

Minimizing graph of the connected graphs whose complements are bicyclic with two cycles

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Abstract: In a certain class of graphs, a graph is called minimizing if the least eigenvalue of its adjacency matrix attains the minimum. A connected graph containing two or three cycles is called a bicyclic graph if its number of edges is equal to its number of vertices plus one. In this paper, we characterize the minimizing graph among all the connected graphs that belong to a class of graphs whose complements are bicyclic with two cycles.

Key words: Adjacency matrix, least eigenvalue, bicyclic graphs

1. Introduction

Let G be a finite, simple, and undirected graph with the vertex-set $V(G) = \{v_i : 1 \leq i \leq n\}$ and the edge-set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$ are the order and size of the graph G , respectively. The adjacency matrix $\mathbf{A}(G) = [a_{i,j}]$ of the graph G is a matrix of order n , where $a_{i,j} = 1$ if v_i is adjacent to v_j and $a_{i,j} = 0$ otherwise. The zeros of $\det(\mathbf{A}(G) - \lambda \mathbf{I})$ are called the eigenvalues of $\mathbf{A}(G)$, where \mathbf{I} is an identity matrix of order n . Since $\mathbf{A}(G)$ is real and symmetric, all the eigenvalues say $\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)$ are real and called the eigenvalues of the graph G . If $\lambda_1(G)$ is the least, then one can arrange the eigenvalues as $\lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$ and the eigenvector corresponding to the least eigenvalue is called the first eigenvector. For further study, we refer to [3, 4].

In a certain class of graphs, a graph is called minimizing if the least eigenvalue of its adjacency matrix attains the minimum. Let $\mathcal{G}(m, n)$ denote the class of connected graphs of order n and size m , where $0 < m < \binom{n}{2}$. Bell et al. [2] characterized the minimizing graphs in $\mathcal{G}(m, n)$ as follows.

Theorem 1.1 *Let G be a minimizing graph in $\mathcal{G}(m, n)$. Then G is either*

(i) *a bipartite graph, or*

(ii) *a join of two nested split graphs (not both totally disconnected).*

It is observed that the complements of the minimizing graphs in $\mathcal{G}(m, n)$ are either disconnected or contain a clique of order greater than or equal to half of the order of the graphs. This motivated to discuss the least eigenvalue of the graphs whose complements are connected and contain cliques of small sizes. Fan et al.

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[6] characterized the unique minimizing graph in the class of graphs of order n whose complements are trees. Recently, Wang et al. [11] characterized the unique minimizing graph in the class of graphs whose complements are unicyclic. In this note, we characterize the minimizing graph among all the connected graphs that belong to a class of graphs whose complements are bicyclic with exactly two cycles.

In the literature, the least eigenvalue received less attention compared to the maximum eigenvalue. The results related to the bounds of the least eigenvalue can be found in [5, 7]. For further study, we refer to [1, 8–10, 12, 13]. The rest of the paper is organized as follows: in Section 2, we give some basic definitions and terminologies that are frequently used in the main results. Section 3 deals with different results related to the minimizing graphs in the class of connected graphs whose complements are bicyclic with exactly two cycles and Section 4 includes the characterization of the minimizing graph in the same class.

2. Preliminaries

A star of size n is a tree that is obtained by joining one specific vertex to the remaining n vertices, where the fix vertex is called the center and all the other vertices are called pendent vertices. It is denoted by $\mathcal{K}_{1,n}$ and its vertex-set and edge-set are defined as $V(\mathcal{K}_{1,n}) = \{v_i : 1 \leq i \leq n + 1\}$ and $E(\mathcal{K}_{1,n}) = \{v_1v_i : 2 \leq i \leq n + 1\}$, respectively. Moreover, $\mathcal{S}_{1,n}^1$ is a graph obtained by joining any one pair of pendent vertices of $\mathcal{K}_{1,n}$. If we choose a pair of pendent vertices of $\mathcal{K}_{1,n}$ consisting of v_n and v_{n+1} , then $V(\mathcal{S}_{1,n}^1) = \{v_i : 1 \leq i \leq n + 1\}$ and $E(\mathcal{S}_{1,n}^1) = \{v_1v_i : 2 \leq i \leq n + 1\} \cup \{v_nv_{n+1}\}$ are the vertex-set and the edge-set of the graph $\mathcal{S}_{1,n}^1$, respectively. Similarly, $\mathcal{S}_{1,n}^2$ is a graph obtained by joining any two distinct pairs of pendent vertices of $\mathcal{K}_{1,n}$ such that $V(\mathcal{S}_{1,n}^2) = V(\mathcal{S}_{1,n}^1)$ and $E(\mathcal{S}_{1,n}^2) = E(\mathcal{S}_{1,n}^1) \cup \{v_{n-2}v_{n-1}\}$, where (v_{n-2}, v_{n-1}) is chosen as the second pair of pendent vertices different from (v_n, v_{n+1}) .

Since bicyclic graphs containing two or three cycles are connected graphs in which the number of edges equals the number of vertices plus one, we conclude that $\mathcal{S}_{1,n}^2$ is a bicyclic graph with exactly two cycles and $n - 4$ pendent vertices. In particular, $\mathcal{S}_{1,4}^2$ is a bicyclic graph of order 5 with exactly two cycles. In the following definitions, we define some more graphs that are bicyclic with exactly two cycles.

Definition 2.1 Let $\mathcal{K}_{1,p}$ be a star and $\mathcal{S}_{1,4}^2$ be a bicyclic graph with exactly two cycles and five vertices. The bicyclic graph denoted by $\mathcal{B}'(p)$ is obtained by joining one pendent vertex of $\mathcal{K}_{1,p}$ with a vertex of degree 4 of the graph $\mathcal{S}_{1,4}^2$, where $p \geq 2$. The vertex-set and the edge-set of $\mathcal{B}'(p)$ are defined as $V(\mathcal{B}'(p)) = \{v_1^i : 1 \leq i \leq p - 1\} \cup \{v_i : 2 \leq i \leq 4\} \cup \{v_5^i : 1 \leq i \leq 2\} \cup \{v_6^i : 1 \leq i \leq 2\}$ and $E(\mathcal{B}'(p)) = \{v_1^i v_2 : 1 \leq i \leq p - 1\} \cup \{v_2 v_3, v_3 v_4\} \cup \{v_4 v_5^i, v_4 v_6^i : 1 \leq i \leq 2\} \cup \{v_5^1 v_5^2, v_6^1 v_6^2\}$.

