# On oscillation of integro-differential equations

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**Abstract:** We study the oscillatory behavior of solutions for integro-differential equations of the form

$$x'(t) = e(t) - \int_0^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) \, ds, \quad t \geq 0,$$

where $0 < \alpha < 1$. Our method is based on the use of the beta function and asymptotic behavior of nonoscillatory solutions. An example is given to illustrate the main result. Equations of this form include Caputo type fractional differential equations, so the results are applicable to some fractional type differential equations as well.

**Key words:** Integro-differential equation, oscillation, singular, Volterra equation

## 1. Introduction

We investigate the oscillatory behavior of solutions of the integro-differential equation

$$x'(t) = e(t) - \int_0^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) \, ds, \quad t \geq 0, \tag{1}$$

where $0 < \alpha < 1$ is a real number and the functions $e$, $k$, and $f$ are continuous in their domain of definitions. We assume that there exist continuous functions $a, h, m : [0, \infty) \to [0, \infty)$ and real numbers $\gamma > 0$ and $0 < \lambda < 1$ such that

$$0 \leq k(t,s) \leq a(t) h(s) \text{ for all } t \geq s \geq 0 \tag{2}$$

and

$$0 < x f(t, x) \leq t^{\gamma-1} m(t) |x|^{\lambda+1} \text{ for all } x \neq 0 \text{ and } t \geq 0. \tag{3}$$

Furthermore, there exist real numbers $M_1 > 0$ and $M_2$ such that

$$|a(t)| \leq M_1 \tag{4}$$

and for every $T \geq 0$,

$$-M_2 \leq \liminf_{t \to \infty} \frac{1}{t} \int_T^t e(s) \, ds \leq \limsup_{t \to \infty} \frac{1}{t} \int_T^t e(s) \, ds \leq M_2. \tag{5}$$

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By an oscillatory solution of (1) we mean a function $x(t)$ having arbitrarily large zeros. That is, there is a sequence $\{t_n\}$ of real numbers such that $t_n \to \infty$ as $n \to \infty$ and $x(t_n) = 0$. Therefore, in the present work we only consider those solutions of equations (1) that are nontrivial in the neighborhood of the infinity, and for existence and uniqueness of such solutions we refer to [10]. As usual, the equation is called oscillatory if all its solutions are oscillatory. If a solution is not oscillatory it is said to be nonoscillatory. Clearly, a nonoscillatory solution is either eventually positive or eventually negative.

Our motivation in the present work stems from the fact that equations of type (1) arise in many problems of science and engineering such as mathematical models in ecology [12], spread of epidemics [5, 13], electric-circuit analysis [2, 19], finance [3], mechanics [4], and plasma physics [8]. Therefore, information on the qualitative behavior of the solutions of (1) is crucial in order to better understand the underlying structure. To this end, the oscillation of solutions for such equations seems to be highly important. However, the research on oscillation theory for integro-differential equations is its early stages due to difficulties encountered in adapting the techniques from differential equations. For some limited results we refer the reader in particular to [6, 7, 11, 14–16, 18].

It should also be mentioned that there is a connection with fractional differential equations. Indeed, by setting

$$k(t,s) = \frac{1}{\Gamma(\alpha)}, \quad e(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,$$

we may write from (1) that

$$x'(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [g(s) - f(s, x(s))]\, ds, \quad 0 < \alpha < 1. \tag{6}$$

Now it is not difficult to see that Eq. (6) is equivalent to the fractional differential equation of the Caputo type

$$^C D^{\alpha+1} x(t) + f(t, x(t)) = g(t), \quad x_0 = x'(0) \tag{7}$$

under some mild conditions on $f(t,x)$ and $g(t)$; see [9, 17]. Therefore, one can easily rewrite the results of this paper for the fractional differential equation (7). Note that if $g$ is a bounded integrable function on an interval, then $e(t)$ becomes continuous there.

In the present work we establish two theorems. Theorem 1 deals with the asymptotic behavior of nonoscillatory solutions, whereas Theorem 2 provides sufficient conditions for oscillation of all solutions of Eq. (1). The novelty of the oscillation theorem lies in the use of Theorem 1.

