Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 181 – 189
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doi:10.3906/mat-1508-87

On the density and transitivity of sets of operators

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Received: 23.08.2015 • **Accepted/Published Online:** 06.04.2017 • **Final Version:** 22.01.2018

Abstract: By the well-known result of Yood, every strictly transitive algebra of operators on a Banach space is WOT-dense. This motivated us to investigate the relationships between SOT and WOT largeness of sets of operators and the transitivity behavior of them. We show that, to obtain Yood's result, strict transitivity may not be replaced by the weaker condition of hypertransitivity. We prove that, for a wide class of topological vector spaces, every SOT-dense set of operators is hypertransitive. The general form of SOT-dense sets that are not strictly transitive is presented. We also describe the form of WOT-dense sets that are not hypertransitive. It is shown that a set is hypertransitive if and only its SOT-closure is hypertransitive. We introduce strong topological transitivity and we show that every separable infinite-dimensional Hilbert space supports an invertible topologically transitive operator that is not strongly topologically transitive.

Key words: SOT-density, WOT-density, strict transitivity, hypertransitivity, strong topological transitivity

1. Introduction

Let X be a topological vector space over the field of complex numbers $\mathbb{C}$. By X^* we mean the space of all continuous linear functionals on X. Denote by $L(X)$ the algebra of all continuous linear operators on X. By an operator, we always mean a continuous linear operator. We write $A \subset B$ to say that A is a subset of B that may equal B. If A is a proper subset of B then we write $A \subsetneqq B$. We say that a set $\Gamma \subset L(X)$ is hypercyclic if there exists some $x \in X$ for which

$$\mathrm{orb}(\Gamma, x) = \{Tx : T \in \Gamma\},$$

the orbit of x under Γ, is a dense subset of X. An operator $T \in L(X)$ is called hypercyclic if $\Gamma = \{T^n : n \in \mathbb{N}_0\}$ is hypercyclic, where $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and $T^0 = I$, the identity operator on X. In this case, we write $\mathrm{orb}(T, x)$ instead of $\mathrm{orb}(\Gamma, x)$. The set of hypercyclic vectors for Γ is denoted by $HC(\Gamma)$ and when $\Gamma = \{T^n : n \in \mathbb{N}_0\}$ we write $HC(T)$ for $HC(\Gamma)$. A set $\Gamma \subset L(X)$ is called topologically transitive if for each pair of nonempty open sets $U, V \subset X$ there exists some $T \in \Gamma$ such that $T(U) \cap V \neq \emptyset$. An operator T is called topologically transitive if $\{T^n : n \in \mathbb{N}_0\}$ is topologically transitive.

Definition 1 *A set $\Gamma \subset L(X)$ is said to be hypertransitive if $HC(\Gamma) = X \backslash \{0\}$. Γ is called strictly transitive if for each pair of nonzero elements x, y in X, there exists some $T \in \Gamma$ such that $Tx = y$.*

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2010 *AMS Mathematics Subject Classification:* 47A16, 46A99

An operator $T \in L(X)$ is hypertransitive if $\{T^n : n \in \mathbb{N}_0\}$ is hypertransitive. An example of a hypertransitive operator is the Read operator [6]. If Γ is a strictly transitive set then $X\backslash\{0\} \subset \text{orb}(\Gamma, x)$ for each nonzero $x \in X$. This shows that $\overline{\text{orb}(\Gamma, x)} = X$ and that Γ is uncountable. Hence, $HC(\Gamma) = X\backslash\{0\}$ and we conclude that every strictly transitive set of operators is hypertransitive but the converse is not true. Meanwhile, it is easy to see that hypertransitivity implies topological transitivity. The operator $T = 2B$, twice the backward shift on $\ell^2(\mathbb{N})$, is topologically transitive [7], but not hypertransitive since $\ker(T) \neq (0)$ (it is easy to see that every hypertransitive operator is injective).

Recall that for a topological vector space X, the strong operator topology (SOT) on $L(X)$ is the topology with respect to which any $T \in L(X)$ has a neighborhood basis consisting of sets of the form $\Omega = \{S \in L(X) : \ Se_i - Te_i \in U, \ i = 1, 2, \cdots, k\}$, where $k \in \mathbb{N}$, $e_1, e_2, \cdots, e_k \in X$ are linearly independent and U is a neighborhood of zero in X.

The weak operator topology (WOT) on $L(X)$ is the topology in which $T_n \to T$ if and only if $f(T_n x) \to f(Tx)$ for all $x \in X$ and every bounded linear functional f on X.

