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A novel kind of AKNS integrable couplings and their Hamiltonian structures

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Abstract: We present a novel hierarchy of AKNS integrable couplings based on a specific semidirect sum of Lie algebras associated with $\mathrm{sl}(2)$. By applying the variational identity, we derive a bi-Hamiltonian structure of the resulting coupling systems, thereby showing their Liouville integrability.

Key words: Integrable couplings, semidirect sum, Hamiltonian structure

1. Introduction

The concept of integrable couplings has been introduced and various integrable couplings have been studied systematically. This originated from an investigation on centerless Virasoro symmetry algebras of integrable systems [4, 10]. Generally, for a given integrable system $u_t = K(u)$, an integrable coupling is a triangular integrable system

$$\begin{cases} u_t = K(u), \\ v_t = S(u,v), \end{cases} \tag{1.1}$$

where the potentials u and v can be either scalar functions or vector functions of the dependent variables x and t, and $\partial S/\partial[u] \neq 0$, such that the whole system is not separated. At the beginning, integrable couplings were constructed by using perturbations [4, 5, 10], or enlarging spectral problems and Lie algebras [6, 21]. Later, it was found that integrable couplings have a close connection with semidirect sums of Lie algebras, and a new method to construct integrable couplings was presented [13, 14]. The existing approaches are just specific examples of the semidirect sum method, or in other words, applications of semidirect sums. On one hand, an integrable coupling enlarges the original integrable system, which adds new equations, and thus it may be more complicated than the original system but it enriches the considered integrable system. On the other hand, for some concrete equations, based on the physical problems that the original equations describe, the supplementary equations can help us understand the underlying physical problems better.

The theory of Hamiltonian structures is important to study integrable systems. A continuous Hamiltonian system of a PDE reads

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u}, \tag{1.2}$$

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where J is a Hamiltonian operator, $\mathcal{H} = \int H dx$ is the Hamiltonian functional, and $\frac{\delta}{\delta u}$ is the variational derivative with respect to u. Magri [16] introduced the celebrated bi-Hamiltonian formulation, which paves the way for generating infinitely many symmetries and conservation laws. The bi-Hamiltonian formulation often implies the Liouville integrability of PDEs. There are several ways to construct Hamiltonian structures of integrable systems. A very powerful and simple technique by using the trace identity was proposed by Tu [17, 18], which heavily relies on the Killing form of Lie algebras. Because the Killing form is nondegenerate if and only if the Lie algebra is semisimple, the trace identity method is valid only for classical integrable systems that are generated from semisimple Lie algebras. For integrable couplings, the underlying Lie algebras are non-semisimple, and the trace identity method is not valid anymore. To solve this problem, a variational identity method was proposed in [9], which aims to construct Hamiltonian structures for integrable couplings. It is a generalization of the trace identity method.

Many examples of hierarchies of integrable couplings have been presented for classical soliton hierarchies, including the KdV hierarchy, the MKdV hierarchy, the Toda hierarchy, the Glachette–Johnson hierarchy, the Jaulent–Miodek hierarchy, the Benjamin–Ono hierarchy, the Tu hierarchy, the AKNS hierarchy, and the Kaup–Newell hierarchy [8, 15, 19]. There are also studies on biintegrable couplings [7] and tri-integrable couplings [12], which aim to classify integrable couplings from the Lie algebra point of view. The supertrace identity is presented to study superintegrable systems [11]. The Darboux transformation for integrable couplings was also presented [20].

In this paper, by using the zero curvature equation of an enlarged AKNS spectral problem, we shall derive a new hierarchy of integrable couplings and then explore its Hamiltonian structure by the variational identity. The paper is structured as follows. In Section 2, we shall construct a novel hierarchy of integrable couplings from a specific semidirect sum of Lie algebra associated with $\mathrm{sl}(2)$. In Section 3, by applying the variational identity, we shall construct a bi-Hamiltonian structure for the resulting hierarchy of coupling systems, thereby showing their Liouville integrability. A few remarks will be given at the end of the paper.