The following inequality is essentially the Young inequality [1].

**Lemma 1** *If $X$ and $Y$ are nonnegative real numbers, then we have*

$$X^\beta - (1-\beta)Y^\beta - \beta X Y^{\beta-1} \le 0 \quad for\ 0 < \beta < 1,$$

*where the equality holds if and only if $X = Y$.*

We will also make use of the simple identity

$$\int_0^t s^{a-1}(t-s)^{b-1}\, ds = t^{a+b+1} B(a,b), \tag{8}$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{dt}$$

is the well-known beta function.

## 2. The main results

For a given continuous function $\eta : [0,\infty) \to (0,\infty)$, we define

$$C(t,\eta) = \eta^{\frac{\lambda}{1-\lambda}}(t)m^{\frac{1}{1-\lambda}}(t)h^{\frac{1}{1-\lambda}}(t), \quad t \geq 0, \tag{1}$$

where $\lambda$, $m$, and $h$ are as in the previous section.

We first show that every nonoscillatory solution of Eq. (1) satisfies

$$x(t) = O(t), \quad t \to \infty$$

under some very mild conditions.

**Theorem 1** *Let $q$ be a conjugate number of $p > 1$, i.e. $q = p/(p-1)$. Suppose that $p < 1/(1-\alpha)$, and the conditions (2)–(5) hold with $\gamma = 1 - \alpha + 1/q$.*

*If there exists continuous function $\eta : [0,\infty) \to [0,\infty)$ for which $t\eta(t), C(t,\eta(t)) \in L^q[0,\infty)$, then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty. \tag{2}$$

**Proof**  Let $x(t)$ be a nonoscillatory solution, say $x(t) > 0$, for all $t \geq t_1$ for some $t_1 \geq 1$. Put

$$k_1 := \max\{|f(t,x(t))| : \ t \in [0,t_1]\} \geq 0 \ \text{ and } \ k_2 := k_1 \int_0^{t_1} (t_1 - s)^{\alpha-1}h(s)\,\mathrm{ds} \geq 0. \tag{3}$$

In view of (2), (3), and (3), we may write from Eq. (1) that

$$
\begin{aligned}
x'(t) &= e(t) - \int_0^{t_1}(t-s)^{\alpha-1}k(t,s)f(s,x(s))\,\mathrm{ds} - \int_{t_1}^t(t-s)^{\alpha-1}k(t,s)f(s,x(s))\,\mathrm{ds} \\[2mm]
&\leq e(t) + \int_0^{t_1}(t_1-s)^{\alpha-1}k(t,s)|f(s,x(s))|\,\mathrm{ds} + \int_{t_1}^t(t-s)^{\alpha-1}k(t,s)f(s,x(s))\,\mathrm{ds} \\[2mm]
&\leq e(t) + k_1 a(t)\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)\,\mathrm{ds} + a(t)\int_{t_1}^t(t-s)^{\alpha-1}s^{\gamma-1}h(s)m(s)x^\lambda(s)\,\mathrm{ds} \\[2mm]
&\leq e(t) + k_2 a(t) + a(t)\int_{t_1}^t(t-s)^{\alpha-1}s^{\gamma-1}\Big(h(s)m(s)x^\lambda(s) - \eta(s)x(s)\Big)\,\mathrm{ds} \\[2mm]
&\quad + a(t)\int_{t_1}^t(t-s)^{\alpha-1}s^{\gamma-1}\eta(s)x(s)\,\mathrm{ds}, \quad t \geq t_1.
\end{aligned}
\tag{4}
$$

By Lemma 1 with the choices

$$X = (hm)^{1/\lambda}x, \quad Y = (\eta(hm)^{-1/\lambda}\lambda^{-1})^{1/(\lambda-1)}, \qquad \beta = \lambda,$$

we see that

$$h(s)m(s)x^\lambda(s) - \eta(s)x(s) \le \lambda_0\, C(s,\eta(s)), \quad s \ge t_1,$$

where $\lambda_0 = (1-\lambda)\lambda^{\lambda/(1-\lambda)}$ and $C(t,\eta)$ is defined in (1). Therefore, inequality (4) gives