In Section 2, we give the general form of SOT-dense subsets of $L(X)$ that are not strictly transitive. It is proved that on every locally convex Hausdorff space X, the semigroup of rank-one operators on X is strictly transitive but not WOT-dense. We show that for every topological vector space X, if $\Gamma \subset L(X)$ then $HC(\Gamma) = HC(\overline{\Gamma})$, where $\overline{\Gamma}$ stands for the SOT-closure of Γ. Thus, if $L(X)$ is hypertransitive then the SOT-density of $\Gamma \subset L(X)$ implies that it is hypertransitive. This condition holds for every locally convex space and, more generally, for every topological vector space X such that X^* separates the points of X. This class of spaces includes the set of all locally convex spaces as a proper subset. For every space X in the mentioned class, we show that there are hypertransitive subsets of $L(X)$ that are not WOT-dense. We also describe the form of WOT-dense sets that are not hypertransitive.

In Section 3, the notion of strong topological transitivity is introduced and the relationships between it and topological transitivity, hypercyclicity, hypertransitivity, and strict transitivity are investigated. We see that strong topological transitivity is strictly stronger than topological transitivity.

2. Density and transitivity

In the following theorem, the proof is also true for norm-density if X is assumed to be a normed linear space.

Theorem 1 *Let X be a topological vector space. Then for each pair of nonzero vectors $x, y \in X$ there exists a SOT-dense set $\Gamma_{xy} \subset L(X)$ that is not strictly transitive. Furthermore, $\Gamma \subset L(X)$ is a dense nonstrictly transitive set if and only if Γ is a dense subset of Γ_{xy} for some $x, y \in X$.*

Proof Fix nonzero vectors $x, y \in X$ and put $\Gamma_{xy} = \{T \in L(X) : Tx \neq y\}$. It is clear that Γ_{xy} is not strictly transitive. Let Ω be a nonempty open set in $L(X)$ and $S \in \Omega$. If $Sx \neq y$ then $S \in \Omega \cap \Gamma_{xy}$. Otherwise, putting $S_n = S + \frac{1}{n}I$ we see that $S_k \in \Omega$ for some k, but $S_k x \neq y$. Hence, $\Omega \cap \Gamma_{xy} \neq \emptyset$ and the proof is completed.

We prove the second assertion of the theorem. Suppose that Γ is a dense subset of $L(X)$ that is not strictly transitive. Then there are nonzero vectors $x, y \in X$ such that $Tx \neq y$ for all $T \in \Gamma$ and hence $\Gamma \subset \Gamma_{xy}$. To show that Γ is dense in Γ_{xy}, assume that Ω_0 is an open subset of Γ_{xy}. Thus, $\Omega_0 = \Gamma_{xy} \cap \Omega$ for some open set Ω in $L(X)$. Then $\Gamma \cap \Omega_0 = \Gamma \cap \Omega \neq \emptyset$.

For the converse, let Γ be a dense subset of Γ_{xy} for some $x, y \in X$. Then Γ is not strictly transitive. Also, since Γ_{xy} is a dense open subset of $L(X)$, we conclude that Γ is also dense in $L(X)$. Indeed, if Ω is any open set in $L(X)$ then $\Omega \cap \Gamma_{xy} \neq \emptyset$ since Γ_{xy} is dense in $L(X)$. On the other hand, $\Omega \cap \Gamma_{xy}$ is open in Γ_{xy} and so it must intersect Γ since Γ is dense in Γ_{xy}. Thus, $\Omega \cap \Gamma \neq \emptyset$ and so Γ is dense in $L(X)$. □

Corollary 1 *Let X be a topological vector space and Γ be a dense subset of $L(X)$. Then there is a subset Γ_1 of Γ such that $\overline{\Gamma}_1 = L(X)$ and Γ_1 is not strictly transitive.*

Proof For nonzero vectors x, y put $\Gamma_1 = \Gamma \cap \Gamma_{xy}$. □

Recall that for a topological vector space X, a set $M \subset X$ is said to be balanced if $DM \subset M$ where $D = \{z \in \mathbb{C} : |z| \leq 1\}$. It is known that there always exists a base at zero consisting of balanced sets [9, Theorem 3.3-E]. M is called bounded if for every open set W containing zero there is some $\epsilon > 0$ such that $\epsilon M \subset W$. It is easy to see that if M is bounded, $(x_j)_j$ is a net in M, and $(a_j)_j$ is a net of scalars such that $a_j \to 0$, then $a_j x_j \to 0$ in X. A topological vector space X is said to be locally bounded if X has a bounded neighborhood of zero.

