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q-counting hypercubes in Lucas cubes

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Abstract: Lucas and Fibonacci cubes are special subgraphs of the binary hypercubes that have been proposed as models of interconnection networks. Since these families are closely related to hypercubes, it is natural to consider the nature of the hypercubes they contain. Here we study a generalization of the enumerator polynomial of the hypercubes in Lucas cubes, which q-counts them by their distance to the all 0 vertex. Thus, our bivariate polynomials refine the count of the number of hypercubes of a given dimension in Lucas cubes and for $q=1$ they specialize to the cube polynomials of Klavžar and Mollard. We obtain many properties of these polynomials as well as the q-cube polynomials of Fibonacci cubes themselves. These new properties include divisibility, positivity, and functional identities for both families.

Key words: Hypercube, Lucas number, Lucas cube, Fibonacci cube, cube enumerator polynomial, q-analogue

1. Introduction

The hypercube graph $Q_n = (V_n, E_n)$ of dimension n is one of the basic models for interconnection networks. The vertex set V_n denotes the processors and the edge set E_n corresponds to the communication links between processors in an ideal interconnection network. The vertices of Q_n are represented by all binary strings of length n with an edge between two vertices if and only if they differ in exactly one position. As the graph distance between two vertices of a graph is the length of the shortest path in the graph connecting these vertices, in Q_n this distance coincides with the Hamming distance.

Due to their symmetry and recursive properties, Fibonacci cubes and Lucas cubes were introduced as a new model of computation for interconnection networks in [3] and [7], respectively. Both families admit useful decompositions that allow for recursive constructions. The Fibonacci cube Γ_n of dimension n is the induced subgraph of Q_n in which the vertices correspond to those in V_n without two consecutive 1s in their string representation. The Lucas cube Λ_n is the induced subgraph of Q_n (and also subgraph of Γ_n), in which the vertices correspond to those without two consecutive 1s when the string representation of the vertex is viewed circularly. For convenience Γ_0, Λ_0 and Q_0 is taken to be the graph with a single vertex and no edges.

In the literature many properties and applications of these families of graphs are presented. In [4] the usage in theoretical chemistry and some results on the structure of Fibonacci cubes, including representations, recursive construction, hamiltonicity, the nature of the degree sequence, and some enumeration results, are given. The characterization of maximal induced hypercubes in Γ_n and Λ_n appears in [6]. Furthermore, the

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maximum number of disjoint hypercube subgraphs isomorphic to Q_k (called subcubes) in Γ_n is considered in [1, 9] and it is shown that asymptotically all vertices of Γ_n are covered by a maximum set of disjoint subcubes in [9]. The cube polynomial of Λ_n and Γ_n, which is the starting point of this paper, is studied in [5]. This is the polynomial whose coefficients enumerate subcubes of the given graph by their dimension. A q-analogue of the cube polynomial of the Fibonacci cubes, which carries additional data on its subcubes, was constructed in [8].

In this paper we q-count subcubes of Lucas cubes by their distance to the all 0 vertex, to be made precise momentarily. The resulting polynomial is what we refer to as the q-analogue of the ordinary cube polynomial, or the q-cube polynomial of Lucas cubes. This q-analogue, denoted by $C(\Lambda_n, x; q)$, not only adds a geometric meaning to the ordinary cube polynomial $C(\Lambda_n, x)$ but also satisfies a simple recursion similar to the one for the cube polynomial. As a consequence, its computation is relatively straightforward.

As an example, consider the Lucas cube Λ_2 in Figure 1. We have

$$C(\Lambda_2, x) = 3 + 2x,$$

indicating that Λ_2 contains three Q_0s and two Q_1s. On the other hand,

$$C(\Lambda_2, x; q) = 1 + 2q + 2x$$

expresses the fact that two of the three Q_0s are at distance 1 from 00 and the other at distance 0, contributing $1 + 2q$;, and both Q_1s in Λ_2 are at distance 0 from 00 (i.e. they contain 00), contributing the term $2x$.

Certain divisibility properties of the cube polynomials for Λ_n and Γ_n were noted in [5]. Our results extend these and also include information about the nature of the quotients. Interestingly, the quotients as polynomials in x have coefficients that are polynomials in q, which have nonnegative integral coefficients themselves. This is parallel to the case of Fibonacci cubes [8, Theorem 2]. Furthermore, we obtain expressions involving convolutions of the generalizations of the Fibonacci and Lucas numbers for the coefficients of the cube polynomials for Λ_n (and for Γ_n) and also show that they have certain derivative properties (see Section 4.3), which are useful in the computation of the quotient polynomials.

The paper is organized as follows: in Section 2 we give some preliminaries. We present our q-cube enumerator polynomial in Section 3 and investigate divisibility, special values, and other properties of the coefficients in Section 4. We note that many of the results presented here extend those of Klavžar and Mollard [5], as $C(\Lambda_n, x; q)$ is a refinement of the cube polynomial $C(\Lambda_n, x)$. Our approach and proofs follow along the lines of the Fibonacci case treated in [8], though the q-analogues of the Lucas cube polynomials have certain interesting properties in their own right.

