A note on the conjugacy problem for finite Sylow subgroups of linear pseudofinite groups

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Abstract: We prove the conjugacy of Sylow 2-subgroups in pseudofinite $\mathfrak{M}_c$ (in particular linear) groups under the assumption that there is at least one finite Sylow 2-subgroup. We observe the importance of the pseudofiniteness assumption by analyzing an example of a linear group with nonconjugate finite Sylow 2-subgroups, which was constructed by Platonov.

1. Introduction

Sylow theory originated and was developed in the world of finite groups. There is also some work on a possible generalization to infinite groups (for a comprehensive survey, see [19]). While in some particular families of infinite groups conjugacy results hold for Sylow subgroups, there are pathological situations (nonconjugate Sylow p-subgroups) even in the case of linear groups. However, the existence of a finite Sylow p-subgroup yields conjugacy results in some classes of groups (e.g., groups of finite Morley rank for $p = 2$ [1, Lemma 6.6] and locally finite groups [6, Proposition 2.2.3]). In this paper, we show that this existence assumption gives the desired conjugacy result for Sylow 2-subgroups in the case of pseudofinite $\mathfrak{M}_c$-groups. We also present an interesting example constructed by Platonov [10, Example 4.11] that shows that having a finite Sylow 2-subgroup does not guarantee conjugacy in the case of linear groups.

The main result of this paper is stated below.

Theorem 3.3. *If one of the Sylow 2-subgroups of a pseudofinite $\mathfrak{M}_c$-group G is finite then all Sylow 2-subgroups of G are conjugate and hence finite.*

The structure of this paper is as follows.

In the second section, we recall some of the basic notions in group theory and we fix our terminology and notation.

In the third section, we emphasize some properties of pseudofinite groups and provide some (non)-examples. Then we state and prove our main result (Theorem 3.3).

In the last section, we analyze an example constructed by Platonov [10, Example 4.11] in detail.

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2. Preliminaries

In this section, we recall definitions of some basic notions in group theory and list some well-known results that will be needed in the sequel.

A group G is called phperiodic if every element of it has finite order and it is called a ph$p$-group for a prime p if each element of G has order p^n for a natural number n. An example of an infinite p-group is the Prüfer p-group, denoted by C_{p^∞}. By a phSylow p-subgroup of a group G, we mean a maximal p-subgroup of G. Note that the existence of Sylow p-subgroups is guaranteed by Zorn's lemma.

A group G is said to satisfy the phnormalizer condition if any proper subgroup is properly contained in its normalizer. It is well known that finite nilpotent groups satisfy the normalizer condition.

Let $\mathcal{P}$ denote a group theoretical property such as solvability, nilpotency, commutativity, finiteness, etc. A group G is called phlocally $\mathcal{P}$ if every finite subset of G generates a subgroup with the property $\mathcal{P}$. A group G is said to be ph$\mathcal{P}$-by-finite if G has a normal subgroup N with the property $\mathcal{P}$ such that the quotient group G/N is finite.

A phlinear group is a subgroup $G \leqslant \mathrm{GL}_n(F)$ for some field F where $\mathrm{GL}_n(F)$ denotes the general linear group over F. A group G is said to satisfy the phdescending chain condition on centralizers (or phminimal condition on centralizers) if every proper chain of centralizers in G stabilizes after finitely many steps and such groups are called ph$\mathfrak{M}_c$-groups. If moreover there is a global finite bound on the length of such chains, then we say that G has phfinite centralizer dimension. It is well known that any linear group has finite centralizer dimension (see for example the remark after Corollary 2.10 in [16]) and hence the class of $\mathfrak{M}_c$-groups contains the class of linear groups.

The following results about $\mathfrak{M}_c$-groups, which generalize the corresponding classical results for linear groups, will be needed in the sequel.