Definition 2.2 Let $\mathcal{K}_{1,p}$ be a star and $\mathcal{S}_{1,q}^2$ be a bicyclic graph with exactly two cycles and $q - 4$ pendent vertices. The bicyclic graph denoted by $\mathcal{B}_1(p, q)$ is obtained by joining a pendent vertex of $\mathcal{K}_{1,p}$ with a pendent vertex of the graph $\mathcal{S}_{1,q}^2$, where $p \geq 2$ and $q \geq 5$. The vertex-set and the edge-set of $\mathcal{B}_1(p, q)$ are defined as $V(\mathcal{B}_1(p, q)) = \{v_1^i : 1 \leq i \leq p - 1\} \cup \{v_i : 2 \leq i \leq 5\} \cup \{v_6^i : 1 \leq i \leq 2\} \cup \{v_7^i : 1 \leq i \leq 2\} \cup \{v_8^i : 1 \leq i \leq q - 5\}$ and $E(\mathcal{B}_1(p, q)) = \{v_1^i v_2 : 1 \leq i \leq p - 1\} \cup \{v_2 v_3, v_3 v_4, v_4 v_5\} \cup \{v_5 v_6^i, v_5 v_7^i : 1 \leq i \leq 2\} \cup \{v_5 v_8^i : 1 \leq i \leq q - 5\} \cup \{v_6^1 v_6^2, v_7^1 v_7^2\}$.

Definition 2.3 Let $\mathcal{S}_{1,p}^1$ be a unicyclic graph with $p - 2$ pendent vertices. The bicyclic graph denoted by $\mathcal{B}^*(p)$

is obtained by joining a pendent vertex of $\mathcal{S}_{1,p}^1$ with a vertex of \mathcal{C}_3 , where \mathcal{C}_3 is a cycle of order 3 and $p \geq 3$. The vertex-set and the edge-set of $\mathcal{B}^*(p)$ are defined as $V(\mathcal{B}^*(p)) = \{v_1^i : 1 \leq i \leq p-3\} \cup \{v_2^i : 1 \leq i \leq 2\} \cup \{v_i : 3 \leq i \leq 5\} \cup \{v_6^i : 1 \leq i \leq 2\}$ and $E(\mathcal{B}^*(p)) = \{v_1^i v_3 : 1 \leq i \leq p-3\} \cup \{v_2^i v_3 : 1 \leq i \leq 2\} \cup \{v_3 v_4, v_4 v_5\} \cup \{v_5 v_6^i : 1 \leq i \leq 2\} \cup \{v_2^1 v_2^2, v_6^1 v_6^2\}$.

Definition 2.4 Let $\mathcal{S}_{1,p}^1$ and $\mathcal{S}_{1,q}^1$ be two unicyclic graphs with $p-2$ and $q-2$ pendent vertices, respectively. The bicyclic graph denoted by $\mathcal{B}_2(p,q)$ is obtained by joining a pendent vertex of $\mathcal{S}_{1,p}^1$ with a pendent vertex of the graph $\mathcal{S}_{1,q}^1$, where $p, q \geq 3$. The vertex-set and the edge-set of $\mathcal{B}_2(p,q)$ are defined as $V(\mathcal{B}_2(p,q)) = \{v_1^i : 1 \leq i \leq p-3\} \cup \{v_2^i : 1 \leq i \leq 2\} \cup \{v_i : 3 \leq i \leq 6\} \cup \{v_7^i : 1 \leq i \leq 2\} \cup \{v_8^i : 1 \leq i \leq q-3\}$ and $E(\mathcal{B}_2(p,q)) = \{v_1^i v_3 : 1 \leq i \leq p-3\} \cup \{v_2^i v_3 : 1 \leq i \leq 2\} \cup \{v_3 v_4, v_4 v_5, v_5 v_6\} \cup \{v_6 v_7^i : 1 \leq i \leq 2\} \cup \{v_6 v_8^i : 1 \leq i \leq q-3\} \cup \{v_2^1 v_2^2, v_7^1 v_7^2\}$.

Let \mathcal{G}_n be a class of bicyclic graphs with order n and exactly two cycles. Let \mathcal{G}_n^c be a class of connected graphs of order n whose complements are bicyclic with exactly two cycles i.e. $\mathcal{G}_n^c = \{G^c : G^c \text{ is connected and } G \in \mathcal{G}_n\}$. Note that $(\mathcal{S}_{1,n-1}^2)^c$ does not belong to \mathcal{G}_n^c as it is disconnected, where $n > 4$.

By interlacing theorem, for a graph G containing at least one edge, we have $\lambda_{\min}(G) \leq -1$. In particular, if G is a complete graph or disjoint union of complete graphs with at least one non-trivial copy, then $\lambda_{\min}(G) = -1$. Moreover, G contains $\mathcal{K}_{1,2}$ as an induced subgraph that verifies that $\lambda_{\min}(G) \leq \lambda_{\min}(\mathcal{K}_{1,2}) = -\sqrt{2}$. Thus, for a graph G (tree), $\lambda_{\min}(G^c) = -1$ if and only if G is a star. Consequently, if G being a tree is not a star then G^c is connected and $\lambda_{\min}(G^c) < -1$. For a unicyclic graph G , $\lambda_{\min}(G^c) \leq -1$, where equality holds if $G \cong \mathcal{C}_4$ (as $(\mathcal{C}_4)^c$ is $2\mathcal{P}_2$, where \mathcal{P}_2 is a path of order 2). Similarly, for a bicyclic graph G with exactly two cycles, $\lambda_{\min}(G^c) \leq -2$, where equality holds if $G \cong \mathcal{S}_{1,n}^2$ for $n > 3$.

A vector $\mathbf{X} \in \mathcal{R}^n$ is said to be defined on the graph G of order n , if there is a 1-1 map ϕ from $V(G)$ to the entries of \mathbf{X} such that $\phi(u) = X_u$ for each $u \in V(G)$. If \mathbf{X} is an eigenvector of $\mathbf{A}(G)$, then it is naturally defined on $V(G)$, i.e. X_u is the entry of \mathbf{X} corresponding to the vertex u . Thus, it is easy to see that

$$\mathbf{X}^T \mathbf{A}(G) \mathbf{X} = 2 \sum_{uv \in E(G)} X_u X_v, \tag{2.1}$$

and λ is an eigenvalue of G corresponding to the eigenvector \mathbf{X} if and only if $\mathbf{X} \neq 0$. For each $v \in V(G)$, we obtain the following eigen-equation of the graph G :

$$\lambda X_v = \sum_{u \in N_G(v)} X_u, \tag{2.2}$$

where $N_G(v)$ is the set of neighbors of v in G . For an arbitrary unit vector $\mathbf{X} \in \mathcal{R}^n$,

$$\lambda_{\min}(G) \leq \mathbf{X}^T \mathbf{A}(G) \mathbf{X}, \tag{2.3}$$

and equality holds if and only if \mathbf{X} is the first eigenvector of G .

Moreover, for a graph G , $\mathbf{A}(G^c) = \mathbf{J} - \mathbf{I} - \mathbf{A}(G)$, where \mathbf{J} is the all-ones matrix, \mathbf{I} is the identity matrix of same size as of the adjacency matrix $\mathbf{A}(G)$, and G^c is complement of the graph G .

