag{5.1}$$

and outgoing eigenfunctions are of the form

$$\left\langle \exp(-i\lambda\xi)e, -\frac{i}{\sqrt{2}}\left((B-\lambda I)^{-1}(B^{*}-\lambda I)-I\right)\mathcal{P}_{*}^{-1}\Psi_{*}e, S^{*}(\lambda)\exp(-i\lambda\zeta)e\right\rangle \tag{5.2}$$

where $\xi\in\mathbb{R}_{-},\ \zeta\in\mathbb{R}_{+},\ d\in E, e\in E_{*}$ and $\lambda\in\mathbb{R}$.

Therefore we have the following theorem.

Theorem 5.2.1. *The incoming eigenfunctions of the dilation* $\mathcal{B}$ is

$$\left\langle \frac{\Delta_{h}(\lambda)}{\Delta_{\overline{h}}(\lambda)}\exp(-i\lambda\xi), \sqrt{2h_{2}}\frac{[\overline{\varphi},u](b)}{[\overline{\varphi},v](b)+h[\overline{\varphi},u](b)}, \exp(-i\lambda\zeta)\right\rangle \tag{5.3}$$

and outgoing eigenfunction of $\mathcal{B}$ *is*

$$\left\langle \exp(-i\lambda\xi), \sqrt{2h_{2}}\frac{[\varphi,u](b)}{[\varphi,v](b)+\overline{h}[\varphi,u](b)}, \frac{\Delta_{\overline{h}}(\overline{\lambda})}{\Delta_{h}(\overline{\lambda})}\exp(-i\lambda\zeta)\right\rangle \tag{5.4}$$

where $\xi\in\mathbb{R}_{-},\ \zeta\in\mathbb{R}_{+},\ d\in E, e\in E_{*}$ *and* $\lambda\in\mathbb{R}$.

Proof Consider the equation

$$\left(\left(\mathcal{L}_{h}^{*}-\overline{\lambda}\mathbf{1}\right)^{-1}\left(\mathcal{L}_{h}-\overline{\lambda}\mathbf{1}\right)-\mathbf{1}\right)\mathcal{P}^{-1}\Psi d = c\widehat{\varphi}(x,\overline{\lambda}),$$

where $\mathcal{Z}-\mathcal{Y}=c\widehat{\varphi}(x,\lambda),\ \mathcal{Z}\in D(\mathcal{L}_{h}^{*})$ and $\mathcal{Y}\in D(\mathcal{L}_{h})$.

On the other hand, one gets

$$c[\overline{\varphi},u](b)=[\overline{z},u](b)-[\overline{y},u](b)=\frac{-2i\sqrt{h_{2}}[\overline{\varphi},u](b)}{[\overline{\varphi},v](b)+h[\overline{\varphi},u](b)}d. \tag{5.5}$$

Therefore from (5.1) and (5.5) we obtain (5.3).

Now consider the equation

$$\left(\left(\mathcal{L}_{h}-\overline{\lambda}\mathbf{1}\right)^{-1}\left(\mathcal{L}_{h}^{*}-\overline{\lambda}\mathbf{1}\right)-\mathbf{1}\right)\mathcal{P}_{*}^{-1}\Psi_{*}e = \widetilde{c}\widehat{\varphi}(x,\lambda),$$

where $\mathcal{Y}-\mathcal{Z}=\widetilde{c}\widehat{\varphi}(x,\lambda),\ \mathcal{Z}\in D(\mathcal{L}_{h}^{*})$ and $\mathcal{Y}\in D(\mathcal{L}_{h})$. A direct calculation gives

$$\widetilde{c}[\varphi,u](b)=[z,u](b)-[y,u](b)=\frac{2i\sqrt{h_{2}}[\varphi,u](b)}{[\varphi,v](b)+\overline{h}[\varphi,u](b)}e. \tag{5.6}$$