Fact 2.1 (Wagner, Corollary 2.4 in [15])**.** *Sylow* 2*-subgroups of* $\mathfrak{M}_c$*-groups are locally finite.*

Fact 2.2 (Bryant, Theorem A in [4])**.** *Periodic, locally nilpotent* $\mathfrak{M}_c$*-groups are nilpotent-by-finite.*

Fact 2.3 (Wagner, Fact 1.3 in [15])**.** *A nilpotent-by-finite and locally nilpotent group has nontrivial center and satisfies the normalizer condition.*

Remark 2.4. *Note that Sylow* 2*-subgroups of* $\mathfrak{M}_c$*-groups are locally finite by Fact 2.1 and hence locally nilpotent since finite* 2*-groups are nilpotent. Therefore, they are nilpotent-by-finite by Fact 2.2 and they satisfy the normalizer condition by Fact 2.3.*

3. A conjugacy result for pseudofinite groups

In this section, we briefly introduce pseudofinite groups without giving precise definitions of the related notions (such as ultrafilters, ultraproducts, and other basic model theoretical concepts) and we emphasize some properties of these groups that will be needed in the proof of the main result of this paper. We refer the reader to the books [3] and [5] for detailed information about the ultraproduct construction, to [13] for a more complete introduction to pseudofinite groups, and to [18] for a more detailed discussion of these groups.

Pseudofinite groups are defined as infinite models of the theory of finite groups. These groups are group theoretical analogs of pseudofinite fields, which were introduced, studied, and algebraically characterized by James Ax (see [2]). Unfortunately, such an algebraic characterization is not known for pseudofinite groups.

One can also describe (up to elementary equivalence) pseudofinite groups as nonprincipal ultraproducts of finite groups (see [18] for details). This description together with Łoś's theorem [9] (which states that a first-order formula is satisfied in the ultraproduct if and only if it is satisfied in the structures indexed by a set belonging to the ultrafilter) allows us to logically characterize pseudofinite groups as infinite groups satisfying the first-order properties shared by phalmost all (depending on the choice of an ultrafilter) of the finite groups.

A well-known example of a pseudofinite group is the additive group of the rational numbers $(\mathbb{Q}, +)$ (see, e.g., [13, Fact 2.2]). However, the additive group of integers, $(\mathbb{Z}, +)$, is not a pseudofinite group, since while all finite groups satisfy the following first-order statement,

the map $x \mapsto x + x$ is one-to-one if and only if it is onto,

the group $(\mathbb{Z}, +)$ does not.

In the following remark, we mention another first-order property shared by all finite groups. This property will be an important ingredient of our proof.

Remark 3.1. *In any finite group G, two involutions g, h are either conjugate or there is an involution y commuting with both g and h. The first-order sentence below shows that this statement can be expressed in a first-order way in the language of groups.*

$$\forall g, h \quad \left[(g \neq 1 \neq h) \wedge (g^2 = 1 = h^2) \right] \longrightarrow$$

$$\left[\left(\exists x \ g^x = h \right) \vee \left(\exists y \ (y \neq 1) \wedge (y^2 = 1) \wedge (g^y = g) \wedge (h^y = h) \right) \right].$$

Since this property is satisfied by all finite groups, pseudofinite groups satisfy it as well.

Although pseudofinite groups are in a way similar to finite groups, there are also many differences. For example, while all finite groups are isomorphic to linear groups, this is not true for pseudofinite groups. To see this, it is enough to construct a pseudofinite group that does not have finite centralizer dimension since all linear groups have finite centralizer dimension. Consider a nonprincipal ultraproduct of alternating groups, $G = \prod A_n / \mathcal{U}$, such that there is no bound on the orders of the alternating groups in the ultraproduct (if there is a bound, then the ultraproduct is finite; that is, G is not a pseudofinite group). By just considering centralizers of a disjoint even number of transpositions, it easy to see that the centralizer dimension of the alternating groups increases as the rank increases. Since having finite centralizer dimension c is a first-order property of groups (see [7]), the ultraproduct $\prod A_n / \mathcal{U}$ has finite centralizer dimension if and only if there is a bound on the orders of the alternating groups in the ultraproduct. However, by our assumption, there is no bound on the orders of the alternating groups. This proves that $\prod A_n / \mathcal{U}$ does not have finite centralizer dimension, and hence it is not linear.

We will need the following result about pseudofinite groups.

Fact 3.2 (Houcine and Point, Lemma 2.16 in [8]). *Let G be a pseudofinite group. Any definable subgroup or any quotient by a definable normal subgroup is (pseudo)finite.*

Note that when we say phdefinable we mean definable in the language of groups and possibly with parameters (for details, see, for example, the book [5]). In particular, finite sets are definable. It is well known that if X is a definable set in a group G then the centralizer and the normalizer of X in G are definable. Moreover, if G is a group of finite centralizer dimension then the centralizer of any set in G is definable.

