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Analysis of periodic and asymptotically periodic solutions in nonlinear coupled Volterra integro-differential systems

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Abstract: In this note, we investigate the existence of periodic and asymptotically periodic solutions of a system of coupled nonlinear Volterra integro-differential equations with infinite delay. We will make use of Schauder fixed point theorem to prove our maps have fixed points.

Key words: Periodic solutions, asymptotic periodic solutions, coupled equations, Volterra integro-differential equations, Schauder fixed point theorem

1. Introduction

Coupled integro-differential equations have many applications in science and engineering. In computational neuroscience, the Wilson–Cowan model describes the dynamics of interactions between populations of very simple excitatory and inhibitory model neurons. It was developed by H.R. Wilson and Jack D. Cowan [16, 17] and extensions of the model have been widely used in modeling neuronal populations [11, 13, 15, 18]. The model is important historically because it was the first of its kind and it did fit the data uniformly. Because the model neurons are simple, only elementary limit cycle behavior, i.e. neural oscillations, and stimulus-dependent evoked responses are predicted. The key findings include the existence of multiple stable states and hysteresis in the population response.

In addition, coupled differential or integro-differential equations have been used in many areas of biological and environmental sciences. Lotka–Volterra models for competitive species are probably the best-known examples of such coupled equations [2, 6]. A particular case where the Lotka–Volterra model has successfully been used is the famous predator–prey problem for two competing species. Two interlocked or coupled equations are required to model such a problem. Coupled equations are also used in other fields to study various qualitative properties of solutions [3, 4, 8, 12, 14]. In [3, 4] the authors studied the existence of asymptotically periodic solutions of linear systems of Volterra difference equations, and in [8, 12, 14] the authors studied oscillation properties and asymptotic or limiting properties of solutions.

In this paper we study the existence of periodic and asymptotically periodic solutions of the following coupled nonlinear Volterra integro-differential equations with infinite delay

$$\begin{cases} x'(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^{t} a(t,s)f(x(s),y(s))ds, \\ y'(t) = p_1(t)y(t) + p_2(t)x(t) + \int_{-\infty}^{t} b(t,s)g(x(s),y(s))ds, \end{cases} \tag{1.1}$$

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2010 *AMS Mathematics Subject Classification:* Primary 34K20; Secondary 45J05, 45D05 where the functions a, b, f, g, h_i, and $p_i, i = 1, 2$ are assumed to be continuous in their arguments throughout the paper.

In the paper [9] the author considered a simpler version of the system given by (1.1) and studied the existence of asymptotically periodic and periodic solutions. For the readers interested in periodic, asymptotically periodic, and almost periodic solutions of Volterra equations, we refer to the partial list [1, 2, 5, 7, 10], and the references therein.

Some of the studies mentioned above deal with the periodicity on systems of Volterra integral equations with infinite delay, but our results are different with respect to assumptions and methods. However, for asymptotic periodicity we can hardly find any study on equations like the one we considered in this paper. In [2], the author considered a forced asymptotic periodicity on a predator–prey system. The equations, assumptions, and the method are very different from ours. Our considered model is very general and hence it encompass all existing models in coupled Volterra integro-differential systems. We show the existence of periodic solutions in Section 2 and the existence of asymptotic periodic solutions in Section 3 and provide an example. In the analysis, we invert both equations in (1.1), transform them into integral equations, and then use Schauder's fixed point theorem.

We assume that there exists a positive real number T, such that

$$\begin{aligned} a(t+T, s+T) &= a(t,s),\ b(t+T, s+T) = b(t,s) \\ p_i(t+T) &= p_i(t),\ h_i(t+T) = h_i(t),\ i = 1, 2 \end{aligned} \tag{1.2}$$

for all $t \in \mathbb{R}$.

