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On the isospectrality of the scalar energy-dependent Schrödinger problems

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Abstract: In this study, we discuss the inverse spectral problem for the energy-dependent Schrödinger equation on a finite interval. We construct an isospectrality problem and obtain some relations between constants in boundary conditions of the problems that constitute isospectrality. Above all, we obtain degeneracy of $K(x,t) - K_0(x,t)$ and $L(x,t) - L_0(x,t)$ by using a different approach. Some of the main results of our study coincide with results reported by Jodeit and Levitan. However, the method to obtain degeneracy is completely different. Furthermore, we consider all above results for the nonisospectral case.

Key words: Energy-dependent Schrödinger equation, isospectrality problem, Gelfand–Levitan equation

1. Introduction

In this study, we consider the energy-dependent Schrödinger equation

$$-y'' + (2\lambda p(x) + q(x))y = \lambda^2 y, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{1.3}$$

on $(0,\pi)$, where λ is a spectral parameter, $p(x) \in W_2^2(0,\pi)$, $q(x) \in W_2^1(0,\pi)$; h and H are real numbers.

Let us denote the spectrum of this problem by λ_n $(n = 0, \pm 1, \pm 2, ...)$. $\{\lambda_n\}_{-\infty}^{\infty}$ are real, nonzero, simple, and satisfy the asymptotic formula [13]

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n}, \tag{1.4}$$

where $c_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} p(t)dt$, $c_1 = \dfrac{1}{\pi}\left(h + H + \displaystyle\int_0^{\pi}\left(q(t) + p^2(t)\right)dt\right)$, $\displaystyle\sum_n |c_{1,n}|^2 < \infty$.

To explain some phases for the results of our study, it will be useful to sketch the main stages of progress and a physical point of view on the classical inverse spectral problem for the energy-dependent Schrödinger

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equation. Inverse spectral problems frequently appear in classical and quantum mechanics. The most extensive sample is modeling of the motion of massless particles such as photons whereby the Klein–Gordon equations, which can be reduced to energy-dependent Schrödinger equations. The corresponding evolution equations are used to express interactions of colliding spinless particles. The equations under consideration also occur in modeling of mechanical systems vibrations in viscous media; see [33]. The terminology "energy-dependent potentials" is widely accepted in the physical and mathematical literature since the spectral parameter is related to the energy of the system.

Eq. (1.1) appeared earlier in the context of inverse scattering. For instance, Jaulent and Jean studied inverse scattering problems for energy-dependent Schrödinger equations (see [18–21]). The inverse spectral problem for this equation has attracted the attention of many mathematicians. Such types of problems with $p(x) \in W_2^2(0,\pi)$, $q(x) \in W_2^1(0,\pi)$ and with Robin boundary conditions were discussed by Gasymov and Guseinov in [5] containing no proofs. The inverse spectral problem for Eq. (1.1) with $p(x) \in W_2^1(0,1)$ and $q(x) \in L_2(0,1)$ and under (quasi)-periodic boundary conditions or interior given data were considered in [12,14,29]. This kind of problem was considered in various studies (see [1,2,9,11–16,25–27,30,32,34,35]).

Let us consider the following isospectrality problem for real valued energy dependent Schrödinger problems (1.1)–(1.3) and

$$-y'' + (2\lambda p(x) + q_0(x))y = \lambda^2 y, \tag{1.5}$$

$$y'(0) - h_0 y(0) = 0, \quad (h \neq h_0), \tag{1.6}$$

$$y'(\pi) + H_0 y(\pi) = 0, \quad (H \neq H_0), \tag{1.7}$$

on $(0,\pi)$, where $q_0(x) \in W_2^1(0,\pi)$; h_0 and H_0 are real numbers.

Usually, one says that two problems are isospectral when they have the same spectrum (including multiplicities). Using the Gelfand–Levitan integral equation and transmutation operator, the isospectrality problem for Sturm–Liouville operator was studied in scalar and vectorial cases [4,17,22–24]. In this study, we give some results on the isospectrality problem for the energy-dependent Schrödinger equation on a finite interval. Furthermore, we obtain some formulas for the kernels $K(x,t) - K_0(x,t)$ and $L(x,t) - L_0(x,t)$ by using Gelfand–Levitan integral equations with a different perspective. As a result of these formulas, we get the degeneracy of the kernels. An enormous number of papers have been published about isospectrality problems (see [3,6–8,10,31]), but, as far as we know, none similar to our study has appeared.

This study is organized as follows. In next section, we consider an isospectrality problem and give some well-known results about the solutions and kernels of the above problems. In section 3, we prove degeneracy of $K(x,t) - K_0(x,t)$ and $L(x,t) - L_0(x,t)$. In section 4, we obtain a new formula for norming constants to get some important relations between the constants in boundary conditions. Finally, we give some conclusions in section 5.

2. Preliminaries

Before turning to the isospectrality problem for the energy-dependent Schrödinger equation, we need to recall some important theorems and results about this type of problem to use in our main results.