In the proof of the following proposition, we will use the well-known results about definability mentioned above as well as some ideas from the proof of a similar result in the context of groups of finite Morley rank (see Lemma 6.6 in [1]).

Theorem 3.3. *If one of the Sylow 2-subgroups of a pseudofinite $\mathfrak{M}_c$-group G is finite then all Sylow 2-subgroups of G are conjugate and hence finite.*

Proof   Let P be a finite Sylow 2-subgroup of G. Assume that there is a Sylow 2-subgroup Q of G that is not conjugate to P and let $D = P \cap Q$. Without loss of generality, we may assume that Q is chosen so that $|D|$ is maximal (for this fixed finite Sylow 2-subgroup P). Since D is finite, both D and $N_G(D)$ are definable and hence $N_G(D)/D$ is a (pseudo)finite group by Fact 3.2. Moreover, as both P and Q are locally finite (Fact 2.1), locally nilpotent, and nilpotent-by-finite, they satisfy the normalizer condition (see Remark 2.4). Therefore, we have

$$N_G(D) \cap P = N_P(D) > D \quad \text{and} \quad N_G(D) \cap Q = N_Q(D) > D.$$

Claim. *There are nonconjugate Sylow 2-subgroups P_1, Q_1 of G such that $|P_1 \cap Q_1| > |P \cap Q|$, and P and P_1 are conjugate.*

Proof   [Proof of the claim] Take two involutions $\bar{i} = iD, \bar{j} = jD$ from the nontrivial 2-subgroups $N_P(D)/D$ and $N_Q(D)/D$ of $N_G(D)/D$. We know that they are either conjugate or commute with another involution in $N_G(D)/D$ since $N_G(D)/D$ is (pseudo-)finite (see Remark 3.1).

Case 1. *Assume that $\bar{i}, \bar{j}$ are conjugate in $N_G(D)/D$.*

In this case, we have $\bar{i}^{\bar{x}} = \bar{j}$ for some $\bar{x} \in N_G(D)/D$. This means that $xix^{-1}D = jD$; that is, $j = xix^{-1}d$ for some $d \in D$. We get $j = xix^{-1}dxx^{-1} = xid_1x^{-1}$ for some $d_1 \in D$. However, since $id_1 \in P$, we get $j \in P^x \cap Q$. Moreover, since x normalizes D, we get $D \leqslant P^x$ and hence $D \leqslant P^x \cap Q$. Thus, we have $D < \langle D, j \rangle \leqslant P^x \cap Q$ and so we take $P_1 = P^x$ and $Q_1 = Q$.

Case 2. *Assume that there is an involution $\bar{k} \in N_G(D)/D$ such that $\bar{i}$ and $\bar{j}$ commute with $\bar{k}$.*

Now consider the 2-groups $\langle D, i, k \rangle$ and $\langle D, j, k \rangle$ and let R_i and R_j denote the Sylow 2-subgroups of G containing them, respectively. Clearly we have the following inclusions:

$$D < \langle D, i \rangle \leqslant P \cap R_i, \quad D < \langle D, k \rangle \leqslant R_i \cap R_j, \quad D < \langle D, j \rangle \leqslant R_j \cap Q.$$

If P is not conjugate to R_i then take $P_1 = P$ and $Q_1 = R_i$. If P is conjugate to R_i but not conjugate to R_j then take $P_1 = R_i$ and $Q_1 = R_j$. If P is conjugate to both R_i and R_j then take $P_1 = R_j$ and $Q_1 = Q$.

The claim follows.

Let P_1, Q_1 be nonconjugate Sylow 2-subgroups of G as in the claim so that $P_1 = P^g$ for some $g \in G$. Now we have

$$|D| = |P \cap Q| < |P_1 \cap Q_1| = |P^g \cap Q_1| = |(P^g \cap Q_1)^{g^{-1}}| = |P \cap Q_1^{g^{-1}}|.$$

Clearly, $Q_1^{g^{-1}}$ is a Sylow 2-subgroup of G. By the maximality of $|D|$, we conclude that P is conjugate to $Q_1^{g^{-1}}$ and hence conjugate to Q_1. This contradicts the fact that $P_1 = P^g$ is not conjugate to Q_1.