Consequently, (5.2) and (5.6) give (5.4). Therefore the proof is completed. □

6. More on the contraction $\mathcal{C}_h$

6.1. Multiplicity of the contraction $\mathcal{C}_h$

Let $A : X \to X$, $B : U \to X$, $C = X \to Y$, and $D : U \to Y$, where X, Y, and U are Banach spaces, be the linear transformations. For $x(t) \in X$ and $u(t) \in U$, consider the following linear dynamic system [28, 29]:

$$x'(t) = Ax(t) + Bu(t),\ t \geq 0, \tag{6.1}$$

$$y(t) = Cx(t) + Du(t),\ t \geq 0, \tag{6.2}$$

with $x(0) = x_0$, $t \geq 0$. The operators A and B are called the generator operator and control operator, respectively, while C and D are called observation operators. X is called state space and $x(t)$ is the state of the system at time t. Finally, u is the input function, y is the output function, and x_0 is the initial state.

The system (6.1) is called approximately controllable if for every $x_0, x_1 \in X$ and arbitrary $\epsilon > 0$ there exists $\tau \in [0, \infty)$ and $u \in L^2(0, \tau)$ such that $\|x(\tau) - x_1\| < \epsilon$ with $x(0) = x_0$.

Let the system (6.1) be controllable. The system (6.1) is approximately controllable on $[0, \infty)$ if and only if

$$span\,\{S(t)BU : t \geq 0\} = X, \tag{6.3}$$

where $S(.)$ is the semigroup associated with A. In the case that the generator A is bounded, then (6.3) is satisfied if and only if

$$span\,\{A^n BU : n \geq 0\} = X.$$

Therefore, it is important to find the least possible dimension of the control subspace $\dim BU$ (6.1) is approximately controllable. Namely, one should find the following:

$$\min\{\dim BU : (A, B) \text{ is approximately controllable}\}.$$

Multiplicity of the spectrum of an arbitrary bounded operator $T : X \to X$ is defined as

$$\boldsymbol{\mu}_T = \min\{\dim C : span\,(T^n C : n \geq 0) = X\}.$$

For $\boldsymbol{\mu}_T = 1$, T is called multiplicity-free.

It is well known that T is unitary equivalent to the model operator $Z : H_\Theta \to H_\Theta$, where

$$H_\Theta = \left(H^2 \oplus clos\Lambda L^2\right) \ominus (\Theta \oplus \Lambda)\, H^2,$$

$L^2 = L^2(\boldsymbol{C})$, H^2 is the Hardy space, $\Lambda = (1 - |\Theta|^2)^{1/2}$

$$Zf = P_\Theta z f,\ f \in H_\Theta,$$

$P_\Theta = \Theta P_- \overline{\Theta}$ and P_- is the projection of L^2 into H^2_- (the Hardy space in the lower half plane).

The multiplicity of a c.n.u. contraction T may be computed with the help of the characteristic function [28]. In the case that the characteristic function is not identical to zero, then the following theorem can be introduced ([28], p. 247).

Theorem 6.1.1. *C.n.u. contraction $\mathcal{C}_h$ is multiplicity-free.*

In general, the adjoint of a multiplicity-free operator is not generally multiplicity-free. However, since $\mathcal{C}_h \in C_0$ we can find the multiplicity of $\mathcal{C}_h$. Before this, we shall give some definitions.

Definition 6.1.2. *Let V be an isometry in the Hilbert space H. A subspace $\mathcal{L}$ of H is called a wandering space for V if $V^p\mathcal{L} \perp V^q\mathcal{L}$ for every pair of integers $p, q \geq 0,\ p \neq q$. An isometry V on H is called a unilateral shift if there exists in H a subspace $\mathcal{L}$ that is wandering for V and such that*

$$H = \bigoplus_{0}^{\infty} V^n \mathcal{L}.$$

The dimension of $H \ominus V\mathcal{L}$ is called the multiplicity of the unilateral shift V.

Let S denote the unilateral shift of the multiplicity one acting on H^2.