Theorem 2.1 *[13] Let $\phi(x,\lambda)$ be the solution of Eq. (1.1) with $\phi(0,\lambda) = 1$, $\phi'(0,\lambda) = h$. Then there exist real-valued functions $K(x,t)$ and $L(x,t)$ having $m+1$ square-integrable derivatives with regard to the both* *variables such that*

$$\phi(x,\lambda)=\cos\left(\lambda x-\alpha(x)\right)+\int\limits_0^x K(x,t)\cos(\lambda t)dt+\int\limits_0^x L(x,t)\sin(\lambda t)dt, \tag{2.1}$$

$$\alpha(x)=xp(0)+2\int\limits_0^x\left(K(\xi,\xi)\sin\alpha(\xi)-L(\xi,\xi)\cos\alpha(\xi)\right)d\xi, \tag{2.2}$$

$$\partial_{xx}K(x,t)-2p(x)\partial_t L(x,t)-q(x)K(x,t)=\partial_{tt}K(x,t), \tag{2.3}$$

$$\partial_{xx}L(x,t)+2p(x)\partial_t K(x,t)-q(x)L(x,t)=\partial_{tt}L(x,t), \tag{2.4}$$

$$q(x)=-p^2(x)+2\left[K(x,x)\cos\alpha(x)+L(x,x)\sin\alpha(x)\right]', \tag{2.5}$$

$$K(0,0)=h,\ \ L(x,0)=0,\ \ \partial_t K(x,t)\left.\right|_{t=0}=0, \tag{2.6}$$

$$K(x,-t)=0,\ \ L(x,-t)=0\quad for\ \ t>0, \tag{2.7}$$

$$L(x,t)=0,\ \alpha(x)=\int\limits_0^x p(t)dt,\quad for\ \ t>x. \tag{2.8}$$

Conversely, if given $K(x,t)$ and $L(x,t)$ have second order square-integrable partial derivatives satisfying Eqs. (2.3) and (2.4) and conditions (2.2) and (2.5)–(2.8), then the function $\phi(x,\lambda)$, which is constructed by formula (2.1), is the solution of Eq. (1.1) with (1.2).

Similarly, let $\phi_0(x,\lambda)$ be the solution of (1.5) with $\phi_0(0,\lambda)=1,\ \phi_0'(0,\lambda)=h_0$. Therefore, we have [13]

$$\phi_0(x,\lambda)=\cos\left(\lambda x-\alpha(x)\right)+\int\limits_0^x K_0(x,t)\cos(\lambda t)dt+\int\limits_0^x L_0(x,t)\sin(\lambda t)dt, \tag{2.9}$$

where kernels $K_0(x,t)$ and $L_0(x,t)$ have properties similar to those of $K(x,t)$ and $L(x,t):$

$$\partial_{xx}K_0(x,t)-2p(x)\partial_t L_0(x,t)-q_0(x)K_0(x,t)=\partial_{tt}K_0(x,t), \tag{2.10}$$

$$\partial_{xx}L_0(x,t)+2p(x)\partial_t K_0(x,t)-q_0(x)L_0(x,t)=\partial_{tt}L_0(x,t), \tag{2.11}$$

$$q_0(x)=-p^2(x)+2\left[K_0(x,x)\cos\alpha(x)+L_0(x,x)\sin\alpha(x)\right]', \tag{2.12}$$

$$K_0(0,0)=h_0,\ \ L_0(x,0)=0,\ \ \partial_t K_0(x,t)\left.\right|_{t=0}=0, \tag{2.13}$$

$$K_0(x,-t)=0,\ \ L_0(x,-t)=0\quad for\ \ t>0,$$

$$L_0(x,t) = 0, \;\; \alpha(x) = \int_0^x p(t)dt, \quad for \quad t > x.$$

Now we shall express the Gelfand–Levitan integral equations proved by [13], where

$$\alpha_n = \int_0^\pi \phi^2(x,\lambda_n)dx - \frac{1}{\lambda_n}\int_0^\pi p(x)\phi^2(x,\lambda_n)dx, \tag{2.14}$$

$$\alpha_{n,0} = \int_0^\pi \phi_0^2(x,\lambda_n)dx - \frac{1}{\lambda_n}\int_0^\pi p(x)\phi_0^2(x,\lambda_n)dx, \tag{2.15}$$

are normalizing constants of the problems (1.1)–(1.3) and (1.5)–(1.7), respectively.

Theorem 2.2 *[13] The kernels $K(x,t)$ and $L(x,t)$ involved in the representation (2.1) satisfy the following system of Gelfand–Levitan integral equations:*

$$F_{11}(x,t)\cos\alpha(x) + F_{12}(x,t)\sin\alpha(x) + K(x,t) + \int_0^x K(x,\xi)F_{11}(\xi,t)d\xi + \int_0^x L(x,\xi)F_{12}(\xi,t)d\xi = 0, \tag{2.16}$$

$$F_{21}(x,t)\cos\alpha(x) + F_{22}(x,t)\sin\alpha(x) + L(x,t) + \int_0^x K(x,\xi)F_{21}(\xi,t)d\xi + \int_0^x L(x,\xi)F_{22}(\xi,t)d\xi = 0, \tag{2.17}$$

where $0 \leq t < x$ and

$$F_{11}(x,t) = \frac{1}{\pi}\cos(c_0 x)\cos(c_0 t) + \sum_n \left\{ \frac{1}{2\alpha_n}\cos(\lambda_n x)\cos(\lambda_n t) - \frac{1}{\pi}\cos\left((n+c_0)x\right)\cos\left((n+c_0)t\right) \right\}, \tag{2.18}$$

$$F_{12}(x,t) = \frac{1}{\pi}\sin(c_0 x)\cos(c_0 t) + \sum_n \left\{ \frac{1}{2\alpha_n}\sin(\lambda_n x)\cos(\lambda_n t) - \frac{1}{\pi}\sin\left((n+c_0)x\right)\cos\left((n+c_0)t\right) \right\}, \tag{2.19}$$

$$F_{21}(x,t) = \frac{1}{\pi}\cos(c_0 x)\sin(c_0 t) + \sum_n \left\{ \frac{1}{2\alpha_n}\cos(\lambda_n x)\sin(\lambda_n t) - \frac{1}{\pi}\cos\left((n+c_0)x\right)\sin\left((n+c_0)t\right) \right\}, \tag{2.20}$$