To have a well behaved mapping we must assume that

$$\int_0^T h_1(s)ds \neq 0, \text{ and } \int_0^T p_1(s)ds \neq 0. \tag{1.3}$$

Define $P_T = \{(\varphi, \psi) : (\varphi, \psi)(t+T) = (\varphi, \psi)(t)\}$, where both ϕ and ψ are real valued continuous functions on $\mathbb{R}$. Then P_T is a Banach space when endowed with the maximum norm

$$||(x,y)|| = \max\left\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)|\right\}.$$

Lemma 1.1 *Assume (1.2) and (1.3). If $x, y \in P_T$, then $x(t)$ and $y(t)$ is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) &= \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \int_{-\infty}^u a(u,s) f(x(s), y(s))\, dsdu \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} h_2(u) y(u)\, du, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} y(t) &= \int_t^{t+T} \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} \int_{-\infty}^u b(u,s) g(x(s), y(s))\, dsdu \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} p_2(u) x(u)\, du. \end{aligned} \tag{1.5}$$

Proof Let $x, y \in P_T$ be a solution of (1.1). Next we multiply both sides of the first equation in (1.1) with $e^{-\int_0^t h_1(s)ds}$, and then integrate from t to $t+T$ to obtain

$$\begin{aligned} x(t+T)e^{-\int_0^{t+T} h_1(s)ds} \quad & -x(t)e^{-\int_0^t h_1(s)ds} \\ = & \int_t^{t+T} \int_{-\infty}^{u} a(u,s)f(x(s),y(s))\, ds e^{-\int_0^u h_1(s)(s)ds} du \\ & + \int_t^{t+T} h_2(u)y(u)e^{-\int_0^{t+T} h_1(s)ds} du. \end{aligned}$$

Multiply both sides with $e^{\int_0^{t+T} h_1(s)ds}$ and then use the fact that $x(t+T) = x(t)$ and $e^{\int_t^{t+T} h_1(s)ds} = e^{\int_0^T h_1(s)ds}$ to arrive at Eq. (1.4). The proof is complete by reversing every step. The proof of Eq. (1.5) is similar and hence we omit it. □

2. Periodic solutions

Theorem 2.1 (***Schauder's fixed point theorem***) *Let X be a Banach space and K be a closed, bounded, and convex subset of X. If $T : K \to K$ is completely continuous then T has a fixed point in K.*

A map is completely continuous if it is continuous and it maps bounded sets into relatively compact sets.

Let L_1 and L_2 be positive constants such that $0 < L_i < 1,\ i = 1, 2$. Moreover, assume the existence of positive constants M_1, M_2, K_1, and K_2 such that

$$|f(x,y)| \le M_1, \tag{2.1}$$

$$|g(x,y)| \le M_2, \tag{2.2}$$

$$\int_t^{t+T} \Big| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \Big| \int_{-\infty}^{u} |a(u,s)|\, dsdu \le K_1, \tag{2.3}$$

$$\int_t^{t+T} \Big| \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} \Big| \int_{-\infty}^{u} |b(u,s)|\, dsdu \le K_2, \tag{2.4}$$

$$\int_t^{t+T} \Big| \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \Big| |h_2(u)|\, du \le L_1, \tag{2.5}$$

and

$$\int_t^{t+T} \Big| \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} \Big| |p_2(u)|\, du \le L_2. \tag{2.6}$$

Set

$$M = \max\left\{ \frac{M_1K_1}{1-L_1}, \frac{M_2K_2}{1-L_2} \right\}. \tag{2.7}$$

We define a subset $\Omega_{x,y}$ of P_T as follows: $\Omega_{x,y} = \{(x,y) : (x,y) \in P_T \text{ with } ||(x,y)|| \leq M\}$. Then Ω_{xy} is a bounded, closed, and convex subset of P_T. Now for $(x,y) \in \Omega_{xy}$ we can define an operator $E : \Omega_{xy} \to P_T$ by

$$E(x,y)(t) = \left(E_1(x,y)(t), E_2(x,y)(t)\right),$$

where

$$\begin{aligned} E_1(x,y)(t) &= \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \int_{-\infty}^u a(u,s) f(x(s), y(s))\, dsdu \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} h_2(u) y(u)\, du, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} E_2(x,y)(t) &= \int_t^{t+T} \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} \int_{-\infty}^u b(u,s) g(x(s), y(s))\, dsdu \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} p_1(s)ds}}{1 - e^{\int_0^T p_1(s)ds}} p_2(u) x(u)\, du. \end{aligned} \tag{2.9}$$

Theorem 2.2 *Suppose (1.2), (1.3), and (2.1)–(2.6) hold. Then (1.1) has a T-periodic solution.*

