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Bounds for radii of starlikeness and convexity of some special functions

İbrahim AKTAŞ[1,*], **Árpád BARICZ**[2,3,*], **Halit ORHAN**[4]
[1]Department of Mathematical Engineering, Gümüşhane University, Gümüşhane, Turkey
[2]Department of Economics, Babeş-Bolyai University, Cluj-Napoca, Romania
[3]Institute of Applied Mathematics, Óbuda University, Budapest, Hungary
[4]Department of Mathematics, Atatürk University, Erzurum, Turkey

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Abstract: In this paper we consider some normalized Bessel, Struve, and Lommel functions of the first kind and, by using the Euler–Rayleigh inequalities for the first positive zeros of a combination of special functions, we obtain tight lower and upper bounds for the radii of starlikeness of these functions. By considering two different normalizations of Bessel and Struve functions we give some inequalities for the radii of convexity of the same functions. On the other hand, we show that the radii of univalence of some normalized Struve and Lommel functions are exactly the radii of starlikeness of the same functions. In addition, by using some ideas of Ismail and Muldoon we present some new lower and upper bounds for the zeros of derivatives of some normalized Struve and Lommel functions. The Laguerre–Pólya class of real entire functions plays an important role in our study.

Key words: Lommel, Struve, and Bessel functions, univalent, starlike, and convex functions, radius of univalence, starlikeness, and convexity, zeros of Lommel, Struve, and Bessel functions, Mittag–Leffler expansions, Laguerre–Pólya class of entire functions

1. Introduction

It is known that special functions, like Bessel, Struve, and Lommel functions of the first kind, have some beautiful geometric properties. Recently, the geometric properties of the above special functions were investigated, motivated by some earlier results. In the 1960s Brown, Kreyszig and Todd, and Wilf (see [13–15, 18, 22]) considered the univalence and starlikeness of Bessel functions of the first kind, while in recent years the radii of univalence, starlikeness, and convexity for the normalized forms of Bessel, Struve, and Lommel functions of the first kind were obtained; see the papers [1–3, 7–12, 19, 20] and the references therein. In these papers it was shown that the radii of univalence, starlikeness, and convexity are actually solutions of some transcendental equations. On the other hand, it was shown that the obtained radii satisfy some interesting inequalities. In addition, it was proved that the radii of univalence of some normalized Bessel and Struve functions correspond to the radii of starlikeness of the same functions. In the above works the authors intensively used some properties of the positive zeros of Bessel, Struve, and Lommel functions of the first kind under some conditions. They also utilized the Laguerre–Pólya class $\mathcal{LP}$ of real entire functions. Motivated by the above developments in this topic, in this paper our aim is to give some new results for the radii of univalence, starlikeness, and convexity of

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the normalized Bessel, Struve, and Lommel functions of the first kind. This paper is a direct continuation of the paper [1] and it is organized as follows: Section 1 contains some basic concepts, while in Section 2 we focus on a linear combination of the Struve function and its derivative and the derivative of the Lommel function. Here we give some lower and upper bounds for the smallest positive zeros of these functions. To prove our results we use some ideas from [17]. We also consider two normalized forms of Struve and Lommel functions, respectively. For these functions, we show that the radii of univalence and starlikeness coincide. At the end of this section we obtain some new lower and upper bounds concerning the radii of convexity of four different normalized forms of Bessel and Struve functions of the first kind. The bounds deduced for the radii of convexity are in fact particular cases of some Euler–Rayleigh inequalities and it is possible to show that the lower bounds increase and the upper bounds decrease to the corresponding radii of convexity, and thus the inequalities presented in this paper can be improved by using higher order Euler–Rayleigh inequalities. We restricted ourselves to the third Euler–Rayleigh inequalities since these are already complicated. For more details on Euler–Rayleigh inequalities for zeros of Bessel functions we refer to [17] and to [21, p. 501].