$$
\begin{aligned}
x'(t) \;\le\;& e(t) + k_2 a(t) + \lambda_0\, a(t) \int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}C(s,\eta(s))\,\mathrm{d}s \\[2mm]
& + a(t)\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}\eta(s)x(s)\,\mathrm{d}s, \quad t \ge t_1,
\end{aligned}
$$

and hence by (4),

$$
\begin{aligned}
x'(t) \;\le\;& e(t) + k_2 M_1 + \lambda_0\, M_1 \int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}C(s,\eta(s))\,\mathrm{d}s \\[2mm]
& + M_1 \int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}\eta(s)x(s)\,\mathrm{d}s, \quad t \ge t_1.
\end{aligned}
\tag{5}
$$

Integrating (5) over $[t_1,t]$ and then changing the order of integration, we have

$$
\begin{aligned}
x(t) \;\le\;& x(t_1) + k_2 M_1 t + \int_{t_1}^{t} e(s)\,ds + \frac{\lambda_0 M_1 t}{\alpha}\int_{0}^{t}(t-s)^{\alpha-1}s^{\gamma-1}C(s,\eta(s))\,\mathrm{d}s \\[2mm]
& + \frac{M_1 t}{\alpha}\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}\eta(s)x(s)\,\mathrm{d}s, \quad t \ge t_1.
\end{aligned}
\tag{6}
$$

By using the Hölder inequality and the identity (8), we see that

$$
\begin{aligned}
\int_{0}^{t}(t-s)^{\alpha-1}s^{\gamma-1}C(s,\eta(s))\,\mathrm{d}s &\le \left(\int_{0}^{t}(t-s)^{p(\alpha-1)}s^{p(\gamma-1)}\,\mathrm{d}s\right)^{1/p}\left(\int_{0}^{t}C^q(s,\eta(s)\,\mathrm{d}s\right)^{1/q} \\[2mm]
&\le t^{\theta/p}\,B^{1/p}(p(\gamma-1)+1,p(\alpha-1)+1)\left(\int_{0}^{t}C^q(s,\eta(s)\,\mathrm{d}s\right)^{1/q} \\[2mm]
&\le b_1\left(\int_{0}^{\infty}C^q(s,\eta(s))\,\mathrm{d}s\right)^{1/q} = b_1\,\|C\|_q
\end{aligned}
\tag{7}
$$

where $b_1 = B^{1/p}(p(\gamma-1)+1,p(\alpha-1)+1)$, and by the definition of $\gamma$, $\theta = p(\alpha+\gamma-2)+1 = 0$. Similarly, we estimate the second integral as

$$
\begin{aligned}
\int_{t_1}^{t}(t-s)^{\alpha-1}s^{\gamma-1}\eta(s)x(s)\,\mathrm{d}s &\le \left(\int_{0}^{t}(t-s)^{p(\alpha-1)}s^{p(\gamma-1)}\,\mathrm{d}s\right)^{1/p}\left(\int_{t_1}^{t}\eta^q(s)x^q(s)\,\mathrm{d}s\right)^{1/q} \\[2mm]
&\le b_1\left(\int_{t_1}^{t}\eta^q(s)x^q(s)\,\mathrm{d}s\right)^{1/q}.
\end{aligned}
\tag{8}
$$

In view of (7) and (8), from (6) we get

$$\frac{x(t)}{t} \le \lambda_1 + \lambda_2\left(\int_{t_1}^{t}\eta^q(s)x^q(s)\,ds\right)^{1/q}, \quad t \ge t_1,
\tag{9}$$

where

$$\lambda_1 = \frac{x(t_1)}{t_1} + +k_2\,M_1 + M_2 + \frac{\lambda_0 b_1 M_1}{\alpha}\,||C||_q, \quad \lambda_2 = \frac{b_1 M_1}{\alpha}.$$

Now we employ the elementary inequality $(a+b)^q \le 2^{q-1}(a^q + b^q)$ for $a, b \ge 0$ in (9) to arrive at

$$w(t) \le 2^{q-1}\lambda_1^q + 2^{q-1}\lambda_2^q \int_{t_1}^t s^q \eta^q(s) w(s)\,\mathrm{d}s, \quad t \ge t_1, \tag{10}$$

where $w(t) = x^q(t)/t^q$. Finally, if we apply the Gronwall-Bellman inequality in (10) and then use the assumption that $t\eta(t) \in L^q[0, \infty)$, we obtain

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty \tag{11}$$

as desired.