2. Preliminaries

In this section we present some notation and preliminary notions. We start with the description of a hypercube. The n-dimensional hypercube (or n-cube) Q_n is the simple graph with vertex set

$$V_n = \{b_1 b_2 \cdots b_n \mid b_i \in \{0, 1\},\ 1 \leq i \leq n\}.$$

The number of vertices in Q_n is 2^n. The Fibonacci cube Γ_n is the induced subgraph of Q_n, obtained from Q_n by removing all vertices containing consecutive 1s. The number of vertices of Γ_n is f_{n+2}, where $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ are the Fibonacci numbers. If we remove the vertices with

$b_1 = b_n = 1$ from Γ_n, then we obtain the Lucas cube Λ_n. For $n \geq 1$, Λ_n has L_n vertices, where $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ are the Lucas numbers.

In Figure 1 the first four Lucas cubes are presented with their vertices labeled with the corresponding binary strings in the hypercube graph.

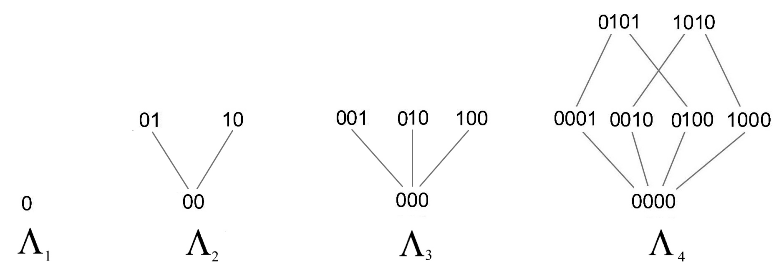

Figure 1. Lucas cubes $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$.

The following decompositions of Γ_n and Λ_n can be obtained easily from the definitions (see, for example, [7]): for the Fibonacci cubes, one can classify the binary strings defining the vertices of Γ_n by whether or not $b_1 = 0$ or $b_1 = 1$. In this way Γ_n decomposes into a subgraph Γ_{n-1} whose vertices are denoted by the strings that start with 0 and a subgraph Γ_{n-2} whose vertices are denoted by the strings that start with 10. This decomposition can be shown as $\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$. Furthermore, Γ_{n-1} in turn has a subgraph Γ'_{n-2} (whose vertices start with 00) isomorphic to Γ_{n-2}, with each vertex of Γ'_{n-2} connected by an edge to its twin in Γ_{n-2}. This is the *fundamental decomposition* of Γ_n. Similarly, Λ_n has a fundamental decomposition that comes from the classification of the binary strings defining the vertices in it: Λ_n has a subgraph Γ_{n-1} whose vertices are denoted by the corresponding strings starting with 0 and a subgraph Γ_{n-3} whose vertices are given by the strings that start with 10 and end with 0 in Λ_n. This decomposition is denoted by $\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0$. Furthermore, in this decomposition Γ_{n-1} has a subgraph Γ'_{n-3} (whose vertices start with 00 and end with 0) isomorphic to Γ_{n-3}, and each vertex of Γ'_{n-3} is connected by an edge to its twin in Γ_{n-3}. In the fundamental decomposition for Fibonacci cubes, there are f_n edges between Γ'_{n-2} and Γ_{n-2}. For Lucas cubes, there are f_{n-1} edges between Γ'_{n-3} and Γ_{n-3}.

Since there is a close relationship between hypercubes, Fibonacci cubes, and Lucas cubes it is natural to consider the number of k-dimensional hypercubes in Λ_n and Γ_n. The enumerator of these subgraphs was considered in [5] and generalized in [8]. In this paper, we consider a generalization of the hypercube enumerator polynomials of the Lucas cubes. Our polynomials involve an extra variable q (see Section 3), from which the known numerical cases given in [5] can be obtained by the specialization $q = 1$.

Starting with the q-analogue of the Fibonacci numbers given by $f_0(q) = 0, f_1(q) = 1$, and

$$f_n(q) = f_{n-1}(q) + q f_{n-2}(q) \tag{1}$$

for $n \geq 2$, the first few $f_n(q)$ for $n \geq 2$ are computed as:

$$1,\ 1+q,\ 1+2q,\ 1+3q+q^2,\ 1+4q+3q^2,\ 1+5q+6q^2+q^3,\ \ldots$$

We define $F_n(q) = f_{n+2}(q)$ for all $n \geq 0$.