Now we would like to present some basic concepts regarding geometric function theory. Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ be the open disk, where $r > 0$. Also, let $f : \mathbb{D}_r \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n\geq 2} a_n z^n. \tag{1.1}$$

The function f, defined by (1.1), is called starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r.$$

The real number

$$r^*(f) = \sup\left\{r > 0 \,\middle|\, \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r\right\}$$

is called the radius of starlikeness of the function f.

The function f, defined by (1.1), is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r.$$

The radius of convexity of the function f is defined by the real number

$$r^c(f) = \sup\left\{r > 0 \,\middle|\, \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for all } z \in \mathbb{D}_r\right\}.$$

Finally, we recall that the radius of univalence of the analytic function f in the form of (1.1) is the largest radius r such that f maps $\mathbb{D}_r$ univalently into $f(\mathbb{D}_r)$.

2. Bounds for the zeros of some special functions

In this paper we consider three classical special functions, the Bessel function of the first kind J_ν, the Struve function of the first kind $\mathbf{H}_\nu$, and the Lommel function of the first kind $s_{\mu,\nu}$. It is known that the Bessel function has the infinite series representation [4, p.8]

$$J_\nu(z)=\sum_{n\geq 0}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu},$$

where $z,\nu\in\mathbf{C}$ such that $\nu\neq -1,-2,\ldots$. Also, the Struve and Lommel functions can be represented as the infinite series

$$\mathbf{H}_\nu(z)=\sum_{n\geq 0}\frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2n+\nu+1}, -\nu-\frac{3}{2}\notin\mathbb{N},$$

and

$$s_{\mu,\nu}=\frac{(z)^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\sum_{n\geq 0}\frac{(-1)^n}{(\frac{\mu-\nu+3}{2})_n(\frac{\mu+\nu+3}{2})_n}\left(\frac{z}{2}\right)^{2n}, \frac{1}{2}(-\mu\pm\nu-3)\notin\mathbb{N},$$

where $z,\mu,\nu\in\mathbf{C}$. In addition, we know that the Bessel function is a solution of the homogeneous Bessel differential equation

$$zw''(z)+zw'(z)+(z^2-\nu^2)w(z)=0,$$

while the Struve and Lommel functions are solutions of the inhomogeneous Bessel differential equations

$$zw''(z)+zw'(z)+(z^2-\nu^2)w(z)=\frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}$$

and

$$zw''(z)+zw'(z)+(z^2-\nu^2)w(z)=z^{\mu+1},$$

respectively. We refer to Watson's treatise [21] for comprehensive information about these functions. On the other hand, the Laguerre–Pólya class $\mathcal{LP}$ of real entire functions plays an important role in our proofs. Recall that a real entire function Ψ belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\Psi(x)=cx^m e^{-ax^2+bx}\prod_{n\geq 1}\left(1+\frac{x}{x_n}\right)e^{-\frac{x}{x_n}},$$

with $c,b,x_n\in\mathbb{R}, a\geq 0, m\in\mathbb{N}_0$, and $\sum 1/{x_n}^2<\infty$.

We note that the class $\mathcal{LP}$ consists of entire functions, which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros. For more details on the class $\mathcal{LP}$ we refer to [16, p. 703] and to the references therein.

2.1. Zeros of linear combination of Struve function and its derivative

In this subsection by considering the Struve function $\mathbf{H}_\nu$ and its derivative $\mathbf{H}'_\nu$ we define the function $\mathcal{H}_\nu$ as follows:

$$\mathcal{H}_\nu(z)=\alpha\mathbf{H}_\nu(z)+z\mathbf{H}'_\nu(z).$$

The function $\mathcal{H}_\nu$ can be written as

$$\mathcal{H}_\nu(z)=\sum_{n\geq 0}\frac{(-1)^n(2n+\nu+\alpha+1)}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+\nu+1}.$$

Let $\alpha+\nu\neq-1$. Here we focus on the following normalized form:

$$h_\nu(z)=(\alpha+\nu+1)^{-1}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)z^{-\frac{\nu+1}{2}}2^{\nu+1}\mathcal{H}_\nu(\sqrt{z})=\sum_{n\geq0}\frac{(-1)^n(2n+\nu+\alpha+1)}{2^{2n}(\nu+\alpha+1)(\frac{3}{2})_n(\nu+\frac{3}{2})_n}z^n.$$

Our first main result is related to the function $\mathcal{H}_\nu$.