Remark 3.4. *The situation is quite complicated when we remove the assumption (in Theorem 3.3) on the existence of a finite Sylow 2-subgroup, even in the linear case (work in progress).*

When we restrict ourselves to $\mathrm{GL}_n(K)$, there is a criterion given by Vol'vachev (for an arbitrary field K) about the conjugacy of the Sylow p-subgroups (see [14]). This criterion implies in particular that nonconjugacy can occur only for Sylow 2-subgroups and only when the characteristic of the field K is zero.

4. On Platonov's example

In this section, we analyze in detail an example constructed by Platonov (Example 4.11 in [10]). The reason for the detailed presentation below is the fact that some computational arguments were skipped in Platonov's original article [10].

For each $i \in \mathbb{N}$, consider the following elements in the group $\mathrm{SL}_2(\mathbb{Q})$:

$$g_i = \begin{pmatrix} 0 & -p_i \\ p_i^{-1} & 0 \end{pmatrix},$$

where $(p_i)_{i \in \mathbb{N}}$ is a sequence of distinct primes of the form $4k + 3$, $k \in \mathbb{N}$.

We will observe that $S_i := \langle g_i \rangle$ is a Sylow 2-subgroup of $\mathrm{SL}_2(\mathbb{Q})$ of order 4 for each i; however, S_i is not conjugate to S_j if $i \neq j$.

Clearly, for each i we have $|S_i| = 4$.

Claim 1. *For each i, the group S_i is a Sylow 2-subgroup of $\mathrm{SL}_2(\mathbb{Q})$.*

Since a proof for this claim is not provided in [10], we list some properties of $\mathrm{SL}_2(\mathbb{Q})$ (some of which are very well known), which lead to a proof of Claim 1.

(1) If $A \in \mathrm{SL}_2(\mathbb{Q})$ has finite order then A is diagonalizable over $\mathbb{C}$.

(2) The group $\mathrm{SL}_2(\mathbb{Q})$ has a unique involution.

(3) There is no element of order 8 in $\mathrm{SL}_2(\mathbb{Q})$.

(4) There is no subgroup of order 8 in $\mathrm{SL}_2(\mathbb{Q})$.

(5) Sylow 2-subgroups of $\mathrm{SL}_2(\mathbb{Q})$ are finite.

Note that properties (1)–(3) follow from basic results in linear algebra. However, since (4) and (5) are more involved, we would like to support them with proofs.

Proof [Proof of (4).] Assume that $H \leqslant \mathrm{SL}_2(\mathbb{Q})$ such that $|H| = 8$. We know that there are only five groups of order 8 up to isomorphism: C_8, $C_2 \times C_2 \times C_2$, $C_4 \times C_2$, D_8 (dihedral group of order 8), and Q_8 (quaternions). Since $\mathrm{SL}_2(\mathbb{Q})$ has a unique involution, we have $H \cong Q_8$ or $H \cong C_8$. However, as $\mathrm{SL}_2(\mathbb{Q})$ has no element of order 8, the latter is not possible and hence $H \cong Q_8$.

Now we will show that Q_8 does not embed in $\mathrm{SL}_2(\mathbb{Q})$ (actually, we can prove more: Q_8 does not embed in $\mathrm{GL}_2(\mathbb{R})$). By the structure of Q_8, it is enough to show that there are no $A, B \in \mathrm{GL}_2(\mathbb{R})$ such that

$$A^2 = B^2 = -I_2 \quad \text{and} \quad AB = -BA. \tag{$*$}$$

We will observe $(*)$ in two steps.

Step 1. *If $A \in \mathrm{GL}_2(\mathbb{R})$ is an element of order 4 then there is $g \in \mathrm{GL}_2(\mathbb{R})$ such that*

$$A^g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

There is $h \in \mathrm{GL}_2(\mathbb{C})$ such that $A^h = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $\lambda_1^4 = \lambda_2^4 = 1$. Without loss of generality we may assume that λ_1 is a primitive 4th root of unity; that is, $\lambda_1 = \pm i$.