Similar to the case of the Fibonacci numbers, a q-analogue of the Lucas numbers can be defined by

$L_0(q) = 2, L_1(q) = 1$, and

$$L_n(q) = L_{n-1}(q) + qL_{n-2}(q) \tag{2}$$

for $n \geq 2$. Using (2), the first few $L_n(q)$ for $n \geq 2$ are:

$$1 + 2q,\ 1 + 3q,\ 1 + 4q + 2q^2,\ 1 + 5q + 5q^2,\ 1 + 6q + 9q^2 + 2q^3,\ \ldots$$

It is well known that (see, for example, [10])

$$\sum_{n\geq 0} f_n t^n = \frac{t}{1-t-t^2} \quad \text{and} \quad \sum_{n\geq 0} L_n t^n = \frac{2-t}{1-t-t^2}\ .$$

Similarly, one can easily obtain the generating functions of $f_n(q)$ and $L_n(q)$ as

$$\sum_{n\geq 0} f_n(q) t^n = \frac{t}{1-t-qt^2} \quad \text{and} \quad \sum_{n\geq 0} L_n(q) t^n = \frac{2-t}{1-t-qt^2}\ . \tag{3}$$

These generating functions will be useful to derive the expressions we have for the coefficients of the q-analogue as convolutions of the polynomials that involve $f_n(q)$ and $L_n(q)$.

Let $h_{n,k}$ denote the number of k-dimensional hypercubes in the Lucas cube Λ_n. The cube polynomial, or the cube enumerator polynomial $C(\Lambda_n, x)$, is defined in [5] as

$$C(\Lambda_n, x) = \sum_{k\geq 0} h_{n,k} x^k\ . \tag{4}$$

A few of these are given below:

$$\begin{aligned}
C(\Lambda_0, x) &= 1,\\
C(\Lambda_1, x) &= 1,\\
C(\Lambda_2, x) &= 3 + 2x,\\
C(\Lambda_3, x) &= 4 + 3x,\\
C(\Lambda_4, x) &= 7 + 8x + 2x^2,\\
C(\Lambda_5, x) &= 11 + 15x + 5x^2,\\
C(\Lambda_6, x) &= 18 + 30x + 15x^2 + 2x^3,\\
C(\Lambda_7, x) &= 29 + 56x + 35x^2 + 7x^3.
\end{aligned}$$

Evidently the constant terms are the number of Q_0s, i.e. the number of vertices of Λ_n. Therefore, for $n \geq 1$ we have $C(\Lambda_n, 0) = L_n$.

Many interesting results on $C(\Lambda_n, x)$ and $h_{n,k}$ in (4) appear in [5]. It is observed in [5] that for $n \geq 3$ the numbers in Table 1 satisfy the recursion

$$h_{n,k} = h_{n-1,k} + h_{n-2,k} + h_{n-2,k-1} \tag{5}$$

where $h_{n,-1} = 0$. The first column entries ($k = 0$) of the table are $1, L_1, L_2, L_3, \ldots$; the diagonal entries are $1, 0, 0, 0, \ldots$ and $h_{2,1} = 2$. After these, the other entries can be filled row by row by using the recursion (5).

Table 1. The table of coefficients of the cube polynomials $C(\Lambda_n, x)$ by rows. The entry in row n, column k is the coefficient $h_{n,k}$, the number of k-dimensional hypercubes in the Lucas cube Λ_n.

$n\backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	0	0	0	0
2	3	2	0	0	0
3	4	3	0	0	0
4	7	8	2	0	0
5	11	15	5	0	0
6	18	30	15	2	0
7	29	56	35	7	0

3. q-counting subcubes in Λ_n

Recall that the distance between two subgraphs of a graph is the smallest distance between pairs of vertices taken one from each. The polynomials $C(\Lambda_n, x; q)$ of the Lucas cube Λ_n will be constructed by keeping track of the distance of each k-dimensional hypercube in Λ_n to the all 0 vertex in Λ_n in the following fashion: $C(\Lambda_n, x; q)$ is defined as the sum of all terms of the form $q^d x^k$, one for each subcube of Λ_n. The exponent k is the dimension of the subcube and the exponent d is the distance of the subcube to the all 0 vertex in Λ_n. Similarly, the q-cube polynomial of Γ_n is defined as the bivariate polynomial $c_n(x; q)$ whose terms are of the form $q^d x^k$ as above [8].

It is useful to think of $C(\Lambda_n, x; q)$ as a polynomial in x whose coefficients are polynomials in q. The first few $C(\Lambda_n, x; q)$ are as follows:

$$\begin{aligned}
C(\Lambda_0, x; q) &= 1, \\
C(\Lambda_1, x; q) &= 1, \\
C(\Lambda_2, x; q) &= 1 + 2q + 2x, \\
C(\Lambda_3, x; q) &= 1 + 3q + 3x, \\
C(\Lambda_4, x; q) &= 1 + 4q + 2q^2 + (4 + 4q)x + 2x^2, \\
C(\Lambda_5, x; q) &= 1 + 5q + 5q^2 + (5 + 10q)x + 5x^2, \\
C(\Lambda_6, x; q) &= 1 + 6q + 9q^2 + 2q^3 + (6 + 18q + 6q^2)x + (9 + 6q)x^2 + 2x^3, \\
C(\Lambda_7, x; q) &= 1 + 7q + 14q^2 + 7q^3 + (7 + 28q + 21q^2)x + (14 + 21q)x^2 + 7x^3.
\end{aligned}$$

Now we illustrate the structure of $C(\Lambda_2, x; q)$ and $C(\Lambda_4, x; q)$ in more detail. Recall that there are three vertices (Q_0) and 2 edges (Q_1) in the graph of Λ_2 as shown in Figure 1. The Q_0s are the vertices of the graph. There is a single vertex having distance 0 to the vertex 00 (i.e. 00 itself) and there are two vertices having distance 1. Therefore, the coefficient of x^0 in $C(\Lambda_2, x; q)$ is $1 + 2q$. Similarly, 1-dimensional hypercubes Q_1 are the edges of the graph. In Λ_2 there are a total of two of those, each having distance zero to the vertex 00. Therefore, the coefficient of x is 2. This gives $C(\Lambda_2, x; q) = 1 + 2q + 2x$.