Definition 6.1.3. *For each inner function $\varphi \in H^\infty$, the Jordan block $S(\varphi)$ is the operator defined on $H(\varphi) = H^2 \ominus \varphi H^2$ by $S(\varphi) = P_{H(\varphi)}S \mid H(\varphi)$ or equivalently, $S(\varphi)^* = S^* \mid H(\varphi)$. By an affinity from H_1 to H_2 it is meant a linear, one-to-one, and bicontinuous transformation X from H_1 onto H_2. Thus bounded operators, say S_1 on H_1 and S_2 on H_2, are said to be similar if there exists an affinity X from H_1 to H_2 such that $XS_1 = S_2X$ (and consequently $X^{-1}S_2 = S_1X$). By a quasi-affinity from H_1 to H_2 it is meant a linear, one-to-one, and continuous transformation X from H_1 onto a dense linear manifold in H_2 if S_1 and S_2 are bounded operators, S_1 on H_1 and S_2 on H_2, it is said that S_1 is a quasi-affine transform of S_2 if there exist a quasi-affinity X from H_1 to H_2 such that $XS_1 = S_2X$. The operators S_1* and *S_2 are called quasi-similar if they are quasi-affine transforms of one another.*

Definition 6.1.4. *Let A be a bounded operator in H, and let $\mathcal{A}$ be a subspace of H. $\mathcal{A}$ is said to be hyperinvariant for A if it is invariant for every bounded operator that commutes with A.*

We are ready to introduce the following theorem [25] (Chap. X, Sect. 4).

Theorem 6.1.5. *i) $\mathcal{C}_h^*$ is multiplicity-free,*

ii) $\mathcal{C}_h$ is quasi-similar to the Jordan block $S\left(\frac{\Delta_h}{\Delta_{\overline{h}}}\right)$,

iii) $\mathcal{C}_h \mid \mathcal{N}$ is multiplicity-free, i.e. where $\mathcal{N}$ is a invariant subspace of $\mathcal{C}_h$,

iv) $\mathcal{M}$ is hyperinvariant, where $\mathcal{M}$ is a invariant subspace of $\mathcal{C}_h$.

6.2. Unitary colligation

Unitary colligation theory has been investigated in recent years by many authors. For example, one may see the book [7] and references therein. It should be noted that Sz-Nagy–Foiaş characteristic function theory is a special case of the unitary colligation theory [14].

Let $\mathfrak{H}, \mathfrak{F}$, and $\mathfrak{G}$ be the separable Hilbert spaces. The set $\mathbf{\Delta} = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S)$ is called a unitary colligation if the following block form

$$\boldsymbol{U} = \begin{pmatrix} T & F \\ G & S \end{pmatrix}$$

is a unitary mapping such that

$$\boldsymbol{U}=\begin{pmatrix} T & F \\ G & S \end{pmatrix}:\mathfrak{H}\oplus\mathfrak{F}\rightarrow\mathfrak{H}\oplus\mathfrak{G}. \tag{6.4}$$

In this case the Hilbert spaces $\mathfrak{H},\mathfrak{G}$, and $\mathfrak{F}$ are called, respectively, the inner, left-outer, and right-outer spaces and $\boldsymbol{U}$ is called the connecting operator. Let P_1 and P_2 denote the orthogonal projections of $\mathfrak{H}\oplus\mathfrak{G}$ onto $\mathfrak{H}$ and $\mathfrak{G}$, respectively. Then the following operators

$$T=P_1\left[\boldsymbol{U}\mid\mathfrak{H}\right],\ F=P_1\left[\boldsymbol{U}\mid\mathfrak{F}\right],\ G=P_2\left[\boldsymbol{U}\mid\mathfrak{H}\right],\ S=P_2\left[\boldsymbol{U}\mid\mathfrak{F}\right]$$

are called the components of $\boldsymbol{\Delta}$ and T,F,G, and S are called the basic, right-channeled, left-channeled, and duplicating operators, respectively. Moreover, the following relations hold:

$$\begin{array}{lll} T^*T+G^*G=I_{\mathfrak{H}}, & F^*F+S^*S=I_{\mathfrak{F}}, & T^*F+G^*S=0, \\ T^*T+F^*F=I_{\mathfrak{H}}, & GG^*+SS^*=I_{\mathfrak{G}}, & TG^*+FS^*=0. \end{array}$$