$$F_{22}(x,t) = \frac{1}{\pi}\sin(c_0 x)\sin(c_0 t) + \sum_n \left\{ \frac{1}{2\alpha_n}\sin(\lambda_n x)\sin(\lambda_n t) - \frac{1}{\pi}\sin\left((n+c_0)x\right)\sin\left((n+c_0)t\right) \right\}, \tag{2.21}$$

where $n = \pm 0, \pm 1, \pm 2, \ldots$. Similarly, we obtain

$$\tilde{F}_{11}(x,t)\cos\alpha(x) + \tilde{F}_{12}(x,t)\sin\alpha(x) + K_0(x,t) + \int_0^x K_0(x,\xi)\tilde{F}_{11}(\xi,t)d\xi + \int_0^x L_0(x,\xi)\tilde{F}_{12}(\xi,t)d\xi = 0, \tag{2.22}$$

$$\tilde{F}_{21}(x,t)\cos\alpha(x)+\tilde{F}_{22}(x,t)\sin\alpha(x)+L_0(x,t)+\int\limits_0^x K_0(x,\xi)\tilde{F}_{21}(\xi,t)d\xi+\int\limits_0^x L_0(x,\xi)\tilde{F}_{22}(\xi,t)d\xi=0, \tag{2.23}$$

for the problem (1.5)–(1.7), where $0\le t<x$ *and*

$$\tilde{F}_{11}(x,t)=\frac{1}{\pi}\cos(c_0x)\cos(c_0t)+\sum_n\left\{\frac{1}{2\alpha_{n,0}}\cos(\lambda_nx)\cos(\lambda_nt)-\frac{1}{\pi}\cos\left((n+c_0)x\right)\cos\left((n+c_0)t\right)\right\}, \tag{2.24}$$

$$\tilde{F}_{12}(x,t)=\frac{1}{\pi}\sin(c_0x)\cos(c_0t)+\sum_n\left\{\frac{1}{2\alpha_{n,0}}\sin(\lambda_nx)\cos(\lambda_nt)-\frac{1}{\pi}\sin\left((n+c_0)x\right)\cos\left((n+c_0)t\right)\right\}, \tag{2.25}$$

$$\tilde{F}_{21}(x,t)=\frac{1}{\pi}\cos(c_0x)\sin(c_0t)+\sum_n\left\{\frac{1}{2\alpha_{n,0}}\cos(\lambda_nx)\sin(\lambda_nt)-\frac{1}{\pi}\cos\left((n+c_0)x\right)\sin\left((n+c_0)t\right)\right\}, \tag{2.26}$$

$$\tilde{F}_{22}(x,t)=\frac{1}{\pi}\sin(c_0x)\sin(c_0t)+\sum_n\left\{\frac{1}{2\alpha_{n,0}}\sin(\lambda_nx)\sin(\lambda_nt)-\frac{1}{\pi}\sin\left((n+c_0)x\right)\sin\left((n+c_0)t\right)\right\}. \tag{2.27}$$

Theorem 2.3 *[13] The system of Gelfand–Levitan integral equations (2.16), (2.17) ((2.22), (2.23)) has only a unique solution* $K(x,t)$ *and* $L(x,t)$ *(*$K_0(x,t)$ *and* $L_0(x,t)$*) for any continuous function* $\alpha(x)$*.*

3. Isospectrality problem and degeneracy of $K(x,t)-K_0(x,t)$ and $L(x,t)-L_0(x,t)$

In this section, we prove that $\phi(x,\lambda)$, which depends on $\phi_0(x,\lambda)$, is a solution of Eq. (1.1). Moreover, we obtain some important relations between h, H and h_0, H_0, which includes norming constants. Then we prove degeneracy of $K(x,t)-K_0(x,t)$ and $L(x,t)-L_0(x,t)$ by using Gelfand–Levitan integral equations, which is different from the classical Levitan method [28].

Theorem 3.1 *If* $K(x,t)$ *and* $L(x,t)$ *(*$K_0(x,t)$ *and* $L_0(x,t)$*) are solutions of the system of Gelfand–Levitan integral equations (2.16), (2.17) ((2.22), (2.23)), then they are also the solutions of the problem (2.3)–(2.6) ((2.10)–(2.13)), respectively and the following statements hold: (3.1)*

$$\phi(x,\lambda)=\phi_0(x,\lambda)+\int\limits_0^x(K(x,t)-K_0(x,t))\cos(\lambda t)dt+\int\limits_0^x(L(x,t)-L_0(x,t))\sin(\lambda t)dt, \tag{3.1}$$

is a solution of Eq. (1.1) for every complex λ*, where*

$$q(x)=q_0(x)+2\frac{d}{dx}\left\{(K(x,x)-K_0(x,x))\cos\alpha(x)+(L(x,x)-L_0(x,x))\sin\alpha(x)\right\}.$$

2) The conditions $\phi(0,\lambda)=1$, $\phi'(0,\lambda)=h$ *are satisfied for* $\phi(x,\lambda)$, *where*

$$h=h_0-\frac{1}{2}\sum_n\left(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n,0}}\right).$$

3) The relation

$$H=H_0+\sum_n\frac{(-1)^n}{2}\left(\frac{\phi(\pi,\lambda_n)}{\alpha_n}-\frac{\phi_0(\pi,\lambda_n)}{\alpha_{n,0}}\right) \tag{3.2}$$

holds when $x=\pi$.

Conclusion 3.2 *Since* $\{\phi(x,\lambda_n)\}$ *implies a complete system of eigenfunctions of the problem (1.1)–(1.3) induced by the problem (1.5)–(1.7), these problems are isospectral.*

To give the proof of Theorem 3.1, we need to obtain some new formulas for kernels $K(x,t)$, $K_0(x,t)$, $L(x,t)$, and $L_0(x,t)$ by using $F_{ij}(x,t)\,(i,j=1,2)$.