Similarly, to construct $C(\Lambda_4, x; q)$ we consider all hypercubes in Λ_4 having dimension $k < 3$ and their distances to the 0000. Note that there are no higher dimensional hypercubes in Λ_4. For $k = 0$ we know that there are seven vertices in the graph giving 0-dimensional hypercubes. The vertex 0000 has distance 0; the vertices 0001, 0010, 0100, and 1000 each have distance 1; and the vertices 0101 and 1010 have distance 2 to 0000. Thus, the coefficient of x^0 is $1 + 4q + 2q^2$.

Now consider $k = 1$, that is, 1-dimensional hypercubes in the graph. We know that they are the edges of the graph and from Figure 2 we see that there are four edges with distance 0 and four edges with distance 1 to the vertex 0000. Thus, the coefficient of x in $C(\Lambda_4, x; q)$ is $4 + 4q$.

Finally, there are only two 2-dimensional hypercubes in Λ_4 and these hypercubes contain the vertex 0000. Thus, the total contribution from these 2-dimensional subcubes is $2x^2$. Adding these contributions, we get $C(\Lambda_4, x; q) = (1 + 4q + 2q^2) + (4 + 4q)x + 2x^2$. A graphical presentation of these hypercubes in Λ_4 and their individual contributions to $C(\Lambda_4, x; q)$ is presented in Figure 2.

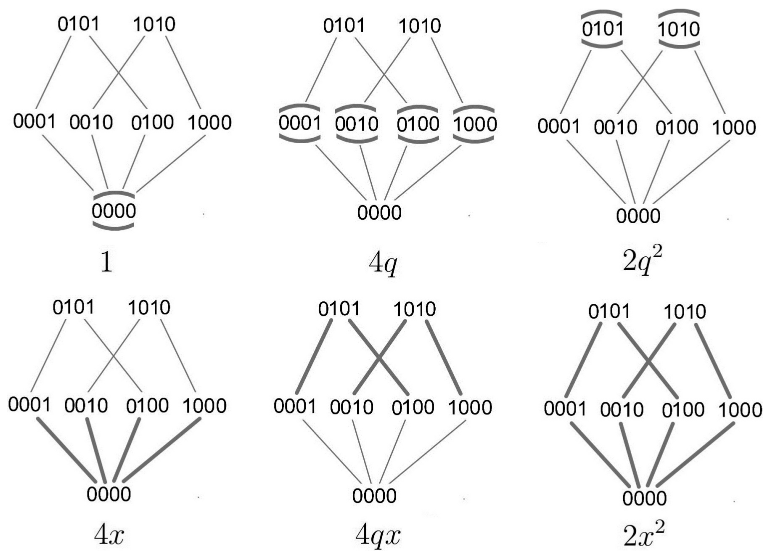

Figure 2. The elements of the q-cube polynomial $C(\Lambda_4, x; q) = 1 + 4q + 2q^2 + (4 + 4q)x + 2x^2$.

Next we present the basic recursion that allows for the calculation of the q-cube polynomials and is central to what follows. We remark that the proof of this result is similar to the proof of [8, Lemma 1].

Lemma 1 *For* $n \geq 3$ *the* q*-cube polynomial* $C(\Lambda_n, x; q)$ *satisfies the recursion*

$$C(\Lambda_n, x; q) = C(\Lambda_{n-1}, x; q) + (q + x)C(\Lambda_{n-2}, x; q) \tag{6}$$

with $C(\Lambda_1, x; q) = 1$ *and* $C(\Lambda_2, x; q) = 1 + 2q + 2x$.

Proof From the numerical values it is easy to see that the result is true for $n \leq 4$. For $n \geq 5$, first we show that

$$C(\Lambda_n, x; q) = c_{n-1}(x; q) + (q + x)c_{n-3}(x; q) \tag{7}$$

where $c_n(x; q)$ is the q-cube polynomial of Γ_n. We make use of the fundamental decomposition of Λ_n into Γ_{n-1} and Γ_{n-3} [7]. Γ_{n-1} contains an isomorphic copy of Γ_{n-3}, which is denoted by Γ'_{n-3}, with unique edges between the corresponding vertices of the subgraph Γ_{n-3} and this copy Γ'_{n-3}. It follows that there are only three kinds of hypercubes in Λ_n:

Case 1: A k-dimensional hypercube in Γ_{n-1} remains a k-dimensional hypercube in Λ_n and the distances of these cubes to the all 0 vertex remain unchanged. By the induction hypothesis, these are enumerated by $c_{n-1}(x;q)$.