Proof It is clear from Lemma 1.1 that $E_1(y)(t+T) = E_1(y)(t)$ and $E_2(x)(t+T) = E_2(x)(t)$. Therefore, $E(x,y)(t+T) = E(x,y)(t)$. Moreover, if $(x,y) \in \Omega_{xy}$, then

$$\begin{aligned} \Big|E_1(x,y)(t)\Big| &\leq \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}}\Big| \int_{-\infty}^u |a(u,s)||f(x(s), y(s))|\, dsdu \\ &+ \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} h_2(u) y(u)\Big|\, du \\ &\leq M_1 K_1 + M L_1. \end{aligned}$$

As a consequence of (2.7),

$$\frac{M_1 K_1}{1 - L_1} \leq M, \quad \text{we have} \quad M_1 K_1 \leq (1 - L_1) M.$$

This implies that

$$\Big|E_1(x,y)(t)\Big| \leq M_1 K_1 + M L_1 \leq (1 - L_1) M + M L_1 = M.$$

In a similar way one can easily show that

$$\Big|E_2(x,y)(t)\Big| \leq M.$$

Thus, E maps Ω_{xy} into itself, i.e. $E(\Omega_{xy}) \subseteq \Omega_{xy}$. Now we have to show that E is continuous. Let $\{(x^l, y^l)\}$ be a sequence in $\Omega_{x,y}$ such that

$$\lim_{l \to \infty} ||(x^l, y^l) - (x,y)|| = 0.$$

Since $\Omega_{x,y}$ is closed, we have $(x,y) \in \Omega_{x,y}$. Then by the definition of E we have

$$\big\|E(x^l,y^l) - E(x,y)\big\| = \max\Big\{\max_{t\in[0,T]}\big|E_1(x^l,y^l)(t) - E_1(x,y)(t)\big|$$

$$, \max_{t\in[0,T]}\big|E_2(x^l,y^l)(t) - E_2(x,y)(t)\big|\Big\},$$

in which

$$\begin{aligned}
\Big|E_1(x^l,y^l)(t) - E_1(x,y)(t)\Big| &= \Big|\int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^u a(u,s) f(x^l(s),y^l(s))\, dsdu \\
&\quad - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} \int_{-\infty}^u a(u,s) f(x(s),y(s))\, dsdu \\
&\quad + \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} h_2(u) y^l(u)\, du \\
&\quad - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} h_2(u) y(u)\, du\Big| \\
&\le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\Big| \left(\int_{-\infty}^u |a(u,s)||f(x^l(s),y^l(s)) - f(x(s),y(s))| ds\right) du \\
&\quad + \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}} h_2(u)\Big| \Big|y^l(u) - y(u)\Big|\, du
\end{aligned}$$

The continuity of f along with the Lebesgue dominated convergence theorem implies that

$$\lim_{l\to\infty} \max_{t\in[0,T]} |E_1(x^l,y^l)(t) - E_1(x,y)(t)| = 0.$$

By a similar argument one can easily argue that

$$\lim_{l\to\infty} \max_{t\in[0,T]} |E_2(x^l,y^l)(t) - E_2(x,y)(t)| = 0.$$

Thus,

$$\lim_{l\to\infty} ||E(x^l,y^l) - E(x,y)|| = 0.$$

This shows that E is a continuous map. To show that the map E is completely continuous, we will show that $E(\Omega_{x,y})$ is relatively compact. We already know from Theorem 2.2 that $E(\Omega_{xy}) \subseteq \Omega_{xy}$, which means $E(\Omega_{xy})$ is uniformly bounded because Ω_{xy} is uniformly bounded. It is an easy exercise to show that for all $(x,y) \in \Omega_{xy}$, there exists a constant $L > 0$ such that $|\frac{d}{dt}E_1(x,y)(t)| \le L$, and $|\frac{d}{dt}E_2(x,y)(t)| \le L$. This means $|\frac{d}{dt}E(x,y)(t)| \le L$. Therefore the set $E(\Omega_{xy})$ is equicontinuous, and hence by Arzela–Ascoli's theorem, it is relatively compact.

By Schauder's fixed point theorem, we conclude that there exist $(x,y) \in \Omega_{x,y}$ such that $(x,y) = E(x,y)$. □

In the next theorem we relax condition (2.2).