Thus, for any vector $\mathbf{X} \in \mathcal{R}^n$

$$\mathbf{X}^T \mathbf{A}(G^c) \mathbf{X} = \mathbf{X}^T (\mathbf{J} - \mathbf{I}) \mathbf{X} - \mathbf{X}^T \mathbf{A}(G) \mathbf{X}. \tag{2.4}$$

Now we state the following result, which is used in the proof of the main theorem of this paper.

Lemma 2.5 [6, 11] *Let \mathcal{T} be a tree and \mathcal{U} be a unicyclic graph with nonnegative or nonpositive real vectors $\mathbf{X} = (X_1, X_2, X_3, \dots, X_n)^T$ and $\mathbf{Y} = (Y_1, Y_2, Y_3, \dots, Y_n)^T$ defined on \mathcal{T} and \mathcal{U} , respectively. The entries of \mathbf{X} and \mathbf{Y} are ordered as $|X_1| \geq |X_2| \geq |X_3| \geq \dots \geq |X_n|$ and $|Y_1| \geq |Y_2| \geq |Y_3| \geq \dots \geq |Y_n|$, where $|V(\mathcal{T})| = |V(\mathcal{U})| = n$. Then*

$$\sum_{uv \in E(\mathcal{T})} X_u X_v \leq \sum_{uv \in E(\mathcal{K}_{1,p})} X_u X_v,$$

where \mathbf{X} is defined on the star $\mathcal{K}_{1,p}$ such that its central vertex of degree $p = n - 1$ has value X_1 , and equality holds if and only if $\mathcal{T} = \mathcal{K}_{1,n-1}$, and

$$\sum_{uv \in E(\mathcal{U})} Y_u Y_v \leq \sum_{uv \in \mathcal{S}_{1,q}^1} Y_u Y_v,$$

where \mathbf{Y} is defined on the unicyclic graph $\mathcal{S}_{1,q}^1$ such that the vertex of degree $q = n - 1$ has value Y_1 and two vertices of degree two have values Y_2 and Y_3 , and equality holds if and only if $\mathcal{U} = \mathcal{S}_{1,q}^1$.

3. Minimizing graphs

In this section, we find some minimizing graphs among the connected graphs whose complements are bicyclic with exactly two cycles under certain conditions.

Let \mathbf{X}' be the first eigenvector of the graph $\mathcal{B}'(p)^c$ with entries corresponding to the vertices as defined in Definition 2.1. By eigen-equation (2.2), the vertices v_1^i for $1 \leq i \leq p - 1$, v_2, v_3, v_4, v_5^i for $1 \leq i \leq 2$ and v_6^i for $1 \leq i \leq 2$ have values in \mathbf{X}' , say X_1, X_2, X_3, X_4, X_5 , and X_6 , respectively. Moreover, if $\lambda_{\min}(\mathcal{B}'(p)^c) = \lambda'$, then we have

$$\begin{cases} \lambda' X_1 = (p - 2)X_1 + X_3 + X_4 + 2X_5 + 2X_6, \\ \lambda' X_2 = X_4 + 2X_5 + 2X_6, \\ \lambda' X_3 = (p - 1)X_1 + 2X_5 + 2X_6, \\ \lambda' X_4 = (p - 1)X_1 + X_2, \\ \lambda' X_5 = (p - 1)X_1 + X_2 + X_3 + 2X_6, \\ \lambda' X_6 = (p - 1)X_1 + X_2 + X_3 + 2X_5. \end{cases} \tag{3.1}$$

Take $\mathbf{X}' = (X_1, X_2, X_3, X_4, X_5, X_6)^T$; then the matrix equation of the above system of equations is $(\mathbf{A} - \lambda' \mathbf{I}) \mathbf{X}' = 0$, where \mathbf{A} is a matrix of order 6. Thus, λ' is the least root of the polynomial

$$\begin{aligned} f'(\lambda, p) &= \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= (20 - 12p) + (8 + 8p)\lambda + (-25 + 17p)\lambda^2 + (-26 - 3p)\lambda^3 \\ &\quad + (-7 - 6p)\lambda^4 - (2 - p)\lambda^5 + \lambda^6. \end{aligned} \tag{3.2}$$

Let \mathbf{X}_1 be the first eigenvector of the graph $\mathcal{B}_1(p, q)^c$ with entries corresponding to the vertices as defined in Definition 2.2. By eigen-equation (2.2), the vertices v_1^i for $1 \leq i \leq p-1$, $v_2, v_3, v_4, v_5, v_6^i$ for $1 \leq i \leq 2$, v_7^i for $1 \leq i \leq 2$ and v_8^i for $1 \leq i \leq q-5$ have values in \mathbf{X}_1 , say $X_1, X_2, X_3, X_4, X_5, X_6, X_7$, and X_8 , respectively. Moreover, if $\lambda_{\min}(\mathcal{B}_1(p, q)^c) = \lambda_1$, then we have

$$\left\{ \begin{array}{l} \lambda_1 X_1 = (p-2)X_1 + X_3 + X_4 + X_5 + 2X_6 + 2X_7 + (q-5)X_8, \\ \lambda_1 X_2 = X_4 + X_5 + 2X_6 + 2X_7 + (q-5)X_8, \\ \lambda_1 X_3 = (p-1)X_1 + X_5 + 2X_6 + 2X_7 + (q-5)X_8, \\ \lambda_1 X_4 = (p-1)X_1 + X_2 + 2X_6 + 2X_7 + (q-5)X_8, \\ \lambda_1 X_5 = (p-1)X_1 + X_2 + X_3, \\ \lambda_1 X_6 = (p-1)X_1 + X_2 + X_3 + X_4 + 2X_7 + (q-5)X_8, \\ \lambda_1 X_7 = (p-1)X_1 + X_2 + X_3 + X_4 + 2X_6 + (q-5)X_8, \\ \lambda_1 X_8 = (p-1)X_1 + X_2 + X_3 + X_4 + 2X_6 + 2X_7 + (q-6)X_8. \end{array} \right. \quad (3.3)$$

Take $\mathbf{X}_1 = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)^T$; then the matrix equation of the above system of equations is $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{X}_1 = 0$, where \mathbf{A} is a matrix of order 8. Thus, λ_1 is the least root of the polynomial

$$\begin{aligned} f_1(\lambda, p, q) &= \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= (20 - 4p - 4q) + (-32 + 28p + 12q - 8pq)\lambda \\ &\quad + (-65 + 3p + 19q + 4pq)\lambda^2 + (-4 - 36p - 8q + 18pq)\lambda^3 \\ &\quad + (65 - 39p - 31q + 11pq)\lambda^4 + (64 - 24p - 24q + 2pq)\lambda^5 \\ &\quad + (30 - 8p - 8q)\lambda^6 + (8 - p - q)\lambda^7 + \lambda^8. \end{aligned} \quad (3.4)$$