Case 2: Any k-dimensional hypercube in Γ_{n-3} is again a k-dimensional hypercube in Λ_n, and the distances of these cubes in Λ_n to the all 0 vertex go up by 1 due to the edges identifying the corresponding vertices in Γ_{n-3} and Γ'_{n-3}. This increase in the distance to the all 0 vertex by 1 means multiplication by q. The contribution of these hypercubes is $qc_{n-3}(x;q)$.

Case 3: A k-dimensional hypercube in Γ_{n-3} has an isomorphic copy in Γ'_{n-3} and all the corresponding vertices of these k-dimensional hypercubes are connected by edges to their twins. Therefore, these two k-dimensional hypercubes together with the edges connecting them form a $(k+1)$-dimensional hypercube in Λ_n. Also, the distances of these cubes to the all 0 vertex remain unchanged. The contribution of these hypercubes is $xc_{n-3}(x;q)$, since multiplication by x has the effect of increasing the dimension by 1. Adding up these three contributions, we obtain (7).

Now by using (7) we complete the proof as follows. For $n \geq 5$ we have

$$\begin{aligned}
C(\Lambda_{n-1},x;q)+(q+x)C(\Lambda_{n-2},x;q) &= [c_{n-2}(x;q)+(q+x)c_{n-4}(x;q)] \\
&\quad +(q+x)\left[c_{n-3}(x;q)+(q+x)c_{n-5}(x;q)\right] \\
&= [c_{n-2}(x;q)+(q+x)c_{n-3}(x;q)] \\
&\quad +(q+x)\left[c_{n-4}(x;q)+(q+x)c_{n-5}(x;q)\right] \\
&= c_{n-1}(x;q)+(q+x)c_{n-3}(x;q) \\
&= C(\Lambda_n,x;q)
\end{aligned}$$

where we have used $c_n(x;q) = c_{n-1}(x;q)+(q+x)c_{n-2}(x;q)$ given in [8, Lemma 1]. □

We next determine the generating function for the q-cube polynomial $C(\Lambda_n,x;q)$ and relate it to the convolutions involving the q-analogues of the Fibonacci numbers and Lucas numbers in (1) and (2).

Proposition 1 *The generating function of the q-cube polynomial $C(\Lambda_n,x;q)$ is*

$$\sum_{n\geq 0} C(\Lambda_n,x;q)t^n = \frac{1+t^2(q+x)}{1-t-t^2(q+x)} \ .$$

Proof Let $S = \sum_{n\geq 1} C(\Lambda_n,x;q)t^n$. We know that $C(\Lambda_1,x;q)=1$, $C(\Lambda_2,x;q)=1+2q+2x$ and $C(\Lambda_n,x;q)$ satisfies recursion (6). Therefore, S satisfies

$$S-t-t^2(1+2q+2x) = t(S-t)+t^2(q+x)S \ ,$$

which can be solved for S to compute the generating function as $1+S$. □

The recursion in (6) can be solved directly to find $C(\Lambda_n,x;q)$ in explicit form.

Theorem 1 *For any positive integer* n, *the* q-*cube polynomial* $C(\Lambda_n, x; q)$ *of the Lucas cube has degree* $\lfloor \frac{n}{2} \rfloor$ *in* x *and it is given explicitly as*

$$C(\Lambda_n, x; q) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (1 + 4(q + x))^i \,. \tag{8}$$

Proof We know that the characteristic equation of the recursion in (6) is

$$r^2 - r - (q + x) = 0.$$

Using the Binet formula, this equation gives an explicit expression in the form

$$C(\Lambda_n, x; q) = \frac{(1 + \theta)^n + (1 - \theta)^n}{2^n} \tag{9}$$

where $\theta = \sqrt{1 + 4(q + x)}$. Using binomial expansions for $(1 \pm \theta)^n$ and after some algebraic manipulation, we obtain (8). □

In particular, writing

$$C(\Lambda_n, x; q) = \sum_{k \geq 0} h_{n,k}(q) x^k \ ,$$

we obtain the following formula for the coefficient polynomials $h_{n,k}(q)$.

Corollary 1 *For any positive integer* n, *the coefficient polynomials of the* q-*cube polynomial* $C(\Lambda_n, x; q)$ *are given by*

$$h_{n,k}(q) = \frac{1}{2^{n-1}} \left(\frac{4}{1 + 4q} \right)^k \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \binom{i}{k} (1 + 4q)^i \ .$$

In particular,

$$L_n(q) = h_{n,0}(q) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (1 + 4q)^i \ .$$

A few of the polynomials $h_{n,k}(q)$ are given in Table 2.

Using the properties of convolutions we also obtain the following result relating the coefficient polynomials $h_{n,k}(q)$ of the q-cube polynomial $C(\Lambda_n, x; q)$ and the q-analogues of the Fibonacci numbers and Lucas numbers given in (1) and (2).