If one takes $F=D_{T^*},\ G=D_T,\ S=-T^*,\ \mathfrak{F}=\mathfrak{D}_{T^*},\ \mathfrak{G}=\mathfrak{D}_T$ then $\boldsymbol{U}$ also provides a unitary colligation.

The connecting operator $\boldsymbol{U}$ constructed in (6.4) can be introduced with a slightly different form:

$$\boldsymbol{U}=\begin{pmatrix} S & G \\ F & T \end{pmatrix}:\mathfrak{F}\oplus\mathfrak{H}\rightarrow\mathfrak{G}\oplus\mathfrak{H}. \tag{6.5}$$

In this case one can infer that the following block form provides a unitary colligation:

$$\boldsymbol{U}=\begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix}:\mathfrak{D}_{T^*}\oplus H\rightarrow\mathfrak{D}_T\oplus H.$$

Let us consider the subspaces $\mathfrak{H}^{(c)}$ and $\mathfrak{H}^{(o)}$, called controllable and the observable subspaces, respectively, of $\mathfrak{H}$ as follows [10]:

$$\begin{array}{l} \mathfrak{H}^{(c)}=\overline{span}\left\{T^nF\mathfrak{F},n=0,1,...\right\}, \\ \mathfrak{H}^{(o)}=\overline{span}\left\{T^{*n}G^*\mathfrak{G},n=0,1,...\right\}. \end{array}$$

In the case that $\overline{\mathfrak{H}^{(c)}+\mathfrak{H}^{(o)}}=\mathfrak{H}$, where

$$\left(\mathfrak{H}^{(c)}\right)^{\perp}:=\mathfrak{H}\ominus\mathfrak{H}^{(c)},\quad\left(\mathfrak{H}^{(o)}\right)^{\perp}:=\mathfrak{H}\ominus\mathfrak{H}^{(o)}.$$

then the unitary colligation is called prime. A unitary colligation $\boldsymbol{\Delta}=(\mathfrak{F},\mathfrak{G},\mathfrak{H};S,G,F,T)$ associated with (6.5) is prime if and only if T is a c.n.u. contraction. The characteristic function $\Theta_{\boldsymbol{\Delta}}(\zeta)$ is defined by

$$\Theta_{\boldsymbol{\Delta}}(\zeta)=S+\zeta G(I_{\mathfrak{H}}-\zeta T)^{-1}F,\ \zeta\in\boldsymbol{D}.$$

The following theorem describes all unitary colligations with basic operator T.

Theorem 6.2.1. ([10], p. 163) *Let T be a contraction with $\mathfrak{d}_T,\mathfrak{d}_{T^*}<\infty$ acting on Hilbert space H. Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are two Hilbert spaces such that* $\dim\mathfrak{N}=\mathfrak{d}_T$ and $\dim\mathfrak{M}=\mathfrak{d}_{T^*}$. *Then all unitary colligations with the basic operator T and left-outer and right-outer subspaces $\mathfrak{M}$ and $\mathfrak{N}$ take the form $\boldsymbol{\Delta}=(\mathfrak{M},\mathfrak{N},H;-KT^*M,KD_T,D_{T^*}M,T)$ such that*

$$\begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \mathfrak{M} \oplus H \to \mathfrak{N} \oplus H,$$

where $K : \mathfrak{D}_T \to \mathfrak{N}$ *and* $M : \mathfrak{M} \to \mathfrak{D}_{T^*}$ *are unitary operators. The characteristic function of* $\boldsymbol{\Delta}$ *is*

$$\Theta_{\boldsymbol{\Delta}}(\zeta) = K\Theta_{T^*}(\zeta)M, \ \zeta \in \boldsymbol{D}.$$