Let X be a topological vector space for which $X^* = (0)$. Then there is no rank-one operator on X. In fact, every rank-one operator T on X is of the form $Tz = f(z)x$ $(z \in X)$, for some nonzero $x \in X$ and nonzero $f \in X^*$. To prove this, let $\mathrm{Ran}T = \mathbb{C}x$. Then, for every $z \in X$, there is a scalar a_z such that $Tz = a_z x$. Define $f : X \to \mathbb{C}$ by $f(z) = a_z$. Then $z = y$ implies $a_z x = Tz = Ty = a_y x$ and so $f(z) = a_z = a_y = f(y)$, which says that f is well defined. Also, $f(z+y)x = T(z+y) = Tz + Ty = f(z)x + f(y)x = (f(z)+f(y))x$ and hence $f(z+y) = f(z)+f(y)$. Finally, let $(z_j)_j$ be a net in X such that $z_j \to 0$. Then $Tz_j \to 0$ or $f(z_j)x \to 0$. Now, if $f(z_j) \nrightarrow 0$, then for some $\epsilon > 0$ and a subnet $(z_{j_i})_i$ we have $|f(z_{j_i})| \geq \epsilon$. Then $\frac{1}{f(z_{j_i})}$ is bounded and hence $x = \frac{1}{f(z_{j_i})}(f(z_{j_i})x) \to 0$, which is not true. Thus, $f(z_j) \to 0$ and we conclude that f is continuous.

The proof of the following proposition is much easier when we work with a normed linear space. We prove it for almost arbitrary topological vector spaces.

Proposition 1 *Let X be a locally bounded Hausdorff topological vector space with $X^* \neq (0)$ and Γ be the set of all rank-one operators on X. Then, with the SOT-topology on $L(X)$, we have $\overline{\Gamma} = \Gamma \cup \{0\}$.*

Proof Let $(a_j)_j$ be a net of scalars such that $a_j \to 0$. For a nonzero $x \in X$ and a nonzero $f \in X^*$, if $T_j z = a_j f(z) x$ $(z \in X)$, then $T_j \xrightarrow{SOT} 0$ and so $0 \in \overline{\Gamma}$. Now, let $T \in L(X)$ be a nonzero operator and $(T_j)_j$ be a net in Γ such that $T_j \xrightarrow{SOT} T$. For each j let $T_j(X) = \mathbb{C}x_j$ where $x_j \in X \backslash \{0\}$. If U is a bounded neighborhood of zero, we claim that there is a balanced neighborhood V of zero such that for every j there is some $r_j > 0$ for which $r_j x_j \in U \backslash \overline{V}$.

Since X is a regular space [9, Theorem 3.3-G], there is an open set W containing zero such that $\overline{W} \subset U$. Let V be a balanced neighborhood of zero such that $V \subset W$. For a fixed j since $\frac{1}{n}x_j \to 0$ there is some $k \in \mathbb{N}$ for which $\frac{1}{k}x_j \in V$. If we put $B = \mathbb{R}^+ x_j$ then we have $B \cap V \neq \emptyset$. On the other hand, since U is bounded we have $B \cap U^c \neq \emptyset$ ($B \subset U$ implies that $(nx_j)_n$ is a sequence in U and so $x_j = \frac{1}{n}(nx_j) \to 0$, a contradiction). We claim that $B \cap (U \backslash \overline{V}) \neq \emptyset$. Indeed, if $B \subset \overline{V} \cup U^c$ since $\overline{V} \cap U^c = \emptyset$ the connectedness of B implies that

$B \subset \overline{V}$ or $B \subset U^c$, which is not true. Thus, there is some $r_j > 0$ such that $y_j = r_j x_j \in U \backslash \overline{V}$. It is clear that $T_j(X) = \mathbb{C} y_j$.

Now, fix some $x \in X \backslash \ker T$ and let $y \in X$ be arbitrary. Then $T_j x = a_j y_j \to Tx$ and $T_j y = b_j y_j \to Ty$. We show that $Ty = \lambda Tx$ for some $\lambda \in \mathbb{C}$, which gives rank(T)=1. We claim that the nets $(a_j)_j, (b_j)_j$ are bounded. Otherwise, there is a subnet $(a_{j_i})_i$ such that $|a_{j_i}| \to \infty$. Thus, $\frac{1}{|a_{j_i}|}(a_{j_i} y_{j_i} - Tx) \to 0$ since V is bounded (since $a_{j_i} y_{j_i} - Tx \to 0$, there is some i_0 such that $i \geq i_0$ implies $a_{j_i} y_{j_i} - Tx \in V$). This implies that $\frac{a_{j_i}}{|a_{j_i}|} y_{j_i} \to 0$, which is impossible since V is balanced and $y_{j_i} \notin V$ for all i (the proof for $(b_j)_j$ is similar). Hence, passing through subnets, we have $a_j \to a$ and $b_j \to b$ for some scalars $a, b \in \mathbb{C}$. Then it is easy to see that $a y_j \to Tx$ and $b y_j \to Ty$, which gives $Ty = \frac{b}{a} Tx$. □

By a result due to Yood [10], for a Banach space X every strictly transitive subalgebra of $L(X)$ is WOT-dense. The following theorem shows that strictly transitive semigroups need not be WOT-dense.