Theorem 3.3 $K(x,t)-K_0(x,t)$ *and* $L(x,t)-L_0(x,t)$ *are degenerate for the problems (1.1)–(1.3) and (1.5)–(1.7), respectively.*

Proof By (2.16), we get

$$K(x,t)=-F_{11}(x,t)\cos\alpha(x)-F_{12}(x,t)\sin\alpha(x)-\int_0^x K(x,\xi)F_{11}(\xi,t)d\xi-\int_0^x L(x,\xi)F_{12}(\xi,t)d\xi,$$

and by (2.18)–(2.19), we obtain

$$\begin{aligned}K(x,t)=&-\frac{\cos(c_0t)}{\pi}\left\{\cos(c_0x-\alpha(x))+\int_0^x K(x,\xi)\cos(c_0\xi)d\xi+\int_0^x L(x,\xi)\sin(c_0\xi)d\xi\right\}\\&-\sum_n\frac{\cos(\lambda_nt)}{2\alpha_n}\left\{\cos(\lambda_nx-\alpha(x))+\int_0^x K(x,\xi)\cos(\lambda_n\xi)d\xi+\int_0^x L(x,\xi)\sin(\lambda_n\xi)d\xi\right\}\\&+\sum_n\frac{\cos((n+c_0)t)}{\pi}\left\{\cos((n+c_0)x-\alpha(x))+\int_0^x K(x,\xi)\cos((n+c_0)\xi))d\xi\right\}\\&+\sum_n\frac{\cos((n+c_0)t)}{\pi}\left\{\int_0^x L(x,\xi)\sin((n+c_0)\xi))d\xi\right\}.\end{aligned} \tag{3.3}$$

In the same way, by (2.22), (2.24), and (2.25), we get

$$K_0(x,t) = -\frac{\cos(c_0 t)}{\pi}\left\{\cos(c_0 x - \alpha(x)) + \int_0^x K_0(x,\xi)\cos(c_0\xi)d\xi + \int_0^x L_0(x,\xi)\sin(c_0\xi)d\xi\right\} \tag{3.4}$$

$$-\sum_n \frac{\cos(\lambda_n t)}{2\alpha_{n,0}}\left\{\cos(\lambda_n x - \alpha(x)) + \int_0^x K_0(x,\xi)\cos(\lambda_n\xi)d\xi + \int_0^x L_0(x,\xi)\sin(\lambda_n\xi)d\xi\right\}$$

$$+\sum_n \frac{\cos((n+c_0)t)}{\pi}\left\{\cos((n+c_0)x - \alpha(x)) + \int_0^x K_0(x,\xi)\cos((n+c_0)\xi))d\xi\right\}$$

$$+\sum_n \frac{\cos((n+c_0)t)}{\pi}\left\{\int_0^x L_0(x,\xi)\sin((n+c_0)\xi))d\xi\right\}.$$

Subtracting Eqs. (3.3) and (3.4) leads to

$$K(x,t) - K_0(x,t) = -\frac{\cos(c_0 t)}{\pi}\left\{\int_0^x (K-K_0)\cos(c_0\xi)d\xi + \int_0^x (L-L_0)\sin(c_0\xi)d\xi\right\} \tag{3.5}$$

$$-\sum_n \frac{\cos(\lambda_n t)}{2}\left\{\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n,0}}\right)\cos(\lambda_n x - \alpha(x)) + \int_0^x \left(\frac{K}{\alpha_n} - \frac{K_0}{\alpha_{n,0}}\right)\cos(\lambda_n\xi)d\xi\right\}$$

$$-\sum_n \frac{\cos(\lambda_n t)}{2}\left\{\int_0^x \left(\frac{L}{\alpha_n} - \frac{L_0}{\alpha_{n,0}}\right)\sin(\lambda_n\xi)d\xi\right\}$$

$$+\int_0^x (K-K_0)\left\{\sum_n \frac{1}{\pi}\cos((n+c_0)\xi)\cos((n+c_0)t)\right\}d\xi$$

$$+\int_0^x (L-L_0)\left\{\sum_n \frac{1}{\pi}\sin((n+c_0)\xi)\cos((n+c_0)t)\right\}d\xi.$$

With a similar process, by Eqs. (2.17), (2.20), (2.21) and (2.23), (2.26), (2.27) it follows that

$$L(x,t) - L_0(x,t) = -\frac{\sin(c_0 t)}{\pi}\left\{\int_0^x (K-K_0)\cos(c_0\xi)d\xi + \int_0^x (L-L_0)\sin(c_0\xi)d\xi\right\} \tag{3.6}$$

$$-\sum_n \frac{\sin(\lambda_n t)}{2}\left\{\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n,0}}\right)\cos(\lambda_n x - \alpha(x)) + \int_0^x \left(\frac{K}{\alpha_n} - \frac{K_0}{\alpha_{n,0}}\right)\cos(\lambda_n\xi)d\xi\right\}$$

$$-\sum_n \frac{\sin(\lambda_n t)}{2}\left\{\int_0^x \left(\frac{L}{\alpha_n} - \frac{L_0}{\alpha_{n,0}}\right)\sin(\lambda_n\xi)d\xi\right\}$$

$$+\int_0^x (K-K_0)\left\{\sum_n \frac{1}{\pi}\cos((n+c_0)\xi)\sin((n+c_0)t)\right\}d\xi$$
$$+\int_0^x (L-L_0)\left\{\sum_n \frac{1}{\pi}\sin((n+c_0)\xi)\sin((n+c_0)t)\right\}d\xi.$$