Lemma 3.1 *If $n \geq 12$, then $\lambda_{\min}(\mathcal{B}_1(n-7, 5)^c) < \lambda_{\min}(\mathcal{B}'(n-6)^c)$.*

Proof Let $\lambda' = \lambda_{\min} \mathcal{B}'(n-6)^c$ and $\lambda_1 = \lambda_{\min} \mathcal{B}_1(n-7, 5)^c$ be the least roots of $f'(\lambda, n-6)$ and $f_1(\lambda, n-7, 5)$, respectively. Define

$$f(\lambda, n-6) = (\lambda+1)^2 f'(\lambda, n-6).$$

Since $\lambda' < -2$, λ' is the least root of $f(\lambda, n-6)$. From (3.2), $f'(-3.5, n-6) = 9409 - 1128n$ and $f(-3.5, n-6) < 0$ for $n \geq 12$. Moreover, if $\lambda \rightarrow -\infty$, then $f(\lambda, n-6) \rightarrow +\infty$, which implies $\lambda' \leq -3.5$. Now, for $\lambda \leq -3.5$ and $n \geq 12$,

$$\begin{aligned} f(\lambda, n-6) - f_1(\lambda, n-7, 5) &= (-n+8)(8+4\lambda+2\lambda^2+15\lambda^3+11\lambda^4+2\lambda^5) \\ &= -(n-8)(\lambda+1)(\lambda+2)(\lambda+3.2920)(2\lambda^2-1.584\lambda+1.2145) > 0. \end{aligned}$$

Consequently, $f_1(\lambda, n-7, 5) < f(\lambda, n-6)$ for $\lambda \leq -3.5$ and $n \geq 12$. In particular, $\lambda_1 < \lambda'$, which implies $\lambda_{\min}(\mathcal{B}_1(n-7, 5)^c) < \lambda_{\min}(\mathcal{B}'(n-6)^c)$ for $n \geq 12$. \square

Lemma 3.2 *Let p and q be any positive integers such that $p \geq q \geq 6$ and $p+q+2 = n \geq 14$; then*

$$\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_1(p, q)^c),$$

equality holds if and only if $p = \lceil \frac{n-2}{2} \rceil$ and $q = \lfloor \frac{n-2}{2} \rfloor$.

Proof From equation (3.4), we have

$$f_1(-3.5, p, q) = \frac{1}{256}[-10874441 - 10530p + q(53928 - 24192p)]$$

and $f_1(-3.5, p, q) < 0$ for $p \geq q \geq 6$, which implies $\lambda_1 < -3.5$, where λ_1 is the least root of $f_1(\lambda, p, q)$. Moreover,

$$\begin{aligned} f_1(\lambda, p-1, q+1) &= (20 - 4p - 4q) + (-24 + 20p + 20q - 8pq)\lambda \\ &\quad + (-69 + 7p + 15q + 4pq)\lambda^2 + (-22 - 18p - 26q + 18pq)\lambda^3 \\ &\quad + (54 - 28p - 42q + 11pq)\lambda^4 + (62 - 22p - 26q + 2pq)\lambda^5 \\ &\quad + (30 - 8p - 8q)\lambda^6 + (8 - p - q)\lambda^7 + \lambda^8, \end{aligned}$$

and

$$\begin{aligned} f_1(\lambda, p, q) - f_1(\lambda, p-1, q+1) &= -(p-q-1)\lambda(-8 + 4\lambda + 18\lambda^2 + 11\lambda^3 + 2\lambda^4) \\ &= -2(p-q-1)\lambda(\lambda - \frac{1}{2})(\lambda + 2)^3. \end{aligned}$$

We note that if $p > q + 1$ and $\lambda < -3.5$, then $f_1(\lambda, p, q) - f_1(\lambda, p-1, q+1) > 0$. Furthermore, $f(-3.5, p-1, q+1) < 0$. Consequently, $\lambda_{\min}(\mathcal{B}_1(p-1, q+1)^c) < \lambda_{\min}(\mathcal{B}_1(p, q)^c)$. It follows that $\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_1(p, q)^c)$ with equality if and only if $p = \lceil \frac{n-2}{2} \rceil$ and $q = \lfloor \frac{n-2}{2} \rfloor$, where $n \geq 14$.

Let \mathbf{X}^* be the first eigenvector of the graph $\mathcal{B}^*(p)^c$ with entries corresponding to the vertices as defined in Definition 2.3. By eigen-equation (2.2), the vertices v_1^i for $1 \leq i \leq p-3$, v_2^i for $1 \leq i \leq 2$, v_3 , v_4 , v_5 , and v_6^i for $1 \leq i \leq 2$ have values in \mathbf{X}^* , say X_1 , X_2 , X_3 , X_4 , X_5 , and X_6 , respectively. Moreover, if $\lambda_{\min}(\mathcal{B}^*(p)^c) = \lambda^*$, then we have

$$\begin{cases} \lambda^* X_1 = (p-4)X_1 + 2X_2 + X_4 + X_5 + 2X_6, \\ \lambda^* X_2 = (p-3)X_1 + X_4 + X_5 + 2X_6, \\ \lambda^* X_3 = X_5 + 2X_6, \\ \lambda^* X_4 = (p-3)X_1 + 2X_2 + 2X_6, \\ \lambda^* X_5 = (p-3)X_1 + 2X_2 + X_3, \\ \lambda^* X_6 = (p-3)X_1 + 2X_2 + X_3 + X_4. \end{cases} \quad (3.5)$$

Take $\mathbf{X}^* = (X_1, X_2, X_3, X_4, X_5, X_6)^T$; then the matrix equation of the above system of equations is $(\mathbf{A} - \lambda^* \mathbf{I})\mathbf{X}^* = 0$, where \mathbf{A} is a matrix of order 6. Thus, λ^* is the least root of the polynomial

$$f^*(\lambda, p, q) = \det(\mathbf{A} - \lambda \mathbf{I}) = (12 - 4p) + (-12 + 8p)\lambda + (-17 + 7p)\lambda^2 + (-7p)\lambda^3 + (5 - 6p)\lambda^4 + (4 - p)\lambda^5 + \lambda^6. \quad (3.6)$$