Proposition 2 *For* $n \geq 1$ *the coefficient polynomials* $h_{n,k}(q)$ *of the* q-*cube enumerator* $C(\Lambda_n, x; q)$ *are given by*

$$h_{n,k}(q) = \sum_{\substack{i_0, i_1, \ldots, i_k \geq 0 \\ i_0 + i_1 + \cdots + i_k = n - k}} L_{i_0}(q) f_{i_1}(q) \cdots f_{i_k}(q).$$

Table 2. The table of coefficients of the q-cube polynomials $C(\Lambda_n, x; q)$ by rows. The entry in row n, column k is the coefficient polynomial $h_{n,k}(q)$.

$n\backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	0	0	0	0
2	$1+2q$	2	0	0	0
3	$1+3q$	3	0	0	0
4	$1+4q+2q^2$	$4+4q$	2	0	0
5	$1+5q+5q^2$	$5+10q$	5	0	0
6	$1+6q+9q^2+2q^3$	$6+18q+6q^2$	$9+6q$	2	0
7	$1+7q+14q^2+7q^3$	$7+28q+21q^2$	$14+21q$	7	0

Proof From Proposition 1 we know that the generating function of the $C(\Lambda_n, x; q)$ is

$$\sum_{n\geq 0} C(\Lambda_n, x; q)t^n = \frac{1+t^2(q+x)}{1-t-t^2(q+x)} \ . \tag{10}$$

On the other hand, by (3) the convolution of $L_n(q)$ with the k-fold convolutions of $f_n(q)$ has the generating function

$$\frac{(2-t)t^k}{(1-t-qt^2)^{k+1}} \ .$$

Setting

$$g_k(t;q) = \frac{(2-t)t^{2k}}{(1-t-qt^2)^{k+1}}$$

for $k \geq 1$ with

$$g_0(t;q) = \frac{2-t}{(1-t-qt^2)} - 1$$

and calculating directly, we find

$$\begin{aligned} \sum_{k\geq 0} g_k(t;q)x^k &= -1 + \frac{2-t}{(1-t-qt^2)} \sum_{k\geq 0} \left(\frac{xt^2}{1-t-qt^2} \right)^k \\ &= \frac{1+t^2(q+x)}{1-t-t^2(q+x)} \ . \end{aligned}$$

This is identical to the generating function of the $C(\Lambda_n, x; q)$ of (10). It follows that the $g_k(t;q)$ are the generating functions of the columns of Table 2. The proposition now follows by equating the coefficients of $t^n x^k$ in the two expressions as we have

$$\sum_{k\geq 0} g_k(t;q)x^k = \sum_{k\geq 0} \left(\left(\sum_{i_0\geq 0} L_{i_0}(q)t^{i_0} \right) \left(\sum_{i\geq 0} f_i(q)t^i \right)^k \right) t^k x^k = \sum_{n\geq 0} \left(\sum_{k\geq 0} h_{n,k}(q)x^k \right) t^n \ .$$

□

From [7] we know that there are

$$\frac{n}{n-i}\binom{n-i}{i} = 2\binom{n-i}{i} - \binom{n-i-1}{i} \tag{11}$$

different vertices in Λ_n, which contain i 1s (that is, vertices having weight i). Note that the distance between a vertex having weight i and all zero vertex is obviously i. Using this information, the polynomials in the first column ($k = 0$) of Table 2 can be written for $n \geq 1$ as

$$L_n(q) = h_{n,0}(q) \quad = \quad \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i} q^i \ . \tag{12}$$

In general, we have the following expression for the entry in row n, column k:

Proposition 3 *The coefficient polynomials $h_{n,k}(q)$ of the q-cube enumerator $C(\Lambda_n, x; q)$ of the Lucas cube Λ_n are given explicitly by*

$$h_{n,k}(q) = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i}\binom{i}{k} q^{i-k} \ .$$

Proof From (11) we know that there are $\frac{n}{n-i}\binom{n-i}{i}$ different vertices in Λ_n, which contain i 1s. For a given subcube Q_k of Λ_n, let u be the vertex of the Q_k having maximum weight among all the vertices of the Q_k, say i. Then we know that there is set of k positions in u such that the Q_k is induced by the 2^k vertices obtained by varying these k bits. Now, we know that each such vertex u gives $\binom{i}{k}$ different subcubes. Furthermore, there is a vertex of the Q_k having minimum weight among all the vertices of the Q_k, whose weight is $i-k$. That is, the distance of such Q_k s to the all zero vertex is $i-k$. Then the result follows. □

Note that Proposition 3 can also be proved directly from the recurrence in (6), by using induction on k and verifying a binomial identity.

Remark 1 *Equating the two different expressions for $h_{n,k}(q)$ in Corollary 1 and Proposition 3 gives the following identity for $n \geq 1$ and $k \leq \lfloor \frac{n}{2} \rfloor$:*

$$\frac{1}{2^{n-1}}\left(\frac{4}{1+4q}\right)^k \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{i}{k}(1+4q)^i = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i}\binom{i}{k} q^{i-k} \ .$$

4. Further results

In this section we consider the divisibility properties of $C(\Lambda_n, x; q)$ and present some special new results for the case $q = 1$, which is the case considered in [5].