Now consider the unitary colligation $\boldsymbol{\Delta}_0 = (\mathfrak{D}_{\mathcal{C}_h^*}, \mathfrak{D}_{\mathcal{C}_h}, H; -\mathcal{C}_h^*, , D_{\mathcal{C}_h}, D_{\mathcal{C}_h^*}, \mathcal{C}_h)$ with the characteristic function

$$\Theta_{\boldsymbol{\Delta}_0}(\zeta) = \left[-\mathcal{C}_h^* + \zeta D_{\mathcal{C}_h}(\mathbf{1} - \zeta\mathcal{C}_h)^{-1} D_{\mathcal{C}_h^*}\right] \mid \mathfrak{D}_{\mathcal{C}_h^*}.$$

Note that $\Theta_{\boldsymbol{\Delta}_0}(\zeta)$ is also the characteristic function of $\mathcal{C}_h^*$. Therefore one gets

$$\Theta_{\boldsymbol{\Delta}_0}(\mu) = \frac{\Delta_{\overline{h}}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_h\left(-i\frac{1+\mu}{1-\mu}\right)}, \ \mu = \frac{\lambda - i}{\lambda + i}, \ \mathrm{Im}\lambda > 0.$$

Since the defect indices of the contraction $\mathcal{C}_h$ are equal to one, the following isometric mappings $K : \mathfrak{D}_{\mathcal{C}_h} \to \mathbb{C}$ and $M : \mathbb{C} \to \mathfrak{D}_{\mathcal{C}_h^*}$ can be considered. Let $H^{(c)}$ and $H^{(o)}$ be the controllable and observable subspaces in H as follows:

$$\begin{array}{l} H^{(c)} = \overline{span}\left\{\mathcal{T}^n D_{\mathcal{T}^*} M\mathbb{C}, n = 0, 1, ...\right\}, \\ H^{(o)} = \overline{span}\left\{\mathcal{T}^{*n}\left(KD_{\mathcal{T}}\right)^* \mathbb{C}, n = 0, 1, ...\right\}. \end{array}$$

Let $\left(H^{(c)}\right)^{\perp} = H \ominus H^{(c)}$, $\left(H^{(o)}\right)^{\perp} = H \ominus H^{(o)}$. Then using the results of [10] we give the following.

Theorem 6.2.2. $\mathcal{C}_h = (\mathcal{L}_h - i\mathbf{1})(\mathcal{L}_h + i\mathbf{1})^{-1}$ *can be included in the unitary colligation* $\boldsymbol{\Delta}_0 = (\mathbb{C}, \mathbb{C}, H; -K\mathcal{C}_h^*M, KD_{\mathcal{C}_h}, D_{\mathcal{C}_h^*}M, \mathcal{C}_h$ *as*

$$\boldsymbol{U}_0 = \begin{pmatrix} -K\mathcal{C}_h^*M & KD_{\mathcal{C}_h} \\ D_{\mathcal{C}_h^*}M & \mathcal{C}_h \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H. \tag{6.6}$$

Let $\overrightarrow{1} = \binom{1}{\widehat{0}} \in \mathbb{C} \oplus H$, where $\widehat{0} = \binom{\mathbf{0}}{0} \in L_w^2(I) \oplus \mathbb{C}$. *Then* $\left(H^{(c)}\right)^{\perp} = (\mathbb{C} \oplus H) \ominus \overline{span}\left\{\boldsymbol{U}_0^n \overrightarrow{1}, n = 0, 1, ...\right\}$, $\left(H^{(o)}\right)^{\perp} = (\mathbb{C} \oplus H) \ominus \overline{span}\left\{\boldsymbol{U}_0^{*n} \overrightarrow{1}, n = 0, 1, ...\right\}$ *and*

- (i) $\boldsymbol{\Delta}_0$ *is prime,*
- (ii) $\overrightarrow{1}$ *is the cyclic vector for* $\boldsymbol{U}_0 : \overline{span}\left\{\boldsymbol{U}_0^n \overrightarrow{1}, \ n \in \mathbb{Z}\right\} = \mathbb{C} \oplus H$.