If we write λ_n instead of λ in (2.1) and (2.9), we acquire

$$\begin{aligned}\phi(x,\lambda_n) &= \cos(\lambda_n x-\alpha(x)) + \int_0^x K(x,t)\cos(\lambda_n t)dt + \int_0^x L(x,t)\sin(\lambda_n t)dt,\\ \phi_0(x,\lambda_n) &= \cos(\lambda_n x-\alpha(x)) + \int_0^x K_0(x,t)\cos(\lambda_n t)dt + \int_0^x L_0(x,t)\sin(\lambda_n t)dt.\end{aligned} \tag{3.7}$$

Since

$$\sum_n \frac{1}{\pi}\cos((n+c_0)x)\cos((n+c_0)t) = \frac{1}{\pi}\cos(c_0x)\cos(c_0t) + \delta(x-t)\cos c_0(x-t),$$

$$\sum_n \frac{1}{\pi}\cos((n+c_0)x)\sin((n+c_0)t) = \frac{1}{\pi}\cos(c_0x)\sin(c_0t) + \delta(x-t)\sin c_0(x-t), \tag{3.8}$$

$$\sum_n \frac{1}{\pi}\sin((n+c_0)x)\sin((n+c_0)t) = \frac{1}{\pi}\sin(c_0x)\sin(c_0t) + \delta(x-t)\cos c_0(x-t),$$

Eqs. (3.5) and (3.6) can be written as

$$K(x,t)-K_0(x,t) = -\sum_n \frac{\cos(\lambda_n t)}{2}\left(\frac{\phi(x,\lambda_n)}{\alpha_n} - \frac{\phi_0(x,\lambda_n)}{\alpha_{n,0}}\right), \tag{3.9}$$

$$L(x,t)-L_0(x,t) = -\sum_n \frac{\sin(\lambda_n t)}{2}\left(\frac{\phi(x,\lambda_n)}{\alpha_n} - \frac{\phi_0(x,\lambda_n)}{\alpha_{n,0}}\right), \tag{3.10}$$

by using formulas (3.7). This completes the proof. Now we are ready to give the proof of Theorem 3.1. □

Proof of Theorem 3.1 By using (3.1), one can easily obtain that

$$\begin{aligned}\phi'(x,\lambda) =& \phi_0'(x,\lambda) + (K(x,x)-K_0(x,x))\cos(\lambda x) + (L(x,x)-L_0(x,x))\sin(\lambda x)\\ &+\int_0^x \partial_x(K-K_0)\cos(\lambda t)dt + \int_0^x \partial_x(L-L_0)\sin(\lambda t)dt,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\phi''(x,\lambda) =&\phi_0''(x,\lambda) + (K(x,x)-K_0(x,x))'\cos(\lambda x)\\&-\lambda(K(x,x)-K_0(x,x))\sin(\lambda x)+(L(x,x)-L_0(x,x))'\sin(\lambda x)\\&+\lambda(L(x,x)-L_0(x,x))\cos(\lambda x)+\partial_x\left(K-K_0\right)|_{t=x}\cos(\lambda x)\\&+\partial_x(L-L_0)|_{t=x}\sin(\lambda x)+\int_0^x\partial_{xx}\left(K-K_0\right)\cos(\lambda t)dt+\int_0^x\partial_{xx}(L-L_0)\sin(\lambda t)dt,\end{aligned}\tag{3.12}$$

$$\begin{aligned}\left(2\lambda p(x)+q(x)\right)\phi =& \left(2\lambda p(x)+q(x)\right)\phi_0 + \left(2\lambda p(x)+q(x)\right)\int_0^x (K(x,t)-K_0(x,t))\cos(\lambda t)dt\\&+\left(2\lambda p(x)+q(x)\right)\int_0^x (L(x,t)-L_0(x,t))\sin(\lambda t)dt.\end{aligned}\tag{3.13}$$

Using integration by parts in (3.13) and considering (2.6) and (2.13), we get

$$\begin{aligned}\left(2\lambda p(x)+q(x)\right)\phi =& \left(2\lambda p(x)+q(x)\right)\phi_0 + 2p(x)(K(x,x)-K_0(x,x))\sin(\lambda x)\\&-2p(x)\int_0^x\partial_t\left(K-K_0\right)\sin(\lambda t)dt-2p(x)(L(x,x)-L_0(x,x))\cos(\lambda x)\\&+2p(x)\int_0^x\partial_t\left(L-L_0\right)\cos(\lambda t)dt+q(x)\int_0^x(K(x,t)-K_0(x,t))\cos(\lambda t)dt\\&+q(x)\int_0^x(L(x,t)-L_0(x,t))\sin(\lambda t)dt.\end{aligned}\tag{3.14}$$

By multiplying Eq. (3.1) by λ^2, we get

$$\lambda^2\phi = \lambda^2\phi_0 + \lambda^2\int_0^x (K(x,t)-K_0(x,t))\cos(\lambda t)dt + \lambda^2\int_0^x (L(x,t)-L_0(x,t))\sin(\lambda t)dt \tag{3.15}$$

and using integration by parts twice in (3.15) yields

$$\begin{aligned}\lambda^2\phi =&\lambda^2\phi_0 + \lambda(K(x,x)-K_0(x,x))\sin(\lambda x)+\partial_t\left(K-K_0\right)|_{t=x}\cos(\lambda x)\\&-\partial_t\left(K-K_0\right)|_{t=0}-\int_0^x\partial_{tt}\left(K-K_0\right)\cos(\lambda t)dt-\lambda(L(x,x)-L_0(x,x))\cos(\lambda x)\\&+\lambda(L(x,0)-L_0(x,0))+\partial_t\left(L-L_0\right)|_{t=x}\sin(\lambda x)-\int_0^x\partial_{tt}(L-L_0)\sin(\lambda t)dt.\end{aligned}\tag{3.16}$$