Theorem 1 *Let $\alpha+\nu>-1, |\nu|<\frac{1}{2}$ and let $\zeta_{\nu,1}$ be the smallest positive zero of the function $\mathcal{H}_\nu$. Then we have the lower bounds*

$$\zeta_{\nu,1}^2>\frac{3(2\nu+3)(\alpha+\nu+1)}{\alpha+\nu+3},$$

$$\zeta_{\nu,1}^2>\frac{3(2\nu+3)(\alpha+\nu+1)\sqrt{5(2\nu+5)}}{\sqrt{\kappa_1}},$$

$$\zeta_{\nu,1}^2>\frac{3(2\nu+3)(\alpha+\nu+1)\sqrt[3]{35(2\nu+5)(2\nu+7)}}{\sqrt[3]{\kappa_2}}$$

and the upper bounds

$$\zeta_{\nu,1}^2<\frac{15(2\nu+3)(2\nu+5)(\alpha+\nu+1)(\alpha+\nu+3)}{\kappa_1},$$

$$\zeta_{\nu,1}^2<\frac{21(2\nu+3)(2\nu+7)(\alpha+\nu+1)\kappa_1}{\kappa_2},$$

where $\kappa_1=-2\alpha^2\nu+7\alpha^2-4\alpha\nu^2+2\alpha\nu+42\alpha-2\nu^3-5\nu^2+72\nu+135$ and $\kappa_2=-4\alpha^3\nu^2-96\alpha^3\nu+145\alpha^3-12\alpha^2\nu^3-324\alpha^2\nu^2-429\alpha^2\nu+1305\alpha^2-12\alpha\nu^4-360\alpha\nu^3-1689\alpha\nu^2+1170\alpha\nu+6291\alpha-4\nu^5-132\nu^4-1115\nu^3+621\nu^2+12339\nu+14931$.

Proof It is known (see [5]) that the zeros of the function

$$h_\nu(z)=\sum_{n\geq0}\frac{(-1)^n(2n+\nu+\alpha+1)}{2^{2n}(\nu+\alpha+1)(\frac{3}{2})_n(\nu+\frac{3}{2})_n}z^n$$

all are real when $\alpha+\nu>-1$ and $|\nu|<\frac{1}{2}$. As a result of this we can say that the function h_ν belongs to the Laguerre–Pólya class $\mathcal{LP}$ of real entire functions, which are uniform limits of real polynomials whose all zeros are real. Thus, the function $z\mapsto h_\nu(z)$ has only real zeros and having growth order $\frac{1}{2}$ it can be written as the product

$$h_\nu(z)=\prod_{n\geq1}\left(1-\frac{z}{\zeta_{\nu,n}^2}\right),$$

where $\zeta_{\nu,n}>0$ for each $n\in\mathbb{N}$. By considering the Euler–Rayleigh sum $\delta_k=\sum_{n\geq1}\zeta_{\nu,n}^{-2k}$ and the infinite sum representation of the Struve function $\mathbf{H}_\nu$ we have

$$\frac{h_\nu'(z)}{h_\nu(z)}=\sum_{n\geq1}\frac{1}{z-\zeta_{\nu,n}^2}=-\sum_{k\geq0}\left(\sum_{n\geq1}\frac{1}{(\zeta_{\nu,n}^2)^{k+1}}\right)z^k=-\sum_{k\geq0}\delta_{k+1}z^k, |z|<\zeta_{\nu,1}^2, \tag{2.1}$$

$$\frac{h'_\nu(z)}{h_\nu(z)} = \sum_{n\geq 0}\theta_n z^n \Big/ \sum_{n\geq 0}\gamma_n z^n, \tag{2.2}$$

where

$$\theta_n = \frac{(-1)^{n+1}(2n+\nu+\alpha+3)(n+1)}{2^{2n+2}(\nu+\alpha+1)(\frac{3}{2})_{n+1}(\nu+\frac{3}{2})_{n+1}} \text{ and } \gamma_n = \frac{(-1)^n(2n+\nu+\alpha+1)}{2^{2n}(\nu+\alpha+1)(\frac{3}{2})_n(\nu+\frac{3}{2})_n}.$$