2. Novel integrable couplings from an enlarged AKNS spectral problem

The AKNS soliton hierarchy [1] is a general source to construct nonlinear integrable equations. We shall construct a novel hierarchy of integrable couplings for the AKNS soliton hierarchy. The key point is to enlarge the spatial spectral problem based on a non-semisimple matrix loop algebra $\bar{\mathfrak{g}}$. The enlarged spectral problem has an additional matrix block depending on the spectral parameter λ, and such an enlarged spatial spectral problem is a first try to generate hierarchies of integrable couplings.

Let $\tilde{\mathrm{sl}}(2)$ denote the loop algebra associated with the special linear algebra $\mathrm{sl}(2)$:

$$\tilde{\mathrm{sl}}(2) \quad = \quad \{M \in \mathrm{sl}(2) \quad | \quad \text{entries of} \quad M - \text{Laurent series of} \quad \lambda\}. \tag{2.1}$$

In order to construct integrable couplings, we enlarge the simple Lie algebra $\tilde{\mathrm{sl}}(2)$ to a non-semisimple Lie algebra $\bar{\mathfrak{g}}$. Consider the Lie algebra consisting of 2×2 block matrices:

$$\bar{\mathfrak{g}} = \left\{ M(A,B) = \left[\begin{array}{cc} A & B \\ 0 & A \end{array} \right] \middle| A, B \in \tilde{\mathrm{sl}}(2) \right\}. \tag{2.2}$$

It is direct to see that $\bar{\mathfrak{g}}$ is a semidirect sum of the simple Lie algebra $\mathfrak{g}$ and the solvable Lie algebra $\mathfrak{g}_c$, i.e.

$$\bar{\mathfrak{g}} = \mathfrak{g} \in \mathfrak{g}_c, \quad \mathfrak{g} - \text{simple}, \quad \mathfrak{g}_c - \text{solvable}, \tag{2.3}$$

where

$$\mathfrak{g} = \{M(A,0)|A \in \tilde{\text{sl}}(2)\}, \quad \mathfrak{g}_c = \{M(0,B)|B \in \tilde{\text{sl}}(2)\}.$$

The semidirect sum means $[\mathfrak{g}, \mathfrak{g}_c] = \{[A,B]|A \in \mathfrak{g}, B \in \mathfrak{g}_c\} \subseteq \mathfrak{g}_c$, where $[\cdot,\cdot]$ denotes the Lie bracket of $\bar{\mathfrak{g}}$. Therefore, $\mathfrak{g}_c$ is an ideal Lie subalgebra of $\bar{\mathfrak{g}}$.

Then we consider an enlarged special spectral problem:

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{U}(\bar{u},\lambda) = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix} \in \bar{\mathfrak{g}}, \tag{2.4}$$

with

$$U = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix} \in \tilde{\text{sl}}(2), \quad U_a = \begin{bmatrix} -\lambda & r \\ s & \lambda \end{bmatrix} \in \tilde{\text{sl}}(2), \tag{2.5}$$

where λ is the spectral parameter, $\bar{\phi}$ is a four-dimensional column vector of eigenfunctions, and $\bar{u} = (p,q,r,s)^T$ denotes the potential. The complement part U_a in this enlarged Lax operator depends explicitly on the spectral parameter λ, which is the first try to generate integrable couplings.

The stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \quad \bar{W} = \begin{bmatrix} W & W_a \\ 0 & W \end{bmatrix}, \tag{2.6}$$

with

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \tilde{\text{sl}}(2), \quad W_a = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} \in \tilde{\text{sl}}(2), \tag{2.7}$$

induces

$$W_x = [U, W], \tag{2.8}$$

$$W_{a,x} = [U, W_a] + [U_a, W]; \tag{2.9}$$

i.e.