Theorem 2 *For every locally convex Hausdorff space X there is a strictly transitive semigroup $\Gamma \subset L(X)$ that is not WOT-dense.*

Proof Let $\Gamma = \{T \in L(X) : \text{rank}(T) = 1\} \cup \{0\}$. For a pair of nonzero vectors $x, y \in X$ define $T \in L(X)$ by $Tz = f(z) y$ $(z \in X)$ for some $f \in X^*$ satisfying $f(x) = 1$. Then $Tx = y$ and hence Γ is strictly transitive.

If $\dim X < \infty$ then the SOT and WOT topologies agree on $L(X)$. Thus, by the above proposition, we conclude that Γ is not WOT-dense in $L(X)$. Now, suppose that $\dim X = \infty$. Let M be a finite-dimensional subspace of X with $\dim M \geq 2$. Since $\dim M < \infty$, by [1, Lemma 2.21] there is a closed subspace Z of X such that $X = M \oplus Z$. Assume that Γ_M is the set of all rank-one operators on M and let $T \in L(M) \backslash (\Gamma_M \cup \{0\})$. Define $\hat{T} \in L(X)$ by $\hat{T} x = Tm$ where $x = m + z$, $m \in M$ and $z \in Z$. We claim that $\hat{T} \notin \overline{\Gamma}$ (WOT). Let $(\hat{T}_j)_j$ be an arbitrary net in Γ and $P \in L(X)$ be the projection onto M. Then $(P\hat{T}_j|_M)_j$ is a net in $\Gamma_M \cup \{0\}$ and since by Proposition 1 (remember that $\dim M < \infty$ and so M is locally bounded, and meanwhile, the SOT and WOT topologies agree on $L(M)$), $\overline{\Gamma}_M = \Gamma_M \cup \{0\}$ (WOT), we conclude that $P\hat{T}_j|_M \nrightarrow T$ (WOT). Hence, there is some $f \in M^*$ and some $m \in M$ such that $f(P\hat{T}_j|_M m) \nrightarrow f(Tm) = f(P\hat{T} m)$. Thus, $f \circ P(\hat{T}_j|_M m) \nrightarrow f \circ P(\hat{T} m)$, but $f \circ P \in X^*$ and we conclude that $\hat{T}_j \nrightarrow \hat{T}$ (WOT). □

Assume that X is a topological vector space and $\Gamma \subset L(X)$. The following result shows that the SOT-closure of Γ is not large enough to have more hypercyclic vectors than Γ.

Proposition 2 *Let X be a topological vector space and $\Gamma \subset L(X)$. If $\overline{\Gamma}$ stands for the SOT-closure of Γ then $HC(\Gamma) = HC(\overline{\Gamma})$.*

Proof We only need to prove that $HC(\overline{\Gamma}) \subset HC(\Gamma)$. Fix $x \in HC(\overline{\Gamma})$ and let U be an arbitrary open subset of X. Then there is some $T \in \overline{\Gamma}$ such that $Tx \in U$. The set $\Omega = \{S \in L(X) : Sx \in U\}$ is a SOT-neighborhood of T and so it must intersect Γ. Therefore, there is some $S \in \Gamma$ such that $Sx \in U$ and this shows that $x \in HC(\Gamma)$. □

Corollary 2 *Let X be a topological vector space and $\Gamma \subset L(X)$. Then Γ is hypertransitive if and only if $\overline{\Gamma}$ is hypertransitive.*

Corollary 3 *Let X be a topological vector space for which $L(X)$ is hypertransitive. If $\Gamma \subset L(X)$ is SOT-dense then it is hypertransitive.*

Let $\mathcal{S}$ be the class of all topological vector spaces X such that X^*, the dual space of X, separates the points of X. If $\mathcal{LCS}$ denotes the class of all locally convex spaces then $\mathcal{LCS} \subset \mathcal{S}$. Meanwhile, since ℓ^p $(0 < p < 1)$ is not a locally convex space but it belongs to $\mathcal{S}$ [5], we see that $\mathcal{LCS}$ is a proper subset of $\mathcal{S}$. The first paragraph of the proof of Theorem 2 shows that for every $X \in \mathcal{S}$, $L(X)$ is strictly transitive and hence it is hypertransitive. Then we have the following result.

Corollary 4 *Let $X \in \mathcal{S}$ and $\Gamma \subset L(X)$. If Γ is SOT-dense in $L(X)$ then Γ is hypertransitive.*

Example 1 shows that we can not reduce the condition of strict transitivity to hypertransitivity in Yood's result. Before it, we give the following lemmas. Recall that for an operator T, the commutant of T, which is denoted by $\{T\}^{'}$, is the set of all operators that commute with T. In fact, $\{T\}^{'}$ is a subalgebra of $L(X)$.