Let \mathbf{X}_2 be the first eigenvector of the graph $\mathcal{B}_2(p, q)^c$ with entries corresponding to the vertices as defined in Definition 2.4. By eigen-equation (2.2), the vertices v_1^i for $1 \leq i \leq p-3$, v_2^i for $1 \leq i \leq 2$, v_3 , v_4 , v_5 , v_6 , v_7^i for $1 \leq i \leq 2$ and v_8^i for $1 \leq i \leq q-3$ have values in \mathbf{X}_2 , say X_1 , X_2 , X_3 , X_4 , X_5 , X_6 , X_7 , and X_8 ,

respectively. Moreover, if $\lambda_{\min}(\mathcal{B}_2(p, q)^c) = \lambda_2$, then we have

$$\left\{ \begin{array}{l} \lambda_2 X_1 = (p - 4)X_1 + 2X_2 + X_4 + X_5 + X_6 + 2X_7 + (q - 3)X_8, \\ \lambda_2 X_2 = (p - 3)X_1 + X_4 + X_5 + X_6 + 2X_7 + (q - 3)X_8, \\ \lambda_2 X_3 = X_5 + X_6 + 2X_7 + (q - 3)X_8, \\ \lambda_2 X_4 = (p - 3)X_1 + 2X_2 + X_6 + 2X_7 + (q - 3)X_8, \\ \lambda_2 X_5 = (p - 3)X_1 + 2X_2 + X_3 + 2X_7 + (q - 3)X_8, \\ \lambda_2 X_6 = (p - 3)X_1 + 2X_2 + X_3 + X_4, \\ \lambda_2 X_7 = (p - 3)X_1 + 2X_2 + X_3 + X_4 + X_5 + (q - 3)X_8, \\ \lambda_2 X_8 = (p - 3)X_1 + 2X_2 + X_3 + X_4 + X_5 + 2X_7 + (q - 4)X_8. \end{array} \right. \quad (3.7)$$

Take $\mathbf{X}_2 = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)^T$; then the matrix equation of the above system of equations is $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{X}_2 = 0$, where \mathbf{A} is a matrix of order 8. Thus, λ_2 is the least root of the polynomial

$$\begin{aligned} f_2(\lambda, p, q) &= \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= (20 - 4p - 4q) + (-40 + 20p + 20q - 8pq)\lambda \\ &\quad + (-53 + 11p + 11q + 4pq)\lambda^2 + (4 - 22p - 22q + 18pq)\lambda^3 \\ &\quad + (65 - 35p - 35q + 11pq)\lambda^4 + (64 - 24p - 24q + 2pq)\lambda^5 \\ &\quad + (30 - 8p - 8q)\lambda^6 + (8 - p - q)\lambda^7 + \lambda^8. \end{aligned} \quad (3.8)$$

□

Lemma 3.3 *If $n \geq 10$, then $\lambda_{\min}(\mathcal{B}_2(n - 5, 3)^c) < \lambda_{\min}(\mathcal{B}^*(n - 4)^c)$.*

Proof Consider $\lambda^* = \lambda_{\min}(\mathcal{B}^*(n - 4)^c)$ and $\lambda_2 = \lambda_{\min}(\mathcal{B}_2(n - 5, 3)^c)$ are the least roots of $f^*(\lambda, n - 4)$ and $f_2(\lambda, n - 5, 3)$, respectively. Define

$$f(\lambda, n - 4) = (\lambda + 1)^2 f^*(\lambda, n - 4).$$

Since $\lambda^* < -2$, λ^* is the least root of $f(\lambda, n - 4)$. From (3.6), $f^*(-3, n - 4) = 133 - 19n$ and $f(-3, n - 4) < 0$ for $n \geq 10$. Moreover, if $\lambda \rightarrow -\infty$, then $f(\lambda, n - 4) \rightarrow +\infty$, which implies $\lambda^* \leq -3$. Now, for $\lambda \leq -3$ and $n \geq 10$,

$$f(\lambda, n - 4) - f_2(\lambda, n - 5, 3) = -\lambda(\lambda - 1)(\lambda + 2.8507)(\lambda - 0.3507)(\lambda + 2(1 - \frac{1}{n - 6}))$$

As $(1 - \frac{1}{n - 6}) \in [0.75, 1]$ for any integral value $n \geq 10$, $f_2(\lambda, n - 5, 3) < f(\lambda, n - 4)$ for $\lambda \leq -3$ and $n \geq 10$. In particular, $\lambda_2 < \lambda^*$, which implies $\lambda_{\min}(\mathcal{B}_2(n - 5, 3)^c) < \lambda_{\min}(\mathcal{B}^*(n - 4)^c)$ for $n \geq 10$. □

Lemma 3.4 *Let p and q be any positive integers such that $p \geq q \geq 4$ and $p + q + 2 = n \geq 10$; then*

$$\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n - 2}{2} \rceil, \lfloor \frac{n - 2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_2(p, q)^c),$$

equality holds if and only if $p = \lceil \frac{n - 2}{2} \rceil$ and $q = \lfloor \frac{n - 2}{2} \rfloor$.

Proof From equation (3.8), we have

$$f_2(-3, p, q) = 203 - 19(p + q) - 21pq$$

and $f_2(-3, p, q) < 0$ for $p \geq q \geq 4$, which implies $\lambda_2 < -3$, where λ_2 is the least root of $f_2(\lambda, p, q)$. Moreover,

$$\begin{aligned} f_2(\lambda, p - 1, q + 1) &= (20 - 4p - 4q) + (-32 + 12p + 28q - 8pq)\lambda \\ &\quad + (-57 + 15p + 7q + 4pq)\lambda^2 + (-14 - 4p - 40q + 18pq)\lambda^3 \\ &\quad + (54 - 24p - 46q + 11pq)\lambda^4 + (62 - 22p - 26q + 2pq)\lambda^5 \\ &\quad + (30 - 8p - 8q)\lambda^6 + (8 - p - q)\lambda^7 + \lambda^8, \text{ and} \end{aligned}$$

and

$$\begin{aligned} f_2(\lambda, p, q) - f_2(\lambda, p - 1, q + 1) &= -(p - q - 1)\lambda(-8 + 4\lambda + 18)\lambda^2 + 11\lambda^3 + 2\lambda^4 \\ &= -2(p - q - 1)\lambda(\lambda - \frac{1}{2})(\lambda + 2)^3. \end{aligned}$$

We note that if $p > q + 1$ and $\lambda < -3$, then $f_2(\lambda, p, q) - f_2(\lambda, p - 1, q + 1) > 0$. Furthermore, $f(-3, p - 1, q + 1) < 0$. Consequently, $\lambda_{\min}(\mathcal{B}_2(p - 1, q + 1)^c) < \lambda_{\min}(\mathcal{B}_2(p, q)^c)$. It follows that $\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_2(p, q)^c)$ with equality if and only if $p = \lceil \frac{n-2}{2} \rceil$ and $q = \lfloor \frac{n-2}{2} \rfloor$, where $n \geq 10$. \square

Lemma 3.5 (a) If $n \geq 22$ and $n \equiv 0 \pmod{2}$, then $\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) < \lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c)$. (b) If $n \geq 19$ and $n \equiv 1 \pmod{2}$, then $\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) < \lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c)$.