By considering (3.12), (3.14), (3.16) and (2.6), (2.13), we obtain

$$\begin{aligned}
&\phi'' - (2\lambda p(x) + q(x))\,\phi + \lambda^2\phi = \phi_0^{''} - (2\lambda p(x) + q(x))\,\phi_0 + \lambda^2\phi_0 \\
&+ (K(x,x) - K_0(x,x))' \cos(\lambda x) + (L(x,x) - L_0(x,x))' \sin(\lambda x) + \partial_x\,(K - K_0)\,|_{t=x} \cos(\lambda x) \\
&+ \partial_x(L - L_0)\,|_{t=x} \sin(\lambda x) + \partial_t\,(K - K_0)\,|_{t=x} \cos(\lambda x) + \partial_t\,(L - L_0)\,|_{t=x} \sin(\lambda x) \\
&- 2p(x)(K(x,x) - K_0(x,x)) \sin(\lambda x) + 2p(x)(L(x,x) - L_0(x,x)) \cos(\lambda x) \\
&+ \int_0^x \{\partial_{xx}\,(K - K_0) - 2p(x)\partial_t\,(L - L_0) - q(x)\,(K - K_0) - \partial_{tt}\,(K - K_0)\} \cos(\lambda t)dt \\
&+ \int_0^x \{\partial_{xx}(L - L_0) + 2p(x)\partial_t\,(K - K_0) - q(x)\,(L - L_0) - \partial_{tt}(L - L_0)\} \sin(\lambda t)dt.
\end{aligned} \tag{3.17}$$

Thus

$$\begin{aligned}
\phi'' - (2\lambda p(x) + q(x))\,\phi + \lambda^2\phi \;&=\; \phi_0^{''} - (2\lambda p(x) + q(x))\,\phi_0 + \lambda^2\phi_0 \\
&+ 2(K(x,x) - K_0(x,x))' \cos(\lambda x) + 2(L(x,x) - L_0(x,x))' \sin(\lambda x) \\
&- 2p(x)(K(x,x) - K_0(x,x)) \sin(\lambda x) + 2p(x)(L(x,x) - L_0(x,x)) \cos(\lambda x) \\
&+ \int_0^x \{\partial_{xx}\,(K - K_0) - 2p(x)\partial_t\,(L - L_0) - q(x)(K - K_0) - \partial_{tt}\,(K - K_0)\} \cos(\lambda t)dt \\
&+ \int_0^x \{\partial_{xx}(L - L_0) + 2p(x)\partial_t\,(K - K_0) - q(x)(L - L_0) - \partial_{tt}(L - L_0)\} \sin(\lambda t)dt.
\end{aligned} \tag{3.18}$$

Since $\varphi_0(x,\lambda)$ is a solution of Eq. (1.5), we can write

$$\phi_0^{''} - (2\lambda p(x) + q(x))\,\phi_0 + \lambda^2\phi_0 = (q_0(x) - q(x))\phi_0.$$

Considering (2.9), we obtain

$$\begin{aligned}
(q_0(x) - q(x))\phi_0 =&(q_0(x) - q(x)) \cos(\lambda x - \alpha(x)) + (q_0(x) - q(x)) \int_0^x K_0(x,t) \cos(\lambda t)dt \\
&+ (q_0(x) - q(x)) \int_0^x L_0(x,t) \sin(\lambda t)dt.
\end{aligned} \tag{3.19}$$

On the other hand, the following equations are provided [13]:

$$\left[p^2(x) + q(x)\right] \cos\alpha(x) = 2K'(x,x) + 2p(x)L(x,x) - p'(x) \sin\alpha(x), \tag{3.20}$$

$$\left[p^2(x) + q(x)\right] \sin\alpha(x) = 2L'(x,x) - 2p(x)K(x,x) + p'(x) \cos\alpha(x), \tag{3.21}$$

$$\left[p^2(x)+q_0(x)\right]\cos\alpha(x)=2K_0^{'}(x,x)+2p(x)L_0(x,x)-p'(x)\sin\alpha(x), \tag{3.22}$$

$$\left[p^2(x)+q_0(x)\right]\sin\alpha(x)=2L_0^{'}(x,x)-2p(x)K_0(x,x)+p'(x)\cos\alpha(x). \tag{3.23}$$

Subtracting (3.20), (3.22) and (3.21), (3.23) leads to

$$[q(x)-q_0(x)]\cos\alpha(x)=2\left(K'(x,x)-K_0^{'}(x,x)\right)+2p(x)(L(x,x)-L_0(x,x)), \tag{3.24}$$

$$[q(x)-q_0(x)]\sin\alpha(x)=2(L'(x,x)-L_0^{'}(x,x))-2p(x)(K(x,x)-K_0(x,x)). \tag{3.25}$$

Considering (3.24) and (3.25),

$$\begin{aligned}(q(x)-q_0(x))\cos\left[\lambda x-\alpha(x)\right]&=\left\{2\left(K'(x,x)-K_0^{'}(x,x)\right)+2p(x)(L(x,x)-L_0(x,x))\right\}\cos(\lambda x)\\&+\left\{2(L'(x,x)-L_0^{'}(x,x))-2p(x)(K(x,x)-K_0(x,x))\right\}\sin(\lambda x).\end{aligned} \tag{3.26}$$

By (3.19) and (3.26) in (3.18), we get

$$\begin{aligned}\phi''-(2\lambda p(x)+q(x))\,\phi+\lambda^2\phi\quad&=\\=\int_0^x\{\partial_{xx}(K-K_0)-2p(x)\partial_t(L-L_0)-q(x)K+q_0(x)K_0-\partial_{tt}(K-K_0)\}\cos(\lambda t)dt\\+\int_0^x\{\partial_{xx}(L-L_0)+2p(x)\partial_t(K-K_0)-q(x)L+q_0(x)L_0-\partial_{tt}(L-L_0)\}\sin(\lambda t)dt,\end{aligned}$$

and by considering (2.3), (2.4) and (2.10), (2.11) together we have

$$\phi''-(2\lambda p(x)+q(x))\,\phi+\lambda^2\phi\quad=0.$$

Thus $\phi(x,\lambda)$ is a solution of Eq. (1.1).