Lemma 1 *Let $X \in \mathcal{S}$ and $T \in L(X)$. Then $\{T\}'$ is WOT-closed subalgebra of $L(X)$.*

Proof Let $(T_j)_j$ be a net in $\{T\}'$ and $T_j \to S$ (WOT). We show that $S \in \{T\}'$. If $TS \neq ST$ then there is some $x \in X$ such that $TSx \neq STx$. Thus, there is a functional $f \in X^*$ such that $f(TSx - STx) \neq 0$ or $f(TSx) \neq f(STx)$. Since $T_j \to S$ (WOT) we have $f(T_jTx) \to f(STx)$ and hence $f(T_jTx) \nrightarrow f(TSx)$. However, $T_jTx = TT_jx$ and therefore $f(TT_jx) \nrightarrow f(TSx)$. Thus, if we put $g = f \circ T$ then $g(T_jx) \nrightarrow g(Sx)$, which contradicts $T_j \to S$ (WOT). Therefore, we must have $TS = ST$ and so $\{T\}'$ is WOT-closed in $L(X)$. □

Lemma 2 *Let $X \in \mathcal{S}$ and $T \in L(X)$. Then $\{T\}^{'} = L(X)$ if and only if $T = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof It is clear that $\{\lambda I\}^{'} = L(X)$ for each $\lambda \in \mathbb{C}$. For the converse, suppose that $\{T\}^{'} = L(X)$. For every nonzero vector $x \in X$, choose a functional $f \in X^*$ satisfying $f(x) = 1$. Let T_x be the rank-one operator on X defined by $T_x z = f(z)x$ $(z \in X)$. Then $Tx = TT_x x = T_x Tx = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$ (in fact, $\lambda_x = f(Tx)$). We show that $\lambda_x = \lambda_y$ for all nonzero vectors $x, y \in X$ to conclude that $T = \lambda I$ for some $\lambda \in \mathbb{C}$. Let $x, y \neq 0$ and $y = ax$ for some $a \in \mathbb{C}$. Then $a\lambda_y x = Tax = aTx = a\lambda_x x$ and hence $\lambda_y = \lambda_x$. Now, suppose that x, y are linearly independent. Then $\lambda_{x+y}(x + y) = T(x + y) = Tx + Ty = \lambda_x x + \lambda_y y$, which gives $\lambda_x = \lambda_y$. Thus, $T = \lambda I$ for some $\lambda \in \mathbb{C}$. □

Example 1 *Let T be the Read operator on $X = \ell^1$. Then $\{T\}^{'}$ is a proper WOT-closed subalgebra of $L(X)$ by the above lemmas. Hence, it may not be strictly transitive by Yood's result, but it is hypertransitive since $\{T^n : n \in \mathbb{N}_0\} \subset \{T\}^{'}$.*

In the next theorem, for every $X \in \mathcal{S}$ we give a hypertransitive set of operators which is not WOT-dense. Note that Theorem 2 gives such a set for every $X \in \mathcal{LCS}$.

Theorem 3 *Suppose that $X \in \mathcal{S}$. For all linearly independent vectors $y_1, y_2 \in X$ there is a hypertransitive set $\Gamma_{y_1 y_2} \subset L(X)$ that is not WOT-dense.*

Proof For $y \in X \backslash \{0\}$ and $f \in X^* \backslash \{0\}$ let $\Gamma_y = \{T \in L(X) : |f(Ty)| \geq 1\}$. To show that $\Gamma_y \neq \emptyset$, let $x \in X$ satisfy $f(x) = 1$ and $T \in L(X)$ be so that $Ty = x$. Then $|f(Ty)| = 1$ and so $T \in \Gamma_y$.

The set Γ_y is not WOT-dense since it is a WOT-closed proper subset of $L(X)$. Now we prove that $HC(\Gamma_y) = X \backslash \mathbb{C}y$. Fix a vector $z \in X$, which is linearly independent to y, and let $w \in X$ be arbitrary. Let $\mathcal{U}$ be a neighborhood base at w. For each $U \in \mathcal{U}$ choose $w_U \in U \backslash \ker f$. Then we have a net $(w_U)_U$ such that $w_U \to w$. For each $U \in \mathcal{U}$ find $g_U \in X^*$ such that $g_U(z) = g_U(f(w_U)y) = 1$ and define $T_U \in L(X)$ by $T_U x = g_U(x) w_U$. Then $T_U z = w_U \to w$ and since $|f(T_U y)| = |g_U(y) f(w_U)| = 1$, we have $T_U \in \Gamma_y$ for all $U \in \mathcal{U}$. Thus, $w \in \overline{\mathrm{orb}(\Gamma_y, z)}$ and we conclude that $z \in HC(\Gamma_y)$. The proof will be completed if we show that $y \notin HC(\Gamma_y)$. Indeed, if $w \in X$ be so that $|f(w)| < 1$ then it is easy to verify that $w \notin \overline{\mathrm{orb}(\Gamma_y, y)}$.