Proof Using (3.4) and (3.8), we have

$$f_1(\lambda, p, q) - f_2(\lambda, p, q) = [-4(p - q)]\lambda^4 + [-14(p - q) - 8]\lambda^3 + [-8(p - q) - 12]\lambda^2 + [8(p - q) + 8]\lambda.$$

(a) If $n \equiv 0 \pmod{2}$, then $p = \lceil \frac{n-2}{2} \rceil = \frac{n-2}{2} = \lfloor \frac{n-2}{2} \rfloor = q$. Thus,

$$f_1(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) - f_2(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) = -4\lambda(\lambda + 2)(\lambda - \frac{1}{2}).$$

This shows that, for $\lambda < -2$,

$$f_2(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) < f_1(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor).$$

Consequently, for $n \geq 22$,

$$\lambda_{\min} \mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c < \lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c).$$

(b) If $n \equiv 1 \pmod{2}$, then $p = \lceil \frac{n-2}{2} \rceil = \frac{n-1}{2}$ and $q = \lfloor \frac{n-2}{2} \rfloor = \frac{n-3}{2}$. Thus,

$$f_2(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) - f_1(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) = 2\lambda(\lambda + 2)(\lambda - \frac{1}{2})(\lambda + 4).$$

This shows that, for $\lambda < -4$,

$$f_1(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor) < f_2(\lambda, \lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor).t$$

Consequently, for $n \geq 19$,

$$\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) < \lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c).$$

□

4. Characterization of the minimizing graph

In this section, we characterize the minimizing graphs among all the connected graphs whose complements are bicyclic with exactly two cycles.

Lemma 4.1 *Let $\mathcal{B} \in \mathcal{G}_n$ and $\mathbf{X} = (X_1, X_2, X_3, \dots, X_n)^T$ be a nonnegative or nonpositive real vector defined on \mathcal{B} such that the entries of \mathbf{X} are ordered as $|X_1| \geq |X_2| \geq |X_3| \geq \dots \geq |X_n|$. Then*

$$\sum_{uv \in E(\mathcal{B})} X_u X_v \leq \sum_{uv \in E(\mathcal{S}_{1,n-1}^2)} X_u X_v,$$

where \mathbf{X} is defined on $\mathcal{S}_{1,n-1}^2$ such that one vertex of degree $n - 1$ has value X_1 and four vertices of degree 2 have values X_2, X_3, X_4 , and X_5 , respectively. The remaining values X_i for $5 \leq i \leq n$ are assigned to the $n - 5$ pendent vertices. The above equality holds if and only if $\mathcal{B} = \mathcal{S}_{1,n-1}^2$.

Proof Without loss of generality assume that \mathbf{X} is nonnegative; otherwise we consider $-\mathbf{X}$. Let v be a vertex of the bicyclic graph \mathcal{B} with value X_1 assigned by the first eigenvector \mathbf{X} . Suppose that there exists a vertex u that is not adjacent with v . Since \mathcal{B} is a connected graph, there exists a neighbor of u , say w , that is on the path of \mathcal{B} containing v and u . If we delete uw and add a new edge vu in \mathcal{B} , then we have a new bicyclic graph $\tilde{\mathcal{B}}$ with exactly two cycles such that

$$\sum_{uv \in E(\mathcal{B})} X_u X_v \leq \sum_{uv \in E(\tilde{\mathcal{B}})} X_u X_v.$$

This process is repeated on the bicyclic graph $\tilde{\mathcal{B}}$ for the nonneighbor of v . Thus, we obtain a bicyclic graph that is in fact a star $\mathcal{K}_{1,n-1}$ with center v and two edges $u'v'$ and $u''v''$ that are not incident to the vertex v . Thus, we have

$$\sum_{uv \in E(\mathcal{B})} X_u X_v \leq \sum_{uv \in E(\tilde{\mathcal{B}})} X_u X_v \leq \sum_{i=2}^n X_1 X_i + X_{u'} X_{v'} + X_{u''} X_{v''}. \text{ Since } X_2 X_3 + X_4 X_5 \geq X_{u'} X_{v'} +$$

$X_{u''} X_{v''}$ and $\sum_{i=2}^n X_1 X_i + X_2 X_3 + X_4 X_5 = \mathcal{S}_{1,n-1}^2$, we obtain

$$\sum_{uv \in E(\mathcal{B})} X_u X_v \leq \sum_{uv \in E(\mathcal{S}_{1,n-1}^2)} X_u X_v.$$

The equality holds if v is adjacent to all the other vertices and there are two nonincident edges to the vertex v in \mathcal{B} , which implies that $\mathcal{B} = \mathcal{S}_{1,n-1}^2$. \square

Lemma 4.2 *Let \mathcal{B}^c be a minimizing graph in \mathcal{G}_n^c and \mathbf{X} be a first eigenvector of \mathcal{B}^c , where $n \geq 10$. Then \mathbf{X} has at least two positive and two negative entries.*

Proof Suppose in contrast that only one vertex v of \mathcal{B}^c has a positive value assigned by \mathbf{X} . We claim that the degree of v (the number of adjacent vertices) in \mathcal{B}^c is nonzero, i.e. $d_{\mathcal{B}^c}(v) \neq 0$. Otherwise, if $d_{\mathcal{B}^c}(v) = 0$, then $\mathcal{B} = \mathcal{S}_{1,n-1}^2$ which is a contradiction. Consequently, $1 \leq d_{\mathcal{B}^c}(v) \leq n - 1$. Let u be another vertex in \mathcal{B}^c . We claim u is adjacent to v ; otherwise the eigen-equation (2.2) does not hold for u in \mathcal{B}^c as $\lambda X_u > 0$ and $\sum_{w \in N_G(u)} X_w < 0$. Hence, u is adjacent to v . Since u is taken as an arbitrary vertex, our claim is true for each vertex in \mathcal{B}^c . Consequently, v is adjacent with all other vertices in \mathcal{B}^c , i.e. $d_{\mathcal{B}^c}(v) = n - 1$. This shows that \mathcal{B} is disconnected, which is again a contradiction. Similarly, we can obtain a contradiction if a vertex v of \mathcal{B}^c is the only one with a negative value assigned by \mathbf{X} . Consequently, for $n \geq 10$, the first eigenvector of the minimizing graph \mathcal{B}^c in \mathcal{G}_n^c has at least two positive and two negative entries. \square

Lemma 4.3 *Let $\mathcal{B}^c \in \mathcal{G}_n^c$ be a connected graph of order n such that its complement is a bicyclic graph with exactly two cycles.*

(a) *If $n \geq 19$ and $n \equiv 1(\text{mod } 2)$, then $\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, where equality holds iff $\mathcal{B} = \mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)$.*