4.1. Divisibility properties of the q-cube polynomials

Consider the q-cube polynomials $c_n(x; q)$ and $C(\Lambda_n, x; q)$ of the Fibonacci cubes Γ_n and the Lucas cubes Λ_n, respectively. Recall that the Binet formulas for these polynomials are obtained in [8, Equation 8] and (9) as

$$c_n(x; q) = \frac{(1+\theta)^{n+2} - (1-\theta)^{n+2}}{2^{n+2}\theta} \quad \text{and} \quad C(\Lambda_n, x; q) = \frac{(1+\theta)^n + (1-\theta)^n}{2^n} \tag{13}$$

where $\theta = \sqrt{1 + 4(q+x)}$. Using these expressions we immediately obtain the following divisibility results.

Proposition 4 *Let $c_n(x;q)$ and $C(\Lambda_n,x;q)$ be the q-cube polynomials of the Fibonacci cubes Γ_n and the Lucas cubes Λ_n, respectively. Then we have:*

1. *For any $n,m \geq 0$ with m odd, $C(\Lambda_n,x;q)$ divides $C(\Lambda_{mn},x;q)$ as a polynomial in x. Furthermore, the coefficients of powers of x in this quotient are polynomials in q with nonnegative integer coefficients.*
2. *For any $m \geq 1$, $c_{2m}(x;q) = c_{m-1}(x;q)C(\Lambda_{m+1},x;q)$.*
3. *For any $n,m \geq 1$ with m even $C(\Lambda_n,x;q)$ divides $c_{mn-2}(x;q)$ as a polynomial in x.*

Proof Using (13) one can easily obtain all the divisibility parts. Here we only prove the second part of the first result using induction on m (see also Remark 3).

For a fixed n, let us define

$$A = (1+\theta)^n \quad \text{and} \qquad B = (1-\theta)^n \ . \tag{14}$$

By the definition of $C(\Lambda_{tn},x;q)$ and (13) we note that $2^{tn} \,|\, (A^t + B^t)$ for all positive integers t. Hence, for the quotient

$$\frac{C(\Lambda_{mn},x;q)}{C(\Lambda_n,x;q)} = \frac{A^m + B^m}{2^{(m-1)n}(A+B)}$$

we only need to consider the coefficients of powers of x in

$$Q_m(x;q) = \frac{A^m + B^m}{A+B} \ . \tag{15}$$

For $m = 1$ we have $Q_1(x;q) = 1$ and assume that the result is true for $Q_m(x;q)$. We need to consider $Q_{m+2}(x;q)$ and we can write

$$\begin{aligned} Q_{m+2}(x;q) &= \frac{A^{m+2} + B^{m+2}}{A+B} \\ &= \frac{A^m + B^m}{A+B}\left(A^2 + B^2\right) - \frac{A^{m-2} + B^{m-2}}{A+B}A^2B^2 \\ &= Q_m(x;q)\left(A^2+B^2\right) - Q_{m-2}(x;q)A^2B^2 \ , \end{aligned} \tag{16}$$

where

$$\begin{aligned} A^2 + B^2 &= (1+\theta)^{2n} + (1-\theta)^{2n} \\ &= \left(\left(1+\sqrt{1+4(q+x)}\right)^{2n} + \left(1-\sqrt{1+4(q+x)}\right)^{2n}\right) \\ &= \sum_{i\geq 0} 2\binom{2n}{2i}(1+4(q+x))^i \ , \end{aligned} \tag{17}$$

and

$$A^2B^2 = (1-\theta^2)^n = ((-4(q+x))^n \ , \tag{18}$$

which are polynomials in x and the coefficients of powers of x are polynomials in q with integer coefficients. Then, using the induction assumption, we conclude that $Q_{m+2}(x;q)$ is a polynomial in x and the coefficients of powers of x are polynomials in q with integer coefficients. Furthermore, using (16), (17), and (18) the nonnegativity of coefficients is clear for odd positive integers n. For even n, the nonnegativity of coefficients can be obtained using (14) and (15) directly. □

Remark 2 *Letting $q = 1$, the results in Proposition 4 reduce to [5, Corollary 6.3, (i)] and [5, Corollary 6.3, (iii)]. Note that the condition m odd is necessary in the first result in Proposition 4 although it is missing in [5, Corollary 6.3, (iii)]. For example $C(\Lambda_2, x; 1) = 2x + 3$ does not divide $C(\Lambda_4, x; 1) = 2x^2 + 8x + 7$.*

4.2. Results for $q = 1$

In this section we present some special results for the case $q = 1$, which is the case considered in [5]. By using (11) and setting $q = 1$, Proposition 3 reduces to

$$h_{n,k}(1) = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \left[2\binom{n-i}{i} - \binom{n-i-1}{i} \right] \binom{i}{k}$$

where $h_{n,k}(1)$ is the coefficient polynomial of the cube enumerator $C(\Lambda_n, x; 1)$ of the Lucas cube Λ_n considered in [5, Theorem 5.2] and [5, Corollary 5.3].*

The constant term in $C(\Lambda_n, x; 1)$ for $n \geq 1$ is the number of vertices L_n of Λ_n, which is obtained by taking $x = 0$ in (8). This gives the following curious result:

Proposition 5 *The Lucas numbers $L_0 = 2, L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ are given by*

$$L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 5^i .$$

Note that a similar result for Fibonacci numbers is obtained in [8, Proposition 5], which is of the form

$$f_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2i+1} 5^i .$$

By combining these two expressions we obtain the well-known identity $L_n + f_n = 2f_{n+1}$.