All other unitary colligations with basic operator $\mathcal{C}_h$ *and left- and right-outer spaces* $\mathbb{C}$ *are of the form* $\widetilde{\boldsymbol{\Delta}}_0 = (\mathbb{C}, \mathbb{C}, H; -d_1 d_2 \mathcal{C}_h^*, d_1 D_{\mathcal{C}_h}, d_2 D_{\mathcal{C}_h^*}, \mathcal{C}_h)$ *with*

$$\widetilde{\boldsymbol{U}}_0 = \begin{pmatrix} -d_1 d_2 K\mathcal{C}_h^*M & d_1 KD_{\mathcal{C}_h} \\ d_2 D_{\mathcal{C}_h^*}M & \mathcal{C}_h \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

where $|d_1| = |d_2| = 1$.

In the unit disc $\boldsymbol{D}$, if a holomorphic function F has the properties $\mathrm{Re}F > 0$ and $F(0) = 1$, then F is called the Carathéodory function. For example,

$$(F(\boldsymbol{U})e, e) = \int_{\boldsymbol{C}} F(\zeta) d\boldsymbol{m}(\zeta)$$

is a Carathéodory function, where $\boldsymbol{U}$ is a unitary operator with a cyclic vector acting on a Hilbert space and $\boldsymbol{m}$ is a nontrivial probability measure on the unit circle $\boldsymbol{C}$ (that is, not supported on a finite set) [10].

Since $\mathfrak{d}_{\mathcal{T}} = \mathfrak{d}_{\mathcal{T}^*} = 1$, we have the following theorem.

Theorem 6.2.3. *Let*

$$\boldsymbol{U}_0 = \begin{pmatrix} -K\mathcal{C}_h^* M & KD_{\mathcal{C}_h} \\ D_{\mathcal{C}_h^*} M & \mathcal{C}_h \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

be the prime unitary colligation with the characteristic function $\Theta_{\boldsymbol{\Delta}_0}$. Let

$$F(\mu) = \left((\boldsymbol{U}_0 + \mu \boldsymbol{I}) (\boldsymbol{U}_0 - \mu \boldsymbol{I})^{-1} \vec{1}, \vec{1} \right)_{\mathbb{C} \oplus H}, \ \mu \in \boldsymbol{D},$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{1} \end{bmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

is the operator in $\mathbb{C} \oplus H$ such that 1 is the scalar in $\mathbb{C}$ and $\mathbf{1}$ is the identity operator in H. Then

$$F(\zeta) = \frac{1 + \mu \overline{\Theta_{\boldsymbol{\Delta}_0}(\mu)}}{1 - \mu \overline{\Theta_{\boldsymbol{\Delta}_0}(\mu)}}, \ \mu \in \boldsymbol{D}.$$

6.3. Jacobi matrix representation

In the spectral theory of self-adjoint operators acting on Hilbert spaces, the theory of orthogonal polynomials on the real line is an important tool. Similarly, in the study of unitary operators the theory of orthogonal polynomials on the unit circle appears in the same fashion. Cantero et al. introduced a five-diagonal matrix representation of a unitary operator called a CMV matrix with a single spectrum. Now we shall introduce this matrix representation and associated results. Note that one can find several papers including CMV matrix representation [15, 16, 18, 33, 34].