$$\begin{cases} a_x = pc - qb, \\ b_x = -2\lambda b - 2pa, \\ c_x = 2qa + 2\lambda c, \end{cases} \qquad \begin{cases} e_x = pg + rc - qf - sb, \\ f_x = -2\lambda f - 2pe - 2\lambda b - 2ra, \\ g_x = 2qe + 2\lambda g + 2sa + 2\lambda c. \end{cases} \tag{2.10}$$

By setting

$$W = \sum_{i=0}^{\infty} W_i \lambda^{-i}, \quad W_i = \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}, \quad i \geq 0, \tag{2.11}$$

$$W_a = \sum_{i=0}^{\infty} W_{a,i} \lambda^{-i}, \quad W_{a,i} = \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix}, \quad i \geq 0, \tag{2.12}$$

substituting them into (2.10), and comparing the coefficients of λ, we obtain the recursion relations:

$$\begin{cases} a_{i+1,x} = pc_{i+1} - qb_{i+1}, \\ b_0 = 0, \quad b_{i+1} = -\frac{1}{2}b_{i,x} - pa_i, \qquad i \geq 0, \\ c_0 = 0, \quad c_{i+1} = \frac{1}{2}c_{i,x} - qa_i, \end{cases} \tag{2.13}$$

$$\begin{cases} e_{i+1,x} = pg_{i+1} - qf_{i+1} + rc_{i+1} - sb_{i+1}, \\ f_0 = -b_0, \quad f_{i+1} = -\frac{1}{2}f_{i,x} - pe_i - ra_i - b_{i+1}, \qquad i \geq 0. \\ g_0 = -c_0, \quad g_{i+1} = \frac{1}{2}g_{i,x} - qe_i - sa_i - c_{i+1}, \end{cases} \tag{2.14}$$

Let us take the initial values

$$a_0 = \alpha, \quad e_0 = \beta, \tag{2.15}$$

where α and β are arbitrary constants, and we assume that $\alpha \neq 0$. In addition, choose the constants of integration to be zero:

$$a_i|_{u=0} = 0, \quad e_i|_{\bar{u}=0} = 0, \quad i \geq 1. \tag{2.16}$$

Then b_i, c_i, a_i and f_i, g_i, e_i can be computed from (2.13) and (2.14), respectively.

Now let us introduce the temporal spectral problems

$$\bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi}, \quad \bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{bmatrix} \in \bar{\mathfrak{g}}, \tag{2.17}$$

with

$$V^{[m]} = (\lambda^m W)_+ = \sum_{i=0}^{m} W_i \lambda^{m-i}, \tag{2.18}$$

$$V_a^{[m]} = (\lambda^m W_a)_+ = \sum_{i=0}^{m} W_{a,i} \lambda^{m-i}, \tag{2.19}$$

where $m \geq 0$ and P_+ denotes the polynomial part of P in λ. Then the compatibility condition $\bar{U}_t - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0$ yields

$$U_t - V_x^{[m]} + [U, V^{[m]}] = 0, \tag{2.20}$$

$$U_{a,t} - V_{a,x}^{[m]} + [U, V_a^{[m]}] + [U_a, V^{[m]}] = 0. \tag{2.21}$$

Equation (2.20) generates the original AKNS soliton system, while Equation (2.21) results in a supplementary soliton subsystem. Noting the recursion relations (2.13) and (2.14), we obtain a hierarchy of AKNS integrable couplings:

$$p_t = b_{m,x} + 2pa_m, \quad q_t = c_{m,x} - 2qa_m, \tag{2.22}$$

$$r_t = f_{m,x} + 2ra_m + 2pe_m, \quad s_t = g_{m,x} - 2sa_m - 2qe_m, \tag{2.23}$$

the first of which is exactly the AKNS soliton hierarchy. Finally, we can express the whole hierarchy of integrable couplings as follows:

$$\begin{aligned} \bar{u}_{t_m} &= \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2(f_{m+1}+b_{m+1}) \\ 2(g_{m+1}+c_{m+1}). \end{bmatrix} \\ &= \bar{\Phi}^m \begin{bmatrix} -2b_1 \\ 2c_1 \\ -2(f_1+b_1) \\ 2(g_1+c_1) \end{bmatrix}, \quad m \geq 0, \end{aligned} \tag{2.24}$$