Now suppose that y_1, y_2 are linearly independent vectors in X. Then $\mathbb{C}^* y_1 \subset HC(\Gamma_{y_2})$ and $\mathbb{C}^* y_2 \subset HC(\Gamma_{y_1})$ where $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. Thus, the set $\Gamma_{y_1 y_2} = \Gamma_{y_1} \cup \Gamma_{y_2}$ is hypertransitive, but $\Gamma_{y_1 y_2}$ is also a WOT-closed proper subset of $L(X)$. □

Let X be any topological vector space. The following proposition describes WOT-dense subsets of $L(X)$ that are not hypertransitive. We use it in Theorem 4 to construct such a subset on infinite-dimensional normed linear spaces.

Proposition 3 *Let X be a topological vector space. For a nonzero vector $x \in X$ and a nonempty open set $U \subset X$ let $\Gamma_{x,U} = \{T \in L(X) : Tx \notin U\}$. Then:*

(i) $\Gamma \subset L(X)$ is not hypertransitive if and only if $\Gamma \subset \Gamma_{x,U}$ for some pair (x, U);

(ii) If there is a net $(x_j)_j$ such that $x_j \to 0$ weakly, $x_j + U \subset U^c$ for all j, and there is some $f \in X^*$ such that $f(x) = 1$, then $\Gamma_{x,U}$ is WOT-dense in $L(X)$;

(iii) If $\Gamma_{x,U}$ is WOT-dense in $L(X)$ then for every $T \in L(X) \backslash \Gamma_{x,U}$ there is a net $(x_j)_j$ such that $x_j \to 0$ weakly and $x_j + Tx \in U^c$ for every j.

Proof

(i) Since x is not a hypercyclic vector for $\Gamma_{x,U}$ no subset of it is hypertransitive. For the converse, suppose that Γ is not hypertransitive. Thus, there is a nonzero $x \in X$ such that $\overline{\mathrm{orb}(\Gamma, x)} \neq X$ and so there exists a nonempty open set $U \subset X$ such that $\mathrm{orb}(\Gamma, x) \cap U = \emptyset$. Hence, $\Gamma \subset \Gamma_{x,U}$.

(ii) For $T \in L(X) \backslash \Gamma_{x,U}$ define the net $(T_j)_j$ in $L(X)$ by $T_j z = Tz + f(z) x_j$ where $f \in X^*$ satisfies $f(x) = 1$. Then $T_j \overset{WOT}{\longrightarrow} T$ and $T_j x = Tx + x_j \in x_j + U \subset U^c$, which gives $T_j \in \Gamma_{x,U}$ for all j. Hence, $\Gamma_{x,U}$ is WOT-dense in $L(X)$.

(iii) For $T \in L(X) \backslash \Gamma_{x,U}$ let $(T_j)_j$ be a net in $\Gamma_{x,U}$ such that $T_j \overset{WOT}{\longrightarrow} T$. If we put $x_j = T_j x - Tx$ then $(x_j)_j$ is the desired net.

□

Theorem 4 *Let X be an infinite-dimensional normed linear space. Then for each nonzero vector $x \in X$ there exists a WOT-dense set $\Gamma_x \subset L(X)$ that is not hypertransitive.*

Proof Let $U = \{y \in X : \|y\| < \frac{1}{2}\}$ and $x \in X \backslash \{0\}$ be arbitrary. If $(x_j)_j$ is a net of unit vectors in X that converges weakly to zero then $\Gamma_x = \Gamma_{x,U}$ is the desired set by part (ii) of the above proposition. □

Corollary 5 *Let X be an infinite-dimensional normed linear space and Γ be a WOT-dense open subset of $L(X)$. Then there is a WOT-dense subset of Γ that is not hypertransitive.*

Proof Let x be a nonzero vector in X and $U = \{y \in X : \|y\| < \frac{1}{2}\}$. Then $\Gamma_1 = \Gamma \cap \Gamma_{x,U}$ is the desired set. □

Recall from Proposition 2 that for every topological vector space X and any $\Gamma \subset L(X)$, the sets of hypercyclic vectors for Γ and $\overline{\Gamma}$ (SOT) are the same. Theorem 4 shows that this may not hold for WOT-closure. In fact, for the set Γ_x obtained in that theorem, we have $HC(\Gamma_x) \subsetneqq X \backslash \{0\} = HC(L(X)) = HC(\overline{\Gamma_x})$ (WOT).