Theorem 2.3 *Suppose (1.2), (1.3), (2.1), and (2.3)–(2.6) hold. In addition, we assume the existence of continuous nondecreasing function G such that*

$$|g(x,y)| \leq g(|x|,y) \leq QG(|x|) \text{ for some positive constant } Q \tag{2.10}$$

and for $u>0$ we ask that

$$\frac{G(u)}{u} \leq \frac{1-L_2}{K_2 Q}. \tag{2.11}$$

Then (1.1) has a T-periodic solution.

Proof Set

$$M = \max\left\{\frac{M_1K_1}{1-L_1}, \frac{K_2QG(M)}{1-L_2}\right\}. \tag{2.12}$$

Note that due to (2.11) we have

$$M \geq \frac{K_2QG(M)}{1-L_2}$$

and hence (2.11) is well defined. For $(x,y)\in\Omega_{x,y}$, we have by the proof of the previous theorem that

$$\Big|E_1(x,y)(t)\Big| \leq M.$$

Thus,

$$\begin{aligned}
\Big|E_2(x,y)(t)\Big| &\leq \int_t^{t+T}\Big|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}\Big|\int_{-\infty}^{u}|b(u,s)||g(x(s),y(s))|\,dsdu \\
&\quad + \int_t^{t+T}|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}||p_2(u)|x(u)\,du \\
&\leq \int_t^{t+T}\Big|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}\Big|\int_{-\infty}^{u}|b(u,s)|g(|E_1(x(s),y(s))|,y(s))\,dsdu \\
&\quad + \int_t^{t+T}|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}||p_2(u)||E_1(x(s),y(s))|\,du \\
&\leq Q\int_t^{t+T}\Big|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}\Big|\int_{-\infty}^{u}|b(u,s)|G(|E_1(x(s),y(s))|)\,dsdu \\
&\quad + \int_t^{t+T}|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}||p_2(u)||E_1(x(s),y(s))|\,du \\
&\leq Q\int_t^{t+T}\Big|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}\Big|\int_{-\infty}^{u}|b(u,s)|G(M)\,dsdu \\
&\quad +M\int_t^{t+T}|\frac{e^{\int_u^{t+T}p_1(s)ds}}{1-e^{\int_0^T p_1(s)ds}}||p_2(u)|\,du \\
&\leq K_2QG(M)+ML_2 \\
&\leq M(1-L_2)+ML_2 = M.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

In the next theorem we relax condition (2.1).

Theorem 2.4 *Suppose (1.2), (1.3), (2.2), and (2.3)–(2.6) hold. In addition, we assume the existence of continuous nondecreasing function G such that*

$$|f(x,y)| \leq f(x,|y|) \leq RW(|y|) \text{ for some positive constant } R \tag{2.13}$$

and for $u > 0$ we ask that

$$\frac{W(u)}{u} \leq \frac{1-L_1}{K_1 R}. \tag{2.14}$$

Then (1.1) has a T-periodic solution.

Proof Set

$$M = \max\left\{\frac{K_1 RW(M)}{1-L_1}, \frac{M_2 K_2}{1-L_2}\right\}. \tag{2.15}$$

Note that due to (2.14) we have

$$M \geq \frac{K_1 RW(M)}{1-L_1}$$

and hence (2.14) is well defined. For $(x,y) \in \Omega_{x,y}$, we have by the proof of the previous theorem that

$$\Big|E_2(x,y)(t)\Big| \leq M.$$

Thus,

$$\begin{aligned}
\Big|E_1(x,y)(t)\Big| \quad &\leq \quad \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\Big| \int_{-\infty}^{u} |a(u,s)||f(x(s),y(s))|\, dsdu \\
&\quad + \int_t^{t+T} |\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}||h_2(u)|y(u)\, du \\
&\leq \quad \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\Big| \int_{-\infty}^{u} |a(u,s)|f(x(s),|E_2(x(s),y(s))|)\, dsdu \\
&\quad + \int_t^{t+T} |\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}||h_2(u)||E_2(x(s),y(s))|\, du \\
&\leq \quad R\int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\Big| \int_{-\infty}^{u} |a(u,s)|W(|E_2(x(s),y(s))|)\, dsdu \\
&\quad + \int_t^{t+T} |\frac{e^{\int_u^{t+T} h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}||h_2(u)||E_2(x(s),y(s))|\, du
\end{aligned}$$