where the recursion operator $\bar{\Phi}$ is defined by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_c - \Phi & \Phi \end{bmatrix}, \tag{2.25}$$

with

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \tag{2.26}$$

$$\Phi_c = \begin{bmatrix} r\partial^{-1}q + p\partial^{-1}s & r\partial^{-1}p + p\partial^{-1}r \\ -s\partial^{-1}q - q\partial^{-1}s & -s\partial^{-1}p - q\partial^{-1}r \end{bmatrix}. \tag{2.27}$$

The first few equations are computed as follows:

$$\bar{u}_{t_1} = \bar{K}_1(\bar{u}) = \begin{bmatrix} -\alpha p_x \\ -\alpha q_x \\ -\beta p_x - \alpha(r_x - p_x) \\ -\beta q_x - \alpha(s_x - q_x) \end{bmatrix}, \tag{2.28}$$

$$\bar{u}_{t_2} = \bar{K}_2(\bar{u}) = \begin{bmatrix} \alpha\left(\frac{1}{2}p_{xx} - p^2 q\right) \\ -\alpha\left(\frac{1}{2}q_{xx} - q^2 p\right) \\ \beta\left(\frac{1}{2}p_{xx} - p^2 q\right) + \alpha\left(\frac{1}{2}r_{xx} - p_{xx} + 2p^2 q - p^2 s - 2pqr\right) \\ -\beta\left(\frac{1}{2}q_{xx} - pq^2\right) - \alpha\left(\frac{1}{2}s_{xx} - q_{xx} + 2pq^2 - q^2 r - 2pqs\right) \end{bmatrix}, \tag{2.29}$$

If we denote

$$K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix}, \quad S_m = \begin{bmatrix} -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \tag{2.30}$$

then the above integrable coupling hierarchy (2.24) can be written as

$$\bar{K}_m = \begin{bmatrix} K_m \\ S_m + K_m \end{bmatrix} = \bar{\Phi}^m \begin{bmatrix} K_0 \\ S_0 + K_0 \end{bmatrix}, \tag{2.31}$$

and the recursion operator $\bar{\Phi}$ is determined by the recursion relation

$$\bar{K}_{m+1} = \bar{\Phi}\bar{K}_m, \quad m \geq 0. \tag{2.32}$$

A lengthy computation shows that $\bar{\Phi}$ is hereditary [2, 3], i.e. it satisfies

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{K}]\bar{S} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{K}]\bar{S} = \bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{S}]\bar{K} - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{S}]\bar{K} \tag{2.33}$$

for all vector fields $\bar{K}$ and $\bar{S}$.

We point out that a case with an additional matrix block

$$U_a = \begin{bmatrix} -v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix} \tag{2.34}$$

was investigated in [9], where $v_i, i = 1, 2, 3$ are all new dependent variables. Here in our situation, the submatrix operators Φ and Φ_c are the same as the ones in [9], but the recursion operator $\bar{\Phi}$ in (2.25) is different from the one in [9], and we introduce a new soliton hierarchy of AKNS integrable couplings.

3. Hamiltonian structures of the resulting AKNS integrable couplings

To construct Hamiltonian structures of the resulting coupling systems, we need to compute a nondegenerate, symmetric, and ad-invariant bilinear form on the enlarged matrix Lie algebra that we adopted above:

$$\bar{\mathfrak{g}} = \left\{ \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \middle| A, B \in \tilde{\mathrm{sl}}(2) \right\}. \tag{3.1}$$

This Lie algebra has been used in [9], and we would like to recall the construction procedure of its bilinear forms made in [9] as follows.