By comparing the coefficients of (2.1) and (2.2) we have the following:

$$\delta_1 = \frac{(\alpha+\nu+3)}{3(2\nu+3)(\alpha+\nu+1)}, \delta_2 = \frac{\kappa_1}{45(2\nu+3)^2(2\nu+5)(\alpha+\nu+1)^2},$$

$$\delta_3 = \frac{\kappa_2}{945(2\nu+3)^3(4\nu^2+24\nu+35)(\alpha+\nu+1)^2},$$

where

$$\kappa_1 = -2\alpha^2\nu + 7\alpha^2 - 4\alpha\nu^2 + 2\alpha\nu + 42\alpha - 2\nu^3 - 5\nu^2 + 72\nu + 135$$

and

$$\begin{aligned}\kappa_2 = &-4\alpha^3\nu^2 - 96\alpha^3\nu + 145\alpha^3 - 12\alpha^2\nu^3 - 324\alpha^2\nu^2 - 429\alpha^2\nu + 1305\alpha^2 - 12\alpha\nu^4 - 360\alpha\nu^3 \\ &- 1689\alpha\nu^2 + 1170\alpha\nu + 6291\alpha - 4\nu^5 - 132\nu^4 - 1115\nu^3 + 621\nu^2 + 12339\nu + 14931.\end{aligned}$$

Now, by using the Euler–Rayleigh inequalities ${\delta_k}^{-\frac{1}{k}} < \zeta^2_{\nu,1} < \frac{\delta_k}{\delta_{k+1}}$ for $\alpha+\nu > -1, |\nu| < \frac{1}{2}$ and $k \in \{1,2,3\}$, we get the following lower bounds:

$$\zeta^2_{\nu,1} > \frac{3(2\nu+3)(\alpha+\nu+1)}{\alpha+\nu+3},$$

$$\zeta^2_{\nu,1} > \frac{3(2\nu+3)(\alpha+\nu+1)\sqrt{5(2\nu+5)}}{\sqrt{\kappa_1}},$$

$$\zeta^2_{\nu,1} > \frac{3(2\nu+3)(\alpha+\nu+1)\sqrt[3]{35(2\nu+5)(2\nu+7)}}{\sqrt[3]{\kappa_2}}$$

and the upper bounds

$$\zeta^2_{\nu,1} < \frac{15(2\nu+3)(2\nu+5)(\alpha+\nu+1)(\alpha+\nu+3)}{\kappa_1},$$

$$\zeta^2_{\nu,1} < \frac{21(2\nu+3)(2\nu+7)(\alpha+\nu+1)\kappa_1}{\kappa_2}.$$

□

In particular, when $\alpha = 0$, Theorem 1 reduces to the following:

Theorem 2 *Let $|\nu| < \frac{1}{2}$ and let $h'_{\nu,1}$ be the smallest positive root of $\mathbf{H}'_\nu$. Then we have the lower bounds*

$$(h'_{\nu,1})^2 > \frac{3(2\nu+3)(\nu+1)}{\nu+3},$$

$$(h'_{\nu,1})^2 > \frac{3(2\nu+3)(\nu+1)\sqrt{5(2\nu+5)}}{\sqrt{-2\nu^3-5\nu^2+72\nu+135}},$$