First assume $\lambda_1 = i$ and let $\vec{z} = \vec{x} + \vec{y}i$ be a corresponding eigenvector (note that $\vec{x}, \vec{y} \in \mathbb{R}^2$). We have

$$A\vec{z} = i\vec{z}$$
$$A(\vec{x} + \vec{y}i) = i(\vec{x} + \vec{y}i)$$
$$A\vec{x} + A\vec{y}i = i\vec{x} - \vec{y}.$$

Therefore, we get $A\vec{x} = -\vec{y}$ and $A\vec{y} = \vec{x}$. Note that $\{\vec{x}, \vec{y}\}$ forms a basis for $\mathbb{R}^2$ since they are linearly independent over $\mathbb{R}$ (if $\vec{y} = \alpha\vec{x}$ for some $\alpha \in \mathbb{R}$, then we get $A\vec{x} = -\alpha\vec{x}$ and $A\alpha\vec{x} = \vec{x}$, which in turn gives $\alpha^2 = -1$, a contradiction). When we represent A with respect to this basis, we can conclude that A is conjugate to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similarly, if $\lambda_1 = -i$, then A is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})$, which is in turn conjugate to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Step 2. *There are no $A, B \in \mathrm{GL}_2(\mathbb{R})$ satisfying the conditions $(*)$.*

Assume that there are $A, B \in \mathrm{GL}_2(\mathbb{R})$ satisfying the conditions $(*)$. Using Step 1, without loss of generality, we can assume that $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $AB = -BA$, we get $B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, but on the other hand, since the order of B is 4, its square, $B^2 = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix}$ is an involution. By the uniqueness of the involution, we get

$$\begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which leads to a contradiction since $a, b \in \mathbb{R}$.

Proof [Proof of (5).] Assume that $\mathrm{SL}_2(\mathbb{Q})$ has an infinite Sylow 2-subgroup P. Since periodic linear groups are locally finite (see [11]), P is locally finite. Therefore, P has a finite subgroup, say X, of order greater than 8 (just consider the subgroup generated by 8 distinct elements of P). Then, by Sylow's first theorem, X has a subgroup of order 8, which is clearly a subgroup of $\mathrm{SL}_2(\mathbb{Q})$. Since this is not possible, (5) follows.

By properties (4) and (5), we conclude that the groups $S_i = \langle g_i \rangle$ defined above are Sylow 2-subgroups of $\mathrm{SL}_2(\mathbb{Q})$.

Claim 2. *For $i \neq j$, the groups S_i, S_j are not conjugate in $\mathrm{SL}_2(\mathbb{Q})$.*

Assume that $S_i = S_j^g$ for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$. Then we either have $gg_ig^{-1} = g_j$ or $gg_ig^{-1} = g_j^3$. We consider the first case:

$$gg_ig^{-1} = g_j$$

$$gg_i = g_jg$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -p_i \\ p_i^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -p_j \\ p_j^{-1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This equation gives the following equalities:

$$bp_i^{-1} = -cp_j, \;\; dp_i^{-1} = ap_j^{-1}, \;\; -ap_i = -dp_j, \;\; -cp_i = bp_j^{-1}.$$

Multiplying both sides of the first and third equations by $-cp_i$ and $-ap_j^{-1}$, respectively, we get

$$-bc = c^2p_jp_i, \;\; ad = a^2p_ip_j^{-1}.$$

By combining this with the fact that $ad - bc = 1$, we have

$$a^2p_ip_j^{-1} + c^2p_jp_i = 1,$$

which in turn gives

$$p_i(a^2 + c^2p_j^2) = p_j. \tag{$\diamond$}$$

We give an argument for the impossibility of ($\diamond$), since it is skipped in [10]. Note that the p_j-adic valuation of the right-hand side of the equation ($\diamond$) is clearly 1. However, the p_j-adic valuation of the left-hand side is even by the following fact, which is folklore.