3. Strong topological transitivity

It is easy to see that a set $\Gamma \subset L(X)$ is topologically transitive if and only if for every nonempty open set $U \subset X$ we have $\overline{\bigcup_{T \in \Gamma} T(U)} = X$. Now we introduce another notion of transitivity and we will see that it is strictly stronger than topological transitivity.

Definition 2 *Let X be a topological vector space. A set $\Gamma \subset L(X)$ is called strongly topologically transitive if for each nonempty open set $U \subset X$, $X \backslash \{0\} \subset \bigcup_{T \in \Gamma} T(U)$.*

An operator T is called strongly topologically transitive if $\Gamma = \{T^n : n \in \mathbb{N}_0\}$ is strongly topologically transitive. Strong topological transitivity implies topological transitivity, but it does not imply hypercyclicity (see Example 2 below). If X is a second countable Baire topological vector space then strong topological transitivity implies hypercyclicity since, in this case, topological transitivity and hypercyclicity are equivalent [2]. It is easy to see that strict transitivity implies strong topological transitivity.

Example 2 *Let $X = C_{00}(\mathbb{N})$, the space of all finitely supported sequences in $\mathbb{C}$, and $T = 2B$, twice the backward shift on X. Then T is not hypercyclic since* $\operatorname{orb}(T, x)$ *is a finite set for all $x \in X$. To show that T is strongly topologically transitive, let U be an open subset of X and $x = (b_1, b_2, \cdots, b_m, 0, 0, \cdots)$ be a nonzero vector in X. Let $y = (a_1, a_2, \cdots, a_k, 0, 0, \cdots) \in U$, and let $\epsilon > 0$ satisfy $D(y; \epsilon) = \{z : \|z - y\| < \epsilon\} \subset U$. Choose $n > k$ large enough such that $2^{-n}\|x\| < \epsilon$. If we set*

$$z = (a_1, a_2, \cdots, a_k, 0, 0, \cdots, 2^{-n}b_1, 2^{-n}b_2, \cdots, 2^{-n}b_m, 0, 0, \cdots),$$

where $2^{-n}b_1$ is in the $(n+1)^{th}$ position, it can be easily seen that $z \in U$ and $T^n z = x$. Thus, $X \backslash \{0\} \subset \bigcup_{n \in \mathbb{N}_0} T^n(U)$.

Proposition 4 *Let X be a topological vector space with nontrivial topology. If $T \in L(X)$ is strongly topologically transitive then T is surjective.*

Proof Let $U \neq \emptyset$ be a proper open subset of X. Since X is a regular space there is an open set W such that $\overline{W} \subset U$ and hence $\overline{W} \neq X$. Fix a nonzero vector $x \in X$. Since $X \backslash \{0\} \subset \bigcup_{n \in \mathbb{N}_0} T^n(W)$, if $x \notin W$ then $x \in T(X)$. On the other hand, if $x \in W$ then $x \notin V = X \backslash \overline{W}$ and hence the equation $X \backslash \{0\} \subset \bigcup_{n \in \mathbb{N}_0} T^n(V)$ shows that $x \in T(X)$. □

The Read operator is not surjective [6] and hence it is not strongly topologically transitive by the above proposition. Thus, hypertransitivity does not imply strong topological transitivity.

In the following example we show that strong topological transitivity is strictly stronger than topological transitivity. Let $\mathbb{D}$ be the open unit disk in the complex plane. Recall that a nonconstant map $\phi : \mathbb{D} \to \mathbb{D}$ is called a linear fractional map if it is of the form $\phi(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$. Such a map can be considered as the restriction to $\mathbb{D}$ of an automorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (still denoted by ϕ) mapping $\mathbb{D}$ into itself. A linear fractional map that has two fixed points, one in $\partial\mathbb{D}$ and the other in $\hat{\mathbb{C}} \backslash \mathbb{D}$, is called hyperbolic.

Example 3 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be defined by $\phi(z) = \frac{z+1}{2}$ and C_ϕ be the composition operator on the Hardy space H^2 defined by $C_\phi(f) = f \circ \phi$ ($f \in H^2$). Clearly, ϕ is a hyperbolic linear fractional map and so C_ϕ is topologically transitive [3]. The operator C_ϕ is one-to-one since ϕ is nonconstant, but C_ϕ is not surjective since C_ϕ is invertible if and only if ϕ is a disk automorphism, i.e. ϕ is a linear fractional map carrying $\mathbb{D}$ onto itself [8]. Thus, Proposition 4 shows that C_ϕ is not strongly topologically transitive.*

The following proposition gives an equivalent definition for hypertransitivity, which makes it easy to investigate the relationships between hypertransitivity and strong topological transitivity. The proof is easy and so it is omitted.