2) *By (3.1), we get $\phi(0,\lambda)=\phi_0(0,\lambda)=1$ for $x=0$. From (3.11) with $x=0$, we get*

$$\phi'(0,\lambda)=\phi_0^{'}(0,\lambda)+(K(0,0)-K_0(0,0)).$$

Then, by (2.16) and (2.22) for $x=t=0$, we have

$$\phi'(0,\lambda)=\phi_0^{'}(0,\lambda)-\left[F_{11}(0,0)-\tilde{F}_{11}(0,0)\right].$$

By (2.18) and (2.24), we get

$$\phi'(0,\lambda)=\phi_0^{'}(0,\lambda)-\frac{1}{2}\sum_n\left(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n,0}}\right).$$

Since $\phi'(0,\lambda)=h$, $\phi_0^{'}(0,\lambda)=h_0$, we have

$$h=h_0-\frac{1}{2}\sum_n\left(\frac{1}{\alpha_n}-\frac{1}{\alpha_{n,0}}\right). \tag{3.27}$$

3) As $n \to \infty$, the following formula is proved in [32]:

$$(K(\pi,\pi) - K_0(\pi,\pi))\cos(c_0\pi) + (L(\pi,\pi) - L_0(\pi,\pi))\sin(c_0\pi) = H_0 - H. \tag{3.28}$$

Taking into consideration Eqs. (3.9) and (3.10) in Theorem 3.2 for $x = \pi$, we get

$$-\sum_n \frac{\cos((\lambda_n - c_0)\,\pi)}{2}\left(\frac{\phi(\pi,\lambda_n)}{\alpha_n} - \frac{\phi_0(\pi,\lambda_n)}{\alpha_{n,0}}\right) = H_0 - H.$$

Since $n \to \infty$ and $\lambda_n \to n + c_0$, we obtain

$$H = H_0 + \sum_n \frac{(-1)^n}{2}\left(\frac{\phi(\pi,\lambda_n)}{\alpha_n} - \frac{\phi_0(\pi,\lambda_n)}{\alpha_{n,0}}\right). \tag{3.29}$$

Hence, the theorem is proved.

Theorem 3.4 *Let λ_n, $n = \pm 0, \pm 1, \pm 2, \ldots$ be the eigenvalues of the problem (1.1)–(1.3). Hence $\phi(x,\lambda_n)$ defined by Eq. (1.1) with λ_n instead of λ can be formulated as*

$$\phi(x,\lambda_n) = \phi_0(x,\lambda_n) - \frac{x}{2}\sum_k \left(\frac{\phi(x,\lambda_k)}{\alpha_k} - \frac{\phi_0(x,\lambda_k)}{\alpha_{k,0}}\right). \tag{3.30}$$

Proof If we collocate (3.1), (3.9), and (3.10) in Theorem 3.3, we have the formula (3.30). □

4. An alternative method

In this part, we obtain a new formula for norming constants to rewrite the relations between h, H and h_0, H_0 that just depends on eigenfunctions without norming constants. Moreover, we consider all above results for the nonisospectrality case.

Lemma 4.1

$$\alpha_n = \frac{1}{2\lambda_n}\left|\phi(\pi,\lambda_n)\right|\left|\dot{D}\left(\lambda_n\right)\right|, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

where the dot over D indicates the derivative of $D(\lambda)$ with respect to λ.

Proof By (1.1), we have

$$-\phi''(x,\lambda) + (2\lambda p(x) + q(x))\,\phi(x,\lambda) = \lambda^2\phi(x,\lambda),$$

$$-\phi''(x,\mu) + (2\mu p(x) + q(x))\,\phi(x,\mu) = \mu^2\phi(x,\mu),$$

for arbitrary λ and μ and with a simple calculation we obtain

$$\phi''(x,\lambda)\phi(x,\mu) - \phi(x,\lambda)\phi''(x,\mu) = \left(\left(\mu^2 - \lambda^2\right) - 2p(x)\left(\mu - \lambda\right)\right)\phi(x,\lambda)\phi(x,\mu),$$

or

$$\frac{d}{dx}\left(\phi'(x,\lambda)\phi(x,\mu) - \phi(x,\lambda)\phi'(x,\mu)\right) = \left(\mu - \lambda\right)\left(\mu + \lambda - 2p(x)\right)\phi(x,\lambda)\phi(x,\mu).$$

By taking the integral of the last equation with respect to x from 0 to π, we get

$$\int_0^\pi (\lambda+\mu-2p(x))\,\phi(x,\lambda)\phi(x,\mu)dx = \frac{\phi'(\pi,\lambda)\phi(\pi,\mu)-\phi(\pi,\lambda)\phi'(\pi,\mu)}{\mu-\lambda}.$$

Since $\phi'(\pi,\lambda) = -H\phi(\pi,\lambda)$, as $\mu\to\lambda$, we can write

$$\begin{aligned} 2\int_0^\pi [\lambda-p(x)]\,\phi^2(x,\lambda)dx &= \phi'(\pi,\lambda)\dot{\phi}(\pi,\lambda)-\phi(\pi,\lambda)\dot{\phi}'(\pi,\lambda) \\ &= -\phi(\pi,\lambda)\left(\dot{\phi}'(\pi,\lambda)+H\dot{\phi}(\pi,\lambda)\right) \\ &= -\phi(\pi,\lambda)\dot{D}(\lambda). \end{aligned}$$