(b) *If $n \geq 22$ and $n \equiv 0(\text{mod } 2)$, then $\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, where equality holds iff $\mathcal{B} = \mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)$.*

Proof Let \mathbf{X} be the first eigenvector of \mathcal{B}^c with unit length. Define $V_+ = \{v : X_v \geq 0, v \in V(\mathcal{B}^c)\}$ and $V_- = \{v : X_v < 0, v \in V(\mathcal{B}^c)\}$. By Lemma 4.2, both V_+ and V_- contain at least two elements. Suppose that \mathcal{B}_+ and \mathcal{B}_- are subgraphs of \mathcal{B} induced by V_+ and V_- , respectively. Let \mathcal{E}' be the set of edges between \mathcal{B}_+ and \mathcal{B}_- in \mathcal{B} . As \mathcal{B} is connected, \mathcal{E}' is nonempty. Thus, we have

$$\sum_{uv \in E(\mathcal{B})} X_u X_v = \sum_{uv \in \mathcal{B}_+} X_u X_v + \sum_{uv \in \mathcal{B}_-} X_u X_v + \sum_{uv \in \mathcal{E}'} X_u X_v. \tag{4.1}$$

There are two possibilities for the edges of the cycles of \mathcal{B} . Either all the edges of the cycles of \mathcal{B} are in one of \mathcal{B}_+ or \mathcal{B}_- , or in both.

(a) Take $n \geq 19$, $n \equiv 1(\text{mod } 2)$, and $\mathcal{E}' = \mathcal{E}_1$. Without loss of generality, suppose that \mathcal{B}_+ does not include any edge of the cycles of \mathcal{B} ; otherwise we take $-\mathbf{X}$ as a first eigenvector. Let $\bar{\mathcal{B}}$ be a graph obtained from \mathcal{B} such that the subgraphs $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$ of $\bar{\mathcal{B}}$ are induced by the subgraphs \mathcal{B}_+ and \mathcal{B}_- of \mathcal{B} , respectively. Moreover, the subgraph $\bar{\mathcal{B}}_+$ is a tree and the subgraph $\bar{\mathcal{B}}_-$ is bicyclic with exactly two cycles. By the deletion and addition of some edges in the tree $\bar{\mathcal{B}}_+$, we have a star $\mathcal{K}_{1,p}$ with center u' that has a maximum modulus value among all the values of $\bar{\mathcal{B}}_+$ given by \mathbf{X} and $p + 1 = |V_+| \geq 6$. Similarly, in $\bar{\mathcal{B}}_-$, we have $\mathcal{S}_{1,q}^1$ with v' that is adjacent to all other vertices in $\mathcal{S}_{1,q}^1$ such that v' has maximum modulus value among all the values of

$\bar{\mathcal{B}}_-$ and $q + 1 = |V_+| \geq 6$. Thus, by Lemma 2.1 and Lemma 4.1, we have

$$\sum_{uv \in \mathcal{B}_+} X_u X_v \leq \sum_{uv \in \bar{\mathcal{B}}_+} X_u X_v \leq \sum_{uv \in \mathcal{K}_{1,p}} X_u X_v \tag{4.2}$$

and

$$\sum_{uv \in \mathcal{B}_-} X_u X_v \leq \sum_{uv \in \bar{\mathcal{B}}_-} X_u X_v \leq \sum_{uv \in \mathcal{S}_{1,q}^2} X_u X_v. \tag{4.3}$$

Let u'' and v'' be the vertices of $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$ with minimum modulus among all the vertices of $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$, respectively. Then

$$\sum_{uv \in \mathcal{E}_1} X_u X_v \leq X_{u''} X_{v''}. \tag{4.4}$$

Using 4.2, 4.3, and 4.4 in 4.1, we have

$$\sum_{uv \in \mathcal{B}} X_u X_v \leq \sum_{uv \in \mathcal{K}_{1,p}} X_u X_v + \sum_{uv \in \mathcal{S}_{1,q}^2} X_u X_v + X_{u''} X_{v''}. \tag{4.5}$$

Since $p \geq q \geq 6$, the vertices u'' and v'' can be taken from the pendent vertices of $\mathcal{K}_{1,p}$ and $\mathcal{S}_{1,q}^2$, respectively. Thus, 4.5 becomes

$$\sum_{uv \in \mathcal{B}} X_u X_v \leq \sum_{uv \in \mathcal{B}_1(p,q)} X_u X_v. \tag{4.6}$$

Now by (2.4) and (4.6), we have

$$\begin{aligned} \lambda_{\min}(\mathcal{B}^c) &= \mathbf{X}^T \mathbf{A}(\mathcal{B}^c) \mathbf{X} = \mathbf{X}^T (\mathbf{J} - \mathbf{I} - \mathbf{A}(\mathcal{B})) \mathbf{X} \\ &= \mathbf{X}^T (\mathbf{J} - \mathbf{I}) \mathbf{X} - \mathbf{X}^T \mathbf{A}(\mathcal{B}) \mathbf{X} \\ &\geq \mathbf{X}^T (\mathbf{J} - \mathbf{I}) \mathbf{X} - \mathbf{X}^T \mathbf{A}(\mathcal{B}_1(p, q)) \mathbf{X} \\ &= \mathbf{X}^T \mathbf{A}(\mathcal{B}_1(p, q)^c) \mathbf{X} \geq \lambda_{\min}(\mathcal{B}_1(p, q)^c), \end{aligned}$$

where $p \geq q \geq 6$ and $p + q + 2 = n \geq 19$. This implies that

$$\lambda_{\min}(\mathcal{B}_1(p, q)^c) \leq \lambda_{\min}(\mathcal{B}^c). \tag{4.7}$$

By Lemma 3.2, $\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_1(p, q)^c)$, where equality holds if $p = \lceil \frac{n-2}{2} \rceil$, $q = \lfloor \frac{n-2}{2} \rfloor$, $p \geq q \geq 6$, and $p + q + 2 = n \geq 14$. Consequently, for $n \geq 19$, $\lambda_{\min}(\mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, with equality if and only if $\mathcal{B} = \mathcal{B}_1(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)$.

Now, to complete the proof, we prove that the set \mathcal{E}_1 consists of exactly one edge and the set V_+ does not contain any vertex with zero value given by \mathbf{X} . Before this, we prove that $X_3 < X_1 < X_2$ and $X_5 < X_6 = X_7 < X_8 < X_4$.