Proposition 5 *A set $\Gamma \subset L(X)$ is hypertransitive if and only if for every nonempty open set $U \subset X$, $X \backslash \{0\} \subset \bigcup_{T \in \Gamma} T^{-1}(U)$.*

Thus, an operator T is hypertransitive if and only if for every nonempty open set $U \subset X$, $X \backslash \{0\} \subset \bigcup_{n \in \mathbb{N}_0} T^{-n}(U)$. Note that $T^{-n}(U)$ is simply written for $(T^n)^{-1}(U)$. If T is an invertible operator then $T^{-n}(U) = (T^{-1})^n(U)$ and so we have:

Proposition 6 *An invertible operator T is strongly topologically transitive if and only if T^{-1} is hypertransitive.*

Let $\phi : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function and M_ϕ be the multiplication operator on H^2, defined by $M_\phi f = \phi f$. It is known that M_ϕ is never hypercyclic but M_ϕ^*, the adjoint of M_ϕ, is hypercyclic if and only if ϕ is nonconstant and $\phi(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$ [4].

Recall that the a set $M \subset X$ is called an invariant subset for $T \in L(X)$ (or a T-invariant subset) if $T(M) \subset M$. It is clear that T is hypertransitive if and only if T has no closed nontrivial invariant subsets.

We finish this paper by giving the following result to show that there are surjective (in fact invertible) topologically transitive, or equivalently hypercyclic, operators on separable infinite-dimensional Hilbert spaces that are not strongly topologically transitive.

Theorem 5 *Let H be a separable infinite-dimensional Hilbert space and $T \in B(H)$ be invertible. Then:*

(i) T is strongly topologically transitive if and only if T^* is strongly topologically transitive;

(ii) H supports an invertible hypercyclic operator that is not strongly topologically transitive.

Proof

(i) It is easy to verify that M is a T-invariant closed set if and only if $M^{\perp}$ is a T^*-invariant closed set. Thus, T is hypertransitive if and only if T^* is hypertransitive and so the equation $(T^*)^{-1} = (T^{-1})^*$ together with Proposition 6 will complete the proof.

(ii) Let $\phi : \mathbb{D} \to \mathbb{C}$ be defined by $\phi(z) = e^z$. Then ϕ and $\frac{1}{\phi}$ are nonconstant bounded analytic functions on $\mathbb{D}$ whose images intersect the boundary of $\mathbb{D}$. Hence, M_ϕ^* and $M_{\frac{1}{\phi}}^*$ are hypercyclic on H^2, but $M_{\frac{1}{\phi}}^* = (M_\phi^*)^{-1}$ and so M_ϕ^* is an invertible hypercyclic operator whose adjoint is not hypercyclic. Thus, M_ϕ^* is not strongly topologically transitive by (i).

Now let H be any separable infinite-dimensional Hilbert space and $U : H^2 \to H$ be an isometric isomorphism. If we put $T = UM_\phi^* U^{-1}$ then T is an invertible hypercyclic operator. If T is strongly topologically transitive, by (i) so is T^* and hence T^* is hypercyclic. However, $T^* = (U^*)^{-1} M_\phi U^*$ and the hypercyclicity of T^* implies the hypercyclicity of M_ϕ, which is a contradiction. □

References

[1] Bayart F, Matheron É. Dynamics of Linear Operators. New York, NY, USA: Cambridge University Press, 2009.

[2] Birkhoff G. Surface transformations and their dynamical applications. Acta Math 1922; 43: 1-119.

[3] Bourdon P, Shapiro J. Cyclic composition operators on H^2. In: Proceedings of Symposia in Pure Mathematics; 1990. Durham, NH, USA: American Mathematical Society, 1990, pp. 43-53.

[4] Godefroy G, Shapiro J. Operators with dense invariant cyclic vector manifolds. J Funct Anal 1991; 98: 229-269.

[5] Peck N. On non locally convex spaces. Math Ann 1965; 161: 102-115.

[6] Read C. The invariant subspace problem for a class of Banach spaces II. Hypercyclic operators. Israel J Math 1988; 63: 1-40.

[7] Rolewicz S. On orbits of elements. Studia Math 1969; 32: 17-22.

[8] Schwartz H. Composition operators on H^p. PhD, University of Toledo, Toledo, OH, USA, 1969.

[9] Taylor A. Introduction to Functional Analysis. New York, NY, USA: John Wiley & Sons, 1958.

[10] Yood B. Additive groups and linear manifolds of transformations between Banach spaces. Amer J Math 1949; 71: 663-677.