Here $D(\lambda)\equiv\phi'(\pi,\lambda)+H\phi(\pi,\lambda)$. From the formula (2.14) with $\lambda=\lambda_n$, we get

$$\alpha_n = \frac{1}{2\lambda_n}\left|\phi(\pi,\lambda_n)\right|\left|\dot{D}(\lambda_n)\right|.$$

□

Conclusion 4.2 *The formulas (3.27) and (3.29) can be expressed by*

$$h = h_0 - \sum_n \frac{\lambda_n}{\left|\dot{D}(\lambda_n)\right|}\left(\frac{1}{|\phi(\pi,\lambda_n)|}-\frac{1}{|\phi_0(\pi,\lambda_n)|}\right), \tag{4.2}$$

$$H = H_0 + \sum_n \frac{(-1)^n\lambda_n}{\left|\dot{D}(\lambda_n)\right|}\left(\frac{\phi(\pi,\lambda_n)}{|\phi(\pi,\lambda_n)|}-\frac{\phi_0(\pi,\lambda_n)}{|\phi_0(\pi,\lambda_n)|}\right), \tag{4.3}$$

by using Lemma 4.1. These equalities enable us to obtain relations between h, H and h_0, H_0 without norming constants.

Now we assume that the problems (1.1)–(1.3) and (1.5)–(1.7) have different spectrums λ_n and μ_n. In this case, Gelfand–Levitan integral equations have the same forms as (2.16), (2.17) and (2.22), (2.23). However, the functions $F_{ij}(x,t)$ and $\tilde{F}_{ij}(x,t)\,(i,j=1,2)$ are different and can be determined as follows in terms of λ_n and μ_n:

$$F_{11}(x,t)-\tilde{F}_{11}(x,t) = \sum_n\left\{\frac{1}{2\alpha_n}\cos(\mu_n x)\cos(\mu_n t)-\frac{1}{2\alpha_{n,0}}\cos(\lambda_n x)\cos(\lambda_n t)\right\}, \tag{4.4}$$

$$F_{12}(x,t)-\tilde{F}_{12}(x,t) = \sum_n\left\{\frac{1}{2\alpha_n}\sin(\mu_n x)\cos(\mu_n t)-\frac{1}{2\alpha_{n,0}}\sin(\lambda_n x)\cos(\lambda_n t)\right\}, \tag{4.5}$$

$$F_{21}(x,t)-\tilde{F}_{21}(x,t) = \sum_n\left\{\frac{1}{2\alpha_n}\cos(\mu_n x)\sin(\mu_n t)-\frac{1}{2\alpha_{n,0}}\cos(\lambda_n x)\sin(\lambda_n t)\right\}, \tag{4.6}$$

$$F_{22}(x,t)-\tilde{F}_{22}(x,t) = \sum_n\left\{\frac{1}{2\alpha_n}\sin(\mu_n x)\sin(\mu_n t)-\frac{1}{2\alpha_{n,0}}\sin(\lambda_n x)\sin(\lambda_n t)\right\}. \tag{4.7}$$

By (3.27) and (4.1), it follows that

$$h = h_0 - \sum_n \left(\frac{\mu_n}{|\phi(\pi,\mu_n)|\,|\dot{\varepsilon}\,(\mu_n)|} - \frac{\lambda_n}{|\phi_0(\pi,\lambda_n)|\left|\dot{D}\,(\lambda_n)\right|} \right). \tag{4.8}$$

Here

$$D(\lambda) = \phi_0^{'}(\pi,\lambda) + H_0\phi_0(\pi,\lambda), \;\; \varepsilon(\lambda) = \phi'(\pi,\lambda) + H\phi(\pi,\lambda).$$

In this case,

$$K(x,t) - K_0(x,t) = -\sum_n \left\{ \frac{\cos(\mu_n t)}{2\alpha_n}\phi(x,\mu_n) - \frac{\cos(\lambda_n t)}{2\alpha_{n,0}}\phi_0(x,\lambda_n) \right\}, \tag{4.9}$$

$$L(x,t) - L_0(x,t) = -\sum_n \left\{ \frac{\sin(\mu_n t)}{2\alpha_n}\phi(x,\mu_n) - \frac{\sin(\lambda_n t)}{2\alpha_{n,0}}\phi_0(x,\lambda_n) \right\}, \tag{4.10}$$

and the formula (3.28) also is valid. Thus, from (4.9) and (4.10) we get

$$-\sum_n (-1)^n \left(\frac{\phi(\pi,\mu_n)}{2\alpha_n} - \frac{\phi_0(\pi,\lambda_n)}{2\alpha_{n,0}} \right) = H_0 - H, \tag{4.11}$$

$$H = H_0 + \sum_n (-1)^n \left\{ \frac{\mu_n\phi(\pi,\mu_n)}{|\phi(\pi,\mu_n)|\,|\dot{\varepsilon}(\mu_n)\,(\mu_n)|} - \frac{\lambda_n\phi_0(\pi,\lambda_n)}{|\phi_0(\pi,\lambda_n)|\left|\dot{D}\,(\lambda_n)\right|} \right\} \tag{4.12}$$

as $n \to \infty$.

5. Conclusion

In this study, we try to get some important results about the inverse spectral problem for the scalar energy-dependent Schrödinger equation on a finite interval. We consider two boundary value problems and construct an isospectrality problem. In particular, we obtain degeneracy of the kernels $K(x,t) - K_0(x,t)$ and $L(x,t) - L_0(x,t)$ by another approach that is different from classical methods. Furthermore, we obtain some quite different relations between the constants in boundary conditions of the problems. After we studied the scalar case of the energy-dependent Schrödinger operator, we noted that some results in the vector valued case could be obtain relatively, but the proofs require much more time and effort.

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