$$\begin{aligned}
&\leq \quad R\int_t^{t+T}\Big|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}\Big|\int_{-\infty}^u |a(u,s)|W(M)\,dsdu \\
&\quad\ +M\int_t^{t+T}|\frac{e^{\int_u^{t+T}h_1(s)ds}}{1-e^{\int_0^T h_1(s)ds}}||h_2(u)|\,du \\
&\leq \quad K_1RW(M)+ML_1 \\
&\leq \quad M(1-L_1)+ML_1=M.
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

3. Asymptotically periodic solutions

In this section, we show that under mild conditions one obtains asymptotically periodic solutions.

Definition 3.1 *A function $x(t)$ is called asymptotically T-periodic if there exist two functions $x_1(t)$ and $x_2(t)$ such that $x_1(t)$ is T-periodic, $\lim_{t\to\infty} x_2(t)=0$ and $x(t)=x_1(t)+x_2(t)$ for all t.*

In this section we do not assume the periodicity condition on the functions $a(t,s)$ and $b(t,s)$. We only assume $h_1(t)$ and $p_1(t)$ are T-periodic, and

$$\int_0^T h_1(s)ds=0 \text{ and } \int_0^T p_1(s)ds=0. \tag{3.1}$$

Since h and p are T-periodic, there are constants m_k, M_k^*, $k=1,2$, such that

$$m_1\leq e^{\int_0^t h_1(s)ds}\leq M_1^* \text{ and } m_2\leq e^{\int_0^t p_1(s)ds}\leq M_2^*. \tag{3.2}$$

Furthermore, we assume that there are positive numbers A and B such that

$$\int_0^\infty\int_{-\infty}^u |a(u,s)|ds\,du\leq A \text{ and } \int_0^\infty\int_{-\infty}^u |b(u,s)|ds\,du\leq B \tag{3.3}$$

In addition, we suppose that

$$\lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |a(u,s)|ds\,du=\lim_{t\to\infty}\int_t^\infty\int_{-\infty}^u |b(u,s)|ds\,du=0, \tag{3.4}$$

$$\int_t^\infty |h_2(u)|du\to 0 \text{ as } t\to\infty, \tag{3.5}$$

$$\int_t^\infty |p_2(u)|du\to 0 \text{ as } t\to\infty, \tag{3.6}$$

and for positive constants M_3^* and M_4^* we ask that

$$\int_0^\infty |h_2(u)|du\leq M_3^* \tag{3.7}$$

and

$$\int_0^\infty |p_2(u)|du \le M_4^*. \tag{3.8}$$

Finally, we make the assumption that

$$1 - M_3^* M_1^* m_1^{-1} > 0 \tag{3.9}$$

and

$$1 - M_4^* M_2^* m_2^{-1} > 0. \tag{3.10}$$

Theorem 3.2 *Suppose that (2.1), (2.2), and (3.1)–(3.10) hold. Then system (1.1) has asymptotically T-periodic solution (x, y) satisfying*

$$x(t) := x_1(t) + x_2(t)$$
$$y(t) := y_1(t) + y_2(t),$$

where

$$x_1(t) = c_1 e^{\int_0^t h_1(s)ds}, \qquad y_1(t) = c_2 e^{\int_0^t p_1(s)ds},\ t \in \mathbb{R}$$

for arbitrary but fixed nonzero constants c_1, c_2 and

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} y_2(t) = 0.$$

Proof Define $P_T^* = \{(\varphi, \psi) : \varphi = \varphi_1 + \varphi_2, \psi = \psi_1 + \psi_2, (\varphi_1, \psi_1)(t+T) = (\varphi_1, \psi_1)(t),$ and $(\varphi_2, \psi_2)(t) \to (0,0)$ as $t \to \infty\}$. Then P_T^* is a Banach space when endowed with the maximum norm

$$||(x, y)|| = \max\left\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)|\right\}.$$