$$(h'_{\nu,1})^2 > \frac{3(2\nu+3)(\nu+1)\sqrt[3]{35(2\nu+5)(2\nu+7)}}{\sqrt[3]{-4\nu^5-132\nu^4-1115\nu^3+621\nu^2+12339\nu+14931}}$$

and the upper bounds

$$(h'_{\nu,1})^2 < \frac{15(2\nu+3)(2\nu+5)(\nu+1)(\nu+3)}{-2\nu^3-5\nu^2+72\nu+135},$$

$$(h'_{\nu,1})^2 < \frac{21(2\nu+3)(2\nu+7)(\nu+1)(-2\nu^3-5\nu^2+72\nu+135)}{-4\nu^5-132\nu^4-1115\nu^3+621\nu^2+12339\nu+14931}.$$

Here it is worth mentioning that Theorem 2 reobtains and improves some results of [6] regarding the first positive zeros of the derivative of the Struve function. We mention that our approach is a little bit different than the approach in [6].

2.2. Bounds for the zeros of derivative of Lommel functions

We consider the function

$$\mathcal{L}_\mu(z) = z s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \sum_{n\geq 0} \frac{(-1)^n(2n+\mu+\frac{1}{2})}{4^n\mu(\mu+1)(\frac{\mu+2}{2})_n(\frac{\mu+3}{2})_n} z^{2n+\mu+\frac{1}{2}},$$

where $s'_{\mu-\frac{1}{2},\frac{1}{2}}(z)$ stands for the derivative of Lommel function. Let $\mu \in (-1,1), \mu \neq 0$ and $\mu \neq -\frac{1}{2}$. Now we define the following normalized form of the function $\mathcal{L}_\mu$. Let

$$l_\mu(z) = \frac{2\mu(\mu+1)}{(2\mu+1)} z^{-\frac{2\mu+1}{4}} \mathcal{L}_\mu(\sqrt{z}).$$

Clearly, the function l_μ can be written as

$$l_\mu(z) = 1 + \sum_{n\geq 1} \frac{(-1)^n(2n+\mu+\frac{1}{2})}{2^{2n}(\mu+\frac{1}{2})(\frac{\mu+2}{2})_n(\frac{\mu+3}{2})_n} z^n.$$

Theorem 3 *Let $\mu \in (-1,1), \mu \neq 0, \mu \neq -\frac{1}{2}$ and let $\tau_{\mu,1}$ be the smallest positive zero of the function $\mathcal{L}_\mu$. Then we have the lower bounds*

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)}{2\mu+5},$$

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)\sqrt{(\mu+4)(\mu+5)}}{\sqrt{-4\mu^4-24\mu^3+19\mu^2+295\mu+392}},$$

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)\sqrt[3]{(\mu+4)(\mu+5)(\mu+6)(\mu+7)}}{\sqrt[3]{8\mu^7+44\mu^6-554\mu^5-4731\mu^4-7672\mu^3+23551\mu^2+85834\mu+72384}}$$

and the upper bounds

$$(\tau_{\mu,1})^2 < \frac{(\mu+2)(\mu+3)(\mu+4)(\mu+5)(2\mu+1)(2\mu+5)}{-4\mu^4-24\mu^3+19\mu^2+295\mu+392},$$

$$(\tau_{\mu,1})^2 < \frac{(\mu+2)(\mu+3)(\mu+6)(\mu+7)(2\mu+1)(-4\mu^4-24\mu^3+19\mu^2+295\mu+392)}{8\mu^7+44\mu^6-554\mu^5-4731\mu^4-7672\mu^3+23551\mu^2+85834\mu+72384}.$$

Proof The normalized Lommel function

$$l_\mu(z) = \frac{2\mu(\mu+1)}{(2\mu+1)} z^{-\frac{2\mu+1}{4}} \mathcal{L}_\mu(\sqrt{z})$$

has only real zeros for $\mu \in (-1,1), \mu \neq 0$ and $\mu \neq -\frac{1}{2}$ (see [12]). Consequently, the function l_μ belongs to the Laguerre–Pólya class $\mathcal{LP}$