Suppose $\mathcal{B}_1(p, q)$ has labeled vertices as in Definition 2.2; thus $v_2 = u'$, $v_5 = v'$, $v_3 = u''$, and $v_4 = v''$. The vertices v_2 and v_3 are unique in \mathcal{B}_+ with maximum and minimum modulus, and v_4 and v_5 are unique

in \mathcal{B}_- with minimum and maximum modulus, respectively. By Lemma 4.2, as X is the first eigenvector of the minimizing graph $\mathcal{B}_1(p, q)$, X_1, X_2, X_3 are nonnegative and X_4, X_5, X_6, X_7, X_8 are negative values of \mathbf{X} . Now, by (3.3), $\lambda_1(X_2 - X_1) = -(p - 2)X_1 - X_3 < 0$ and $\lambda_1(X_1 - X_3) = -X_1 + X_3 + X_4 < 0$. Thus, $X_2 - X_1 > 0$ and $X_1 - X_3 > 0$, which implies $X_3 < X_1 < X_2$. Similarly, $\lambda_1(X_4 - X_8) = -X_4 - X_3 + X_8 < 0$, $\lambda_1(X_8 - X_7) = 2X_7 - X_8 < 0$, $\lambda_1(X_6 - X_7) = 0$, and $\lambda_1(X_6 - X_5) = X_4 + 2X_7 + (q - 5)X_8 < 0$. Thus $X_5 < X_6 = X_7 < X_8 < X_4$.

By the above discussion and (4.2-4.4), we have $\mathcal{B}_+ = \bar{\mathcal{B}}_+ = \mathcal{K}_{1,p}$ and $\mathcal{B}_- = \bar{\mathcal{B}}_- = \mathcal{S}_{1,q}^2$. Consequently, \mathcal{E}_1 contains exactly one edge $u''v'' = v_3v_4$. Now, if the value of v_2 is zero, i.e. $X_2 = 0$, then $X_1 = X_3 = 0$ because $0 < X_3 < X_1 < X_2$. By (3.3) $X_5 = 0$, which is a contradiction. If the value of v_1 is zero, i.e. $X_1 = 0$, then $X_3 = 0$ as $0 < X_3 < X_1$. Solving the first two equations of (3.3), we have $X_2 = 0$ and hence $X_5 = 0$, which is again a contradiction. If the value of v_3 is zero, i.e. $X_3 = 0$, then delete the edges v_3v_4 and v_4v_5 , and join v_4 with v_2 and one of the pendent vertices of $\mathcal{S}_{1,q}^2$. Thus, we get a graph $\mathcal{B}_1(p + 1, q - 1)$ with the same \mathbf{X} such that $\lambda_{\min}(\mathcal{B}_1(p + 1, q - 1)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, which is again a contradiction if $p \geq q$ by Lemma 3.2. Consequently, V_+ does not contain any vertex with zero value given by \mathbf{X} , which completes the proof.

(b) Take $n \geq 22$, $n \equiv 0 \pmod{2}$, and $\mathcal{E}' = \mathcal{E}_2$. Suppose both \mathcal{B}_+ and \mathcal{B}_- of \mathcal{B} contain the edges of the cycles of \mathcal{B} . Let $\bar{\mathcal{B}}$ be a graph obtained from \mathcal{B} such that both $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$ induced by the subgraphs \mathcal{B}_+ and \mathcal{B}_- of \mathcal{B} are unicyclic. By the deletion and addition of some edges in $\bar{\mathcal{B}}_+$, we have $\mathcal{S}_{1,p}^1$ with u^* , which is adjacent to all other vertices in $\mathcal{S}_{1,p}^1$. Moreover, u^* has a maximum modulus value among all the values of $\bar{\mathcal{B}}_+$ given by \mathbf{X} and $p + 1 = |V_+| \geq 11$. Similarly, in $\bar{\mathcal{B}}_-$, we have $\mathcal{S}_{1,q}^1$ with v^* , which is adjacent to all other vertices in $\mathcal{S}_{1,q}^1$. Moreover, v^* has maximum modulus value among all the values of $\bar{\mathcal{B}}_-$ assigned by \mathbf{X} and $q + 1 = |V_-| \geq 11$. Thus, by Lemma 2.1, we have

$$\sum_{uv \in \mathcal{B}_+} X_u X_v \leq \sum_{uv \in \bar{\mathcal{B}}_+} X_u X_v \leq \sum_{uv \in \mathcal{S}_{1,p}^1} X_u X_v \tag{4.8}$$

and

$$\sum_{uv \in \mathcal{B}_-} X_u X_v \leq \sum_{uv \in \bar{\mathcal{B}}_-} X_u X_v \leq \sum_{uv \in \mathcal{S}_{1,q}^1} X_u X_v. \tag{4.9}$$

Let u^{**} and v^{**} be the vertices of $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$ with minimum modulus among all the vertices of $\bar{\mathcal{B}}_+$ and $\bar{\mathcal{B}}_-$, respectively. Then

$$\sum_{uv \in \mathcal{E}_2} X_u X_v \leq X_{u^{**}} X_{v^{**}}. \tag{4.10}$$

Using 4.8, 4.9, and 4.10 in 4.1, we have

$$\sum_{uv \in \mathcal{B}} X_u X_v \leq \sum_{uv \in \mathcal{S}_{1,p}^1} X_u X_v + \sum_{uv \in \mathcal{S}_{1,q}^1} X_u X_v + X_{u^{**}} X_{v^{**}}. \tag{4.11}$$

Since $p \geq q \geq 6$, the vertices u^{**} and v^{**} can be taken from the pendent vertices of $\mathcal{S}_{1,p}^1$ and $\mathcal{S}_{1,q}^1$, respectively. Thus, 4.11 becomes

$$\sum_{uv \in \mathcal{B}} X_u X_v \leq \sum_{uv \in \mathcal{B}_2(p,q)} X_u X_v. \tag{4.12}$$

Now by (2.4) and (4.12), we have, $\lambda_{\min}(\mathcal{B}_2(p, q)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, where $p \geq q \geq 4$ and $p + q + 2 = n \geq 22$. Furthermore, by Lemma 3.4, $\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}_2(p, q)^c)$, where equality holds if $p = \lceil \frac{n-2}{2} \rceil$, $q = \lfloor \frac{n-2}{2} \rfloor$, $p \geq q \geq 4$, and $p + q + 2 = n \geq 10$. Consequently, for $n \geq 22$, $\lambda_{\min}(\mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)^c) \leq \lambda_{\min}(\mathcal{B}^c)$, with equality if and only if $\mathcal{B} = \mathcal{B}_2(\lceil \frac{n-2}{2} \rceil, \lfloor \frac{n-2}{2} \rfloor)$. Moreover, by the use of (3.7), Definition 2.4, Lemma 2.1, and similar discussion as in (a), we have $\mathcal{B}_+ = \bar{\mathcal{B}}_+ = \mathcal{S}_{1,p}^1$, $\mathcal{B}_- = \bar{\mathcal{B}}_- = \mathcal{S}_{1,q}^1$, \mathcal{E}_2 consists of exactly one edge $u^{**}v^{**}$ and the set V_+ does not contain any vertex with zero value given by \mathbf{X} . This completes the proof. \square

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