If $x(t)$ is eventually negative, we can set $y = -x$ to see that $y$ satisfies Eq. (1) with $e(t)$ replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. It follows in a similar manner that

$$\limsup_{t\to\infty} \frac{-x(t)}{t} < \infty. \tag{12}$$

$\square$

Now we give our oscillation theorem.

**Theorem 2** *In addition to the hypothesis of Theorem 1, suppose that*

$$\lim_{t\to\infty} a(t) = 0. \tag{13}$$

*If for every $\mu \in (0, 1)$ we have*

$$\liminf_{t\to\infty} \left[\mu\, t + \int_0^t e(s)\,\mathrm{d}s\right] = -\infty, \quad \limsup_{t\to\infty} \left[\mu\, t - \int_0^t e(s)\,\mathrm{d}s\right] = \infty, \tag{14}$$

*then Eq. (1) is oscillatory.*

**Proof**  Suppose on the contrary that there is a nonoscillatory solution $x(t)$ of Eq. (1). We may assume that $x(t)$ is eventually positive, i.e. there exists a sufficiently large $t_1 > 1$ such that $x(t) > 0$ for all $t \ge t_1$.

Proceeding as in the proof of Theorem 1 we arrive at (6), so

$$x(t) \le x(t_1) + k_2\,M_1 t + \int_{t_1}^t e(s)\,\mathrm{d}s + t\frac{\lambda_0 M_1 b_1 ||C||_q}{\alpha} + t\frac{M_1 b_1}{\alpha}\left(\int_{t_1}^t \eta^q(s)x^q(s)\,\mathrm{d}s\right)^{1/q}. \tag{15}$$

In view of $t\eta(t) \in L^q[0, \infty)$ and (2) we see that the last integral in (15) is bounded, and because of (13) we can make $M_1$ as small as we please by increasing the size of $t_1$ if necessary. Therefore, it follows from (15) that

$$x(t) \le x(t_1) - \int_0^{t_1} e(s)\,\mathrm{d}s + \int_0^t e(s)\,\mathrm{d}s + t/2, \quad t \ge t_2 \tag{16}$$

for some $t_2 \geq t_1$. Taking liminf as $t \to \infty$ in (16) and using (14) results in a contradiction with the fact that $x(t)$ is eventually positive.

The proof when $x(t)$ is eventually negative is similar. $\qquad\square$

**Example 1** Consider the integro-differential equation

$$x'(t) = t \sin t - \int_0^t (t-s)^{-1/3} \frac{s}{t+s+1} x^{1/3}(s) \, \mathrm{d}s, \quad t \geq 0. \tag{17}$$

Comparing with (1) we may write that

$$e(t) = t \sin t, \quad \alpha = 2/3,$$

$$f(t,x) = x^{1/3} = t^{-1/3} t^{1/3} x^{1/3}, \quad \lambda = 1/3,$$

$$k(t,s) = \frac{s}{t+s+1} \leq \frac{s}{t+1}.$$

Letting $q = 3$ and taking $a(t) = 1/(t+1)$, $h(t) = t$, $m(t) = t^{1/3}$, and $\eta(t) = 1/(t+1)^6$, we calculate that

$$p = 3/2, \quad \gamma = 2/3, \quad C(t,\eta) = \frac{t^2}{(t+1)^3}.$$

Clearly, $p < 1/(1-\alpha)$, (2)-(5) hold, $t\eta(t), C(t,\eta(t)) \in L^3$, $\lim_{t\to\infty} a(t) = 0$, and, in view of

$$\int_{t_1}^t e(s) \, \mathrm{d}s = \sin t - t \cos t - \sin t_1 + t_1 \cos t_1,$$

the conditions (5) and (14) are satisfied. Since all the conditions of Theorem 2 are satisfied, we may conclude that Eq. (17) is oscillatory. We should note that to the best of our knowledge none of the results in the literature are applicable to Eq. (17).

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