Fact 4.1. *Suppose p is a prime such that $p \equiv 3 \pmod 4$ and $\alpha, \beta \in \mathbb{Q}$. Then $v_p(\alpha^2 + \beta^2)$ is even, where v_p denotes the p-adic valuation.*

Proof First, without loss of generality, we may assume that α, β are integers. To see this let $\alpha = \dfrac{\alpha_1}{\alpha_2}$, $\beta = \dfrac{\beta_1}{\beta_2}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$. Clearly, we have

$$v_p(\alpha^2 + \beta^2) = v_p\left(\frac{\alpha_1^2}{\alpha_2^2} + \frac{\beta_1^2}{\beta_2^2}\right) = v_p((\alpha_1\beta_2)^2 + (\beta_1\alpha_2)^2) - v_p((\alpha_2\beta_2)^2).$$

Therefore, $v_p(\alpha^2 + \beta^2)$ is even if and only if $v_p((\alpha_1\beta_2)^2 + (\beta_1\alpha_2)^2)$ is even, since the valuation of a square is always even.

Since $p \equiv 3 \pmod 4$, p is irreducible in the ring of Gaussian integers $\mathbb{Z}[i]$. (If $p = (x + yi)(z + ti)$ in $\mathbb{Z}[i]$, then squaring the norms of both sides we get $p^2 = (x^2 + y^2)(z^2 + t^2)$. Since p can not be sum of two squares (which is $0, 1$ modulo 4), either $x + yi$ or $z + ti$ is a unit; that is, p is irreducible.) Moreover, since $\mathbb{Z}[i]$ is a unique factorization domain p is a prime in $\mathbb{Z}[i]$. Now, if p does not divide $\alpha^2 + \beta^2$, clearly the p-adic valuation is zero. Assume that p divides $\alpha^2 + \beta^2$ (in $\mathbb{Z}$). Then p divides $(\alpha + \beta i)(\alpha - \beta i)$ in $\mathbb{Z}[i]$. Then, as a prime in $\mathbb{Z}[i]$, p divides either $\alpha + \beta i$ or $\alpha - \beta i$, but then both α and β are divisible by p in $\mathbb{Z}$ and so $\alpha^2 = p^2k^2$ and $\beta^2 = p^2l^2$ for some $k, l \in \mathbb{Z}$. Now $\alpha^2 + \beta^2 = p^2(k^2 + l^2)$. Inductively, we can conclude that $v_p(\alpha^2 + \beta^2)$ is even.

The computations for the second case (when $gg_ig^{-1} = g_j^3$) are very similar and we obtain

$$p_i(a^2 + c^2p_j^2) = -p_j.$$

This is clearly not possible. We conclude that S_i and S_j are not conjugate.

As a result, we have observed that $\mathrm{SL}_2(\mathbb{Q})$ has infinitely many pairwise nonconjugate Sylow 2-subgroups of order 4.

Remark 4.2. *Note that Theorem 3.3 together with Platonov's example shows that $\mathrm{SL}_2(\mathbb{Q})$ is not a pseudofinite group. One may also observe this directly by first defining $\mathbb{Q}^\times$, the multiplicative group of $\mathbb{Q}$, as a centralizer in $\mathrm{SL}_2(\mathbb{Q})$ and then using the fact that $\mathbb{Q}^\times$ is not a pseudofinite group (note that the definable endomorphism of $\mathbb{Q}^\times$ that maps x to x^3 is one-to-one but not onto).*

More generally, one can observe that for any infinite field K, the group $\mathrm{SL}_n(K)$ is pseudofinite if and only if K is a pseudofinite field (for $n > 1$). It is well known that if K is a pseudofinite field then $\mathrm{SL}_n(K)$ is a pseudofinite group. To see the other direction, we work with the Chevalley group $\mathrm{PSL}_n(K)$, which is also pseudofinite, as a definable quotient of $\mathrm{SL}_n(K)$ and we will refer to the article of Wilson [17]. In this article, Wilson states the following results obtained by Thomas in his dissertation [12]: the class $\{X(K) \mid K \text{ field}\}$ is an elementary class where X denotes a Chevalley group of untwisted type and moreover whenever $X(K) \equiv X(F)$ and K is pseudofinite it follows that F is pseudofinite. If we apply Wilson's theorem [17, Theorem on p. 471] together with the result of Thomas to $\mathrm{PSL}_n(K)$, we get $\mathrm{PSL}_n(K) \cong PSL_n(F)$ for some pseudofinite field F. By referring to the "moreover" part of Thomas' result, we finally conclude that the field K is pseudofinite.

Acknowledgments

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