We define a subset $\Omega_{x,y}$ of P_T^* as follows. For a constant W^* to be defined later in the proof, let $\Omega_{x,y} = \{(x, y) : (x, y) \in P_T^*$ with $||(x, y)|| \le W^*\}$. Then Ω_{xy} is a bounded, closed, and convex subset of P_T^*. Now for $(x, y) \in \Omega_{xy}$ we can define an operator $F : \Omega_{xy} \to P_T^*$ by

$$F(x, y)(t) = (F_1(y)(t), F_2(x)(t)),$$

where

$$\begin{aligned} F_1(y)(t) &= c_1 e^{\int_0^t h_1(s)ds} - \int_t^\infty h_2(u) \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} y(u)du \\ &\quad - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} a(u, s) f(x(s), y(s))ds\, du, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} F_2(x)(t) &= c_2 e^{\int_0^t p_1(s)ds} - \int_t^\infty p_2(u) \frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} y(u)du \\ &\quad - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} b(u, s) g(x(s), y(s))ds\, du, \end{aligned} \tag{3.12}$$

We will show that the mapping F has a fixed point in Ω_{xy}. Set

$$W^* = \max\{\frac{M_1^* m_1^{-1} M_1 A + c_1 M_1}{1 - M_3^* M_1^* m_1^{-1}}, \frac{M_2^* m_2^{-1} M_2 B + c_2 M_2}{1 - M_4^* M_2^* m_2^{-1}}\}.$$

We note that W^* is well defined due to (3.9) and (3.10). First, we demonstrate that $F(\Omega_{x,y}) \subseteq \Omega_{x,y}$. If $\{(x,y)\} \in \Omega_{x,y}$, then by (3.9) we have

$$\begin{aligned} \left| F_1(y)(t) - c_1 e^{\int_0^t h_1(s)ds} \right| &\leq W^* M_3^* M_1^* m_1^{-1} + M_1^* m_1^{-1} M_1 \int_t^\infty \int_{-\infty}^u |a(u,s)|\, ds\, du \\ &\leq W^* M_3^* M_1^* m_1^{-1} + M_1^* m_1^{-1} M_1 \int_0^\infty \int_{-\infty}^u |a(u,s)|\, ds\, du \\ &= W^* M_3^* M_1^* m_1^{-1} + M_1^* m_1^{-1} M_1 A, \end{aligned} \tag{3.13}$$

and in a similar way we have

$$\left| F_2(x)(t) - c_2 e^{\int_0^t p_1(s)ds} \right| \leq W^* M_4^* M_2^* m_2^{-1} + M_2^* m_2^{-1} M_2 B. \tag{3.14}$$

This implies that

$$|F_1(y)(t)| \leq W^* M_3^* M_1^* m_1^{-1} + M_1^* m_1^{-1} M_1 A + c_1 M_1 \leq W^*$$

and

$$|F_2(x)(t)| \leq W^* M_4^* M_2^* m_2^{-1} + M_2^* m_2^{-1} M_2 B + c_2 M_2 \leq W^*.$$

Hence, $F(\Omega_{x,y}) \subseteq \Omega_{x,y}$ as desired. The work to show that F is completely continuous is similar to the corresponding work in Theorem 2.1, and hence we omit it here. Therefore, by Schauder's fixed point theorem, there exists a fixed point $(x,y) \in \Omega_{xy}$ such that $F(x,y)(t) = (F_1(y)(t), F_2(x)(t)) = (x(t), y(t))$. Now we show that this fixed point is a solution of (1.1). Let

$$\begin{aligned} x(t) &= c_1 e^{\int_0^t h_1(s)ds} - \int_t^\infty h_2(u) \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} y(u) du \\ &\quad - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} a(u,s) f(x(s), y(s)) ds\, du. \end{aligned}$$

Then a differentiation with respect to t gives

$$
\begin{aligned}
x'(t) &= c_1h_1(t)e^{\int_0^t h_1(s)ds} + h_2(t)y(t) + \int_{-\infty}^{t} \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^t h_1(l)dl}} a(t,s)f(x(s),y(s))ds \\
&\quad -h_1(t)\int_t^\infty \int_{-\infty}^{u} \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} a(u,s)f(x(s),y(s))ds\, du \\
&\quad -h_1(t)\int_t^\infty h_2(u)\frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} y(u)du \\
&= h_1(t)\big[c_1e^{\int_0^t h_1(s)ds} - \int_t^\infty \int_{-\infty}^{u} \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} a(u,s)f(x(s),y(s))ds\, du \\
&\quad - \int_t^\infty h_2(u)\frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} y(u)du\big] \\
&\quad + \int_{-\infty}^{t} a(t,s)f(x(s),y(s))ds + h_2(t)y(t) \\
&= h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^{t} a(t,s)f(x(s),y(s))ds.
\end{aligned}
$$