There are four steps to construct the required bilinear forms on the above Lie algebra $\bar{\mathfrak{g}}$:

(1) Construct an isomorphism between the loop algebra $\bar{\mathfrak{g}}$ and a Lie algebra in vector form;

(2) Compute the commutator on the Lie algebra in vector form;

(3) Construct the required bilinear forms on the Lie algebra in vector form;

(4) Compute the corresponding bilinear forms on the original Lie algebra $\bar{\mathfrak{g}}$.

The whole procedure of constructing the required bilinear forms is as follows. First, we transform the Lie algebra $\bar{\mathfrak{g}}$ into a Lie algebra in vector form. Let us define a mapping:

$$\begin{aligned} \delta: \quad & \bar{\mathfrak{g}} \to \mathbb{R}^6, \\ & A \mapsto a = (a_1, a_2, a_3, a_4, a_5, a_6)^T, \end{aligned} \tag{3.2}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & -a_1 \end{bmatrix} \in \bar{\mathfrak{g}}. \tag{3.3}$$

This mapping induces a Lie algebraic structure on $\mathbb{R}^6$, which is isomorphic to the original matrix Lie algebra $\bar{\mathfrak{g}}$. The corresponding commutator on $\mathbb{R}^6$ should satisfy:

$$[a, b] = \left(a^T R(b)\right)^T,$$
$$a = (a_1, \ldots, a_6)^T, \quad b = (b_1, \ldots, b_6)^T,$$

where $R(b)$ can be computed as

$$R(b) = \begin{bmatrix} R_1(b) & R_2(b) \\ 0 & R_1(b) \end{bmatrix}, \tag{3.4}$$

$$R_1(b) = \begin{bmatrix} 0 & 2b_2 & -2b_3 \\ b_3 & -2b_1 & 0 \\ -b_2 & 0 & 2b_1 \end{bmatrix}, \quad R_2(b) = \begin{bmatrix} 0 & 2b_5 & -2b_6 \\ b_6 & -2b_4 & 0 \\ -b_5 & 0 & 2b_4 \end{bmatrix}.$$

Then a bilinear form on $\mathbb{R}^6$ can be determined by

$$\langle a, b\rangle_{\mathbb{R}^6} = a^T F b, \tag{3.5}$$

where F is a constant matrix. The nondegenerate property requires $\det F \neq 0$, the symmetric property $\langle a, b\rangle_{\mathbb{R}^6} = \langle b, a\rangle_{\mathbb{R}^6}$ requires $F^T = F$, and the ad-invariance property

$$\langle a, [b, c]\rangle_{\mathbb{R}^6} = \langle [a, b], c\rangle_{\mathbb{R}^6} \tag{3.6}$$

requires that

$$(R(b)F)^T = -R(b)F, \quad b \in \mathbb{R}^6. \tag{3.7}$$

Solving this system, we obtain

$$F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.8}$$

Thus, the required bilinear forms on the original Lie algebra $\bar{\mathfrak{g}}$ read

$$\begin{aligned} \langle A, B\rangle_{\bar{\mathfrak{g}}} &= \langle \delta^{-1}(A), \delta^{-1}(B)\rangle_{\mathbb{R}^6} \\ &= (a_1, \dots, a_6)F(b_1, \dots, b_6)^T \\ &= (2\,a_1b_1 + a_2b_3 + a_3b_2)\,\eta_1 + (a_2b_6 + a_3b_5 + a_5b_3 + a_6b_2)\,\eta_2 \\ &\quad +2\,(\,a_1b_4 + \,a_4b_1)\,\eta_2. \end{aligned} \tag{3.9}$$

Obviously, such bilinear forms are symmetric and ad-invariant:

$$\langle A, B\rangle_{\bar{\mathfrak{g}}} = \langle B, A\rangle_{\bar{\mathfrak{g}}}, \quad \langle A, [B, C]\rangle_{\bar{\mathfrak{g}}} = \langle [A, B], C\rangle_{\bar{\mathfrak{g}}}, \tag{3.10}$$

where $A, B, C \in \bar{\mathfrak{g}}$, and they are nondegenerate if and only if $\eta_2 \neq 0$.