Given a probability measure $\boldsymbol{m}$ on $\boldsymbol{C}$, define the Carathéodory function by

$$F(z) = F(z, \boldsymbol{m}) := \int_{\boldsymbol{C}} \frac{\zeta + z}{\zeta - z} d\boldsymbol{m}(\zeta) = 1 + 2 \sum_{n=1}^{\infty} \beta_n z^n, \ \beta_n = \int_{\boldsymbol{C}} \zeta^{-n} d\boldsymbol{m}$$

the moments of $\boldsymbol{m}$. The function F is an analytic function in the disc $\boldsymbol{D}$. Moreover, F has the properties: $\mathrm{Re}F > 0$, $F(0) = 1$. Then one can define the following Schur function:

$$f(z) = f(z, \boldsymbol{m}) := \frac{1}{z} \frac{F(z) - 1}{F(z) + 1}, \ F(z) = \frac{1 + z f(z)}{1 - z f(z)}.$$

Schur function f becomes an analytic function in the disc $\boldsymbol{D}$ with $\sup_{\boldsymbol{D}} |f(z)| \leq 1$ [10]. It is well known that there is a connection between probability measures, Carathéodory function, and Schur function. Under this correspondence $\boldsymbol{m}$ is trivial if and only if the associated Schur function is a finite Blaschke product. Let $f = f_0$ be a Schur function and not a finite Blaschke product. Then we let

$$f_{n+1}(z) = \frac{f_n(z) - \boldsymbol{\gamma}_n}{z\left(1 - \overline{\boldsymbol{\gamma}_n} f_n(z)\right)},\ \boldsymbol{\gamma}_n = f_n(0).$$

$\{f_n\}$ be an infinite sequence of Schur functions and neither of its terms is a finite Blaschke product. The numbers $\{\boldsymbol{\gamma}_n\}$ are called the Schur parameters

$$\mathcal{S}f = \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, ...\}.$$

If a Schur function f is not a finite Blaschke product, the connection between the nontangental limit values $f(\zeta)$ and its Schur parameters $\{\boldsymbol{\gamma}_n\}$ is given by

$$\prod_{n=0}^{\infty}\left(1 - |\boldsymbol{\gamma}_n|^2\right) = \exp\left\{\int_{\boldsymbol{C}} \ln\left(1 - |f(\zeta)|^2\right) d\boldsymbol{m}\right\}.$$

Therefore the equation holds $\sum_{n=0}^{\infty} |\boldsymbol{\gamma}_n|^2 = \infty$ if and only if $\ln\left(1 - |f(\zeta)|^2\right) \notin L^1(\boldsymbol{C})$

Then we have the following.

Theorem 6.3.1. *There exists a probability measure $\boldsymbol{m}$ on $\boldsymbol{C}$ such that $\mathcal{C}_h = (\mathcal{L}_h - i\mathbf{1})(\mathcal{L}_h + i\mathbf{1})^{-1}$ is unitary equivalent to the following operator:*

$$\mathbb{L}h(\mu) = P_{\mathfrak{K}}\left(\mu h(\mu)\right),\ h \in \mathfrak{K} := L^2(\boldsymbol{C}, d\boldsymbol{m}) \ominus \mathbb{C},$$

where $P_{\mathfrak{K}}$ is the orthogonal projection in $L^2(\boldsymbol{C}, d\boldsymbol{m})$ onto $\mathfrak{K}$. The Schur function associated with $\boldsymbol{m}$ is the characteristic function $\Theta_{\mathcal{C}_h}(\mu)$ of $\mathcal{L}_h$:

$$f(\mu) = \Theta_{\mathcal{C}_h}(\mu) = \frac{\Delta_h(\lambda)}{\Delta_{\overline{h}}(\lambda)},\ \mu = \frac{\lambda - i}{\lambda + i},\ \mathrm{Im}\lambda > 0.$$

Let $\boldsymbol{m}$ be a nontrivial measure on the unit circle $\boldsymbol{C}$. Then the monic orthogonal polynomials $\Phi_n(z, \boldsymbol{m})$ are uniquely determined by

$$\Phi_n(z) = \prod_{j=1}^{n}(z - z_{n,j}),\ \int_{\boldsymbol{C}} \zeta^{-j}\Phi_n(\zeta) d\boldsymbol{m} = 0,\ j = 0, 1, ..., n-1. \tag{6.7}$$

Consequently one has $(\Phi_n, \Phi_m) = 0,\ n \neq m$ on the Hilbert space $L^2(\boldsymbol{C}, d\boldsymbol{m})$. The functions

$$\phi_n = \frac{\Phi_n}{\|\Phi_n\|}$$

define orthonormal polynomials.