In a similar fashion we can easily show that if

$$
\begin{aligned}
y(t) &= c_2e^{\int_0^t p1(s)ds} - \int_t^\infty p_2(u)\frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} y(u)du \\
&\quad - \int_t^\infty \int_{-\infty}^{u} \frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} b(u,s)g(x(s),y(s))ds\, du,
\end{aligned}
$$

then it is a solution to the second equation in (1.1).

For an arbitrary fixed point $(x,y)\in \Omega_{xy}$ of F, we obtain from (3.4), (3.5), and (3.6),

$$\lim_{t\to\infty} |x(t) - c_1x_1(t)| = \lim_{t\to\infty} |F_1(y)(t) - c_1x_1(t)| = 0,$$

and

$$\lim_{t\to\infty} |y(t) - c_1y_1(t)| = \lim_{t\to\infty} |F_2(x)(t) - c_1y_1(t)| = 0.$$

By letting

$$x_2(t) = -\int_t^\infty h_2(u)\frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} y(u)du - \int_t^\infty \int_{-\infty}^{u} \frac{e^{\int_0^t h_1(l)dl}}{e^{\int_0^u h_1(l)dl}} a(u,s)f(y(s))ds\, du$$

and

$$y_2(t) = -\int_t^\infty p_2(u)\frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} y(u)du - \int_t^\infty \int_{-\infty}^{u} \frac{e^{\int_0^t p_1(l)dl}}{e^{\int_0^u p_1(l)dl}} b(u,s)g(x(s))ds\, du$$

we see that $(x(t),y(t))$ given by

$$x(t) := x_1(t) + x_2(t)$$

$$y(t) := y_1(t) + y_2(t)$$

is an asymptotically T-periodic solution of (1.1). Note that by (3.4), (3.5), and (3.6),

$$\lim_{t\to\infty}|x_2(t)| \leq W^* M_1^* m_1^{-1} \lim_{t\to\infty}\int_t^\infty |h_2(u)|du + M_1^* m_1^{-1} M_1 \lim_{t\to\infty}\int_t^\infty \int_{-\infty}^u |a(u,s)|ds\, du = 0.$$

Hence,

$$\lim_{t\to\infty} x_2(t) = 0.$$

Similarly,

$$\lim_{t\to\infty} y_2(t) = 0.$$

Finally, we show that x_1 and y_1 are T-periodic. From (3.1), one can see

$$\begin{aligned} x_1(t+T) &= c_1 e^{\int_0^{t+T} h_1(s)ds} \\ &= c_1 e^{\int_0^t h_1(s)ds + \int_t^{t+T} h_1(s)ds} \\ &= c_1 e^{\int_0^t h_1(s)ds} e^{\int_t^{t+T} h_1(s)ds} \\ &= c_1 e^{\int_0^t h_1(s)ds} \\ &= x_1(t). \end{aligned}$$

Similarly, $y_1(t)$ is T-periodic. □

We end this paper with a example in which we show the existence of an asymptotically periodic solution.

Example 3.3 *Let $h_1(t) = p_1(t) = cos(t), a(t,s) = b(t,s) = e^{-2t+s}$, and $h_2(t) = p_2(t) = \dfrac{2t}{(t^2+2)^2}$. Also assume that $f(x,y) = sin(x+y)$, and $g(x,y) = cos(x+y)$. Then all conditions of Theorem 3.2 are satisfied and hence the system*

$$\begin{cases} x'(t) = cos(t)x(t) + \frac{2t}{(t^2+2)^2} y(t) + \int_{-\infty}^t e^{-2t+s} sin(x(s)+y(s))ds, \\ y'(t) = cos(t)y(t) + \frac{2t}{(t^2+2)^2} x(t) + \int_{-\infty}^t e^{-2t+s} cos(x(s)+y(s))ds \end{cases}$$

has an asymptotically 2π-periodic solution.

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