Recall that the continuous variational identity [9] reads

$$\frac{\delta}{\delta \bar{u}} \int \langle \frac{\partial \bar{U}}{\partial \lambda}, \bar{W}\rangle_{\bar{\mathfrak{g}}} dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \frac{\partial \bar{U}}{\partial \bar{u}}, \bar{W}\rangle_{\bar{\mathfrak{g}}}, \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle \bar{W}, \bar{W}\rangle_{\bar{\mathfrak{g}}}|, \tag{3.11}$$

where $\langle\cdot,\cdot\rangle_{\bar{\mathfrak{g}}}$ is a nondegenerate, symmetric, and ad-invariant bilinear form on the Lie algebra $\bar{\mathfrak{g}}$. If $\bar{\mathfrak{g}}$ is semisimple, any bilinear form is equal to the Killing form up to a constant multiplier: $\langle\frac{\partial\bar{U}}{\partial\lambda},\bar{W}\rangle_{\bar{\mathfrak{g}}}=C\,\mathrm{tr}(\frac{\partial\bar{U}}{\partial\lambda}\bar{W})$. Otherwise, we need to construct a bilinear form on the underlying Lie algebra. The Lie algebra we considered here has the same form as the one in [9], and thus the required bilinear form reads as in [9]:

$$\begin{aligned}&\langle A,B\rangle_{\bar{\mathfrak{g}}}\\ =\;&(2\,a_1b_1+a_2b_3+a_3b_2)\,\eta_1+(2\,a_1b_4+a_2b_6+a_3b_5+2\,a_4b_1+a_5b_3+a_6b_2)\,\eta_2,\end{aligned}$$

where η_1,η_2 are two arbitrary constants, and

$$A=\begin{bmatrix}a_1&a_2&a_4&a_5\\a_3&-a_1&a_6&-a_4\\0&0&a_1&a_2\\0&0&a_3&-a_1\end{bmatrix}\in\bar{\mathfrak{g}},\quad B=\begin{bmatrix}b_1&b_2&b_4&b_5\\b_3&-b_1&b_6&-b_4\\0&0&b_1&b_2\\0&0&b_3&-b_1\end{bmatrix}\in\bar{\mathfrak{g}}.\tag{3.12}$$

It is symmetric and ad-invariant:

$$\langle A,B\rangle_{\bar{\mathfrak{g}}}=\langle B,A\rangle_{\bar{\mathfrak{g}}},\quad\langle A,[B,C]\rangle_{\bar{\mathfrak{g}}}=\langle[A,B],C\rangle_{\bar{\mathfrak{g}}},\quad A,B,C\in\bar{\mathfrak{g}},\tag{3.13}$$

and the nondegenerate property requires that $\eta_2\neq0$.

In order to apply the variational identity (3.11), we compute that

$$\langle\frac{\partial\bar{U}}{\partial\lambda},\bar{W}\rangle_{\bar{\mathfrak{g}}}=-2a\eta_1-2e\eta_2-2a\eta_2,\tag{3.14}$$

$$\langle\frac{\partial\bar{U}}{\partial\bar{u}},\bar{W}\rangle_{\bar{\mathfrak{g}}}=-(c\eta_1+g\eta_2,b\eta_1+f\eta_2,c\eta_2,b\eta_2)^T,\tag{3.15}$$

where $\bar{u}=(p,q,r,s)^T$ as before. Substituting them into (3.11) and comparing the coefficients of $\lambda^{-(m+1)},m\geq 0$, we obtain

$$\begin{aligned}&\frac{\delta}{\delta\bar{u}}\int[2a_{m+1}\eta_1+2\,(a_{m+1}+e_{m+1})\,\eta_2]\,dx\\ =\;&-(\gamma-m)(c_m\eta_1+g_m\eta_2,b_m\eta_1+f_m\eta_2,c_m\eta_2,b_m\eta_2)^T.\end{aligned}\tag{3.16}$$