It is known that the space of polynomials of degree at most n has dimension $n+1$. Then this fact together with (6.7) implies the following:

$$\deg(P) \leq n,\ P \perp \zeta^j,\ j = 0, 1, ..., n-1 \Rightarrow P = c\Phi_n^*.$$

This shows that $\Phi_{n+1}(z) - z\Phi_n(z)$ is of degree n and orthogonal to $z^j,\ j = 1, 2, ..., n$. Moreover,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n(\boldsymbol{m})\Phi_n^*(z). \tag{6.8}$$

Here the complex numbers $\alpha_n(\boldsymbol{m})$ are called Verblunsky coefficients and the equation (6.8) is known as the Szegö recurrence. Substituting the value $z = 0$ into (6.8), we get

$$\alpha_n(\boldsymbol{m}) = \alpha_n = -\overline{\Phi_{n+1}(0)}.$$

The inverse Szegö recurrence is

$$z\Phi_n(z) = \rho_n^{-2}\left(\Phi_{n+1}(z) + \overline{\alpha}_n\Phi_n^*(z)\right),$$

where

$$\rho_j := \sqrt{1 - |\alpha_j|^2},\ 0 < \rho_j \leq 1,\ \ |\alpha_j|^2 + \rho_j^2 = 1. \tag{6.9}$$

Consequently, the norm $\|\Phi_n\|$ in $L^2(\boldsymbol{C}, d\boldsymbol{m})$ may be determined as

$$\|\Phi_n\| = \prod_{j=0}^{n-1}\rho_j,\ n = 1, 2, ...$$

The CMV basis $\{\chi_n\}$ is obtained by orthonormalizing the sequence $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, ...,$ and the matrix

$$\mathcal{C} = \mathcal{C}(\boldsymbol{m}) = \|c_{n,m}\|_{n,m=0}^{\infty} = \|(\zeta\chi_m, \chi_n)\|,\ m, n \in \mathbb{Z}_+$$

is five-diagonal. The elements of $\{\chi_n\}$ may be expressed as follows:

$$\chi_{2n}(z) = z^{-n}\phi_{2n}^*(z),\ \chi_{2n+1}(z) = z^{-n}\phi_{2n+1}^*(z),\ n \in \mathbb{Z}_+.$$

Therefore one can find the matrix elements in terms of α's and ρ's as

$$\mathbb{C}\left(\{\alpha_n\}\right) = \begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \rho_0 & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Here α's are the Verblunsky coefficients and ρ's are as given in (6.9). $\mathbb{C}\left(\{\alpha_n\}\right)$ is the matrix representation of the unitary operator of multiplication by ζ in $L^2(\boldsymbol{C}, d\boldsymbol{m})$.

Finally we get the following matrix, which is obtained from $\mathbb{C}\left(\{\alpha_n\}\right)$ by deleting the first row and the first column:

$$\mathbb{C}_h\left(\{\alpha_n\}\right) = \begin{pmatrix} -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Because the defect indices are equal to one, one may introduce the following theorem [10].

Theorem 6.3.2. *$\mathcal{C}_h = (\mathcal{L}_h - i\mathbf{1})(\mathcal{L}_h + i\mathbf{1})^{-1}$ is unitary equivalent to the operator acting on H determined by the truncated CMV matrix $\mathbb{C}_h\left(\{\alpha_n\}\right)$, where $\{\alpha_n\}$ are the Schur parameters of the characteristic function $\Theta_{\mathcal{C}_h}$ of $\mathcal{C}_h$.*

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