From the expression (8) for the $C(\Lambda_n, x; q)$ we can immediately get other specializations. For instance, taking $x = -(1 + 4q)/4$ gives

$$C\left(\Lambda_n, -\frac{1+4q}{4}; q\right) = \frac{1}{2^{n-1}} \quad (n \geq 1).$$

This can of course be obtained from the original recursion (6) by setting $x = -(1 + 4q)/4$ and solving the resulting recursion. This is the generalization of $C\left(\Lambda_n, -\frac{5}{4}; 1\right) = \frac{1}{2^{n-1}}$, for $n \geq 1$ as given in [5, p. 103].

*The second combinatorial parts of these results in [5] appear to contain a typo. The correct version of the corresponding parts $\binom{n-a+1}{a}$ and $\binom{n-i+1}{i}$ in these results should be $\binom{n-a-1}{a}$ and $\binom{n-i-1}{i}$.

Incidentally, similar to the Fibonacci cube case, the values $a_n = C(\Lambda_n, 1; 1)$ satisfy the recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

with the initial values $a_0 = 1, a_1 = 1$, giving the shifted Jacobsthal sequence [2]:

$$1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, \ldots$$

4.3. Further properties of the q-cube polynomials

Using Corollary 1 (or Proposition 3) and [8, Proposition 3] it is possible to derive additional properties of the coefficients of q-cube polynomials of $C(\Lambda_n, x; q)$ and $c_n(x; q)$. Write

$$C(\Lambda_n, x; q) = \sum_{k\geq 0} h_{n,k}(q)x^k \quad \text{and} \quad c_n(x; q) = \sum_{k\geq 0} H_{n,k}(q)x^k$$

and let $D : \mathbb{Z}[q] \to \mathbb{Z}[q]$ denote the differentiation operator. We have the following result.

Proposition 6 *The coefficient polynomials $h_{n,k}(q)$ and $H_{n,k}(q)$ of the q-cube polynomials of $C(\Lambda_n, x; q)$ and $c_n(x; q)$ satisfy*

$$\frac{1}{k} D h_{n,k-1}(q) = h_{n,k}(q) \quad \textit{and} \quad \frac{1}{k} D H_{n,k-1}(q) = H_{n,k}(q) \ .$$

In particular,

$$\frac{1}{k!} D^k h_{n,0}(q) = h_{n,k}(q) \quad \textit{and} \quad \frac{1}{k!} D^k H_{n,0}(q) = H_{n,k}(q) \ .$$

Using operator notation we can write

$$C(\Lambda_n, x; q) = e^{xD} L_n(q) \quad \text{and} \quad c_n(x; q) = e^{xD} F_n(q) = e^{xD} f_{n+2}(q) \ .$$

Therefore, Taylor's theorem gives the following expressions for the q-cube polynomials.

Proposition 7 *Let $C(\Lambda_n, x; q)$ and $c_n(x; q)$ be the q-cube polynomials of the Lucas cubes Λ_n and the Fibonacci cubes Γ_n, respectively. Then*

$$C(\Lambda_n, x; q) = L_n(x + q) \quad \textit{and} \quad c_n(x; q) = F_n(x + q) = f_{n+2}(x + q)$$

where $L_n(q)$ and $f_n(q)$ are as defined in (2) and (1).

Remark 3 *From Proposition 4, we have that $C(\Lambda_n, x; q)$ divides $C(\Lambda_{mn}, x; q)$. Assume that the quotient is $Q(x; q) = \sum_{k\geq 0} a_k(q)x^k$, so that $C(\Lambda_{mn}, x; q) = C(\Lambda_n, x; q)Q(x; q)$. Since the coefficients of $C(\Lambda_n, x; q)$ and $C(\Lambda_{mn}, x; q)$ satisfy the derivative properties given in Proposition 6 it is clear that the coefficients of $Q(x; q)$ also satisfy these properties; that is, $\frac{1}{k!} D^k a_0(q) = a_k(q)$. Therefore, to compute the coefficients of the quotient polynomial $Q(x; q)$, and especially to prove nonnegativity of the coefficients, it suffices to find*

$$a_0(q) = \frac{C(\Lambda_{mn}, 0; q)}{C(\Lambda_n, 0; q)} = \frac{L_{mn}(q)}{L_n(q)} \ .$$

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