Taking $m=1$ yields the constant $\gamma=0$. Therefore, we have

$$\begin{aligned}&\frac{\delta}{\delta\bar{u}}\int\frac{2a_{m+1}\eta_1+2\,(a_{m+1}+e_{m+1})\,\eta_2}{m}dx\\ =\;&(c_m\eta_1+g_m\eta_2,b_m\eta_1+f_m\eta_2,c_m\eta_2,b_m\eta_2)^T,\end{aligned}\tag{3.17}$$

where $m\geq1$. Noting the notation (2.30), the right-hand side of (3.17) can be denoted by

$$\begin{aligned}&(c_m\eta_1+g_m\eta_2,b_m\eta_1+f_m\eta_2,c_m\eta_2,b_m\eta_2)^T\\ =\;&-\frac{1}{2}((\eta_1\sigma K_{m-1}+\eta_2\sigma S_{m-1})^T,(\eta_2\sigma K_{m-1})^T)^T,\end{aligned}\tag{3.18}$$

where

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.19}$$

It follows now that the enlarged AKNS coupling systems in (2.24) possess the following Hamiltonian structures:

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \quad m \geq 0, \tag{3.20}$$

where the Hamiltonian operator $\bar{J}$ is given by

$$\bar{J} = \frac{2}{\eta_2}\begin{bmatrix} 0 & \sigma \\ \sigma & \sigma - \frac{\eta_1}{\eta_2}\sigma \end{bmatrix}, \tag{3.21}$$

and the Hamiltonian functionals are given by

$$\bar{\mathcal{H}}_m = \int \frac{2\eta_1 a_{m+2} + 2\eta_2 a_{m+2} + 2\eta_2 e_{m+2}}{m+1}dx, \quad m \geq 0. \tag{3.22}$$

Notice that the recursion operator $\bar{\Phi}$ is hereditary, and we can have a Hamiltonian pair consisting of $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$. Therefore, it follows that the Hamiltonian couplings (3.20) possess a bi-Hamiltonian structure, and furthermore, they are all Liouville integrable. In particular, they have infinitely many commuting symmetries $\{\bar{K}_m | m \geq 0\}$ and conserved functionals $\{\bar{\mathcal{H}}_m | m \geq 0\}$, which form two abelian Lie algebras:

$$[\bar{K}_{m_1}, \bar{K}_{m_2}] := \bar{K}'_{m_1}(\bar{u})[\bar{K}_{m_2}] - \bar{K}'_{m_2}(\bar{u})[\bar{K}_{m_1}] = 0,$$
$$\{\bar{\mathcal{H}}_{m_1}, \bar{\mathcal{H}}_{m_2}\} := \int \left(\frac{\delta \bar{\mathcal{H}}_{m_1}}{\delta \bar{u}}\right)^T \bar{J}\frac{\delta \bar{\mathcal{H}}_{m_2}}{\delta \bar{u}}dx = 0,$$

where $m_1, m_2 \geq 0$.

4. Concluding remarks

In this paper, we constructed a novel hierarchy of integrable couplings for the AKNS soliton equations through a different choice of selecting block matrices, which leads to a novel non-semisimple loop algebra. We showed that the resulting coupling systems have bi-Hamiltonian structures with a recursion operator, which leads to Liouville integrability. The theory of integrable couplings brings other interesting results, such as Lax pairs of block forms and Lax pairs with several spectral parameters, and helps us work towards a complete classification of multiple component integrable systems.

The presented enlarged spatial spectral problem in (2.4) and (2.5) associated with the non-semisimple matrix loop algebra defined in (2.2) is of a completely new type. Such a novelty brings complexities and difficulties in computing soliton hierarchies, but new features on integrable couplings. In particular, the dependence of the spectral parameter λ in the additional matrix block can guarantee the effectiveness of applying the Darboux transformation method to construct soliton solutions of integrable couplings. It is expected that more applications of such a new class of enlarged spectral problems can be explored from different points of view. Moreover, an additional interesting question for us is whether any integrable coupling must be linear, i.e. the added subsystem in the coupling system must be linear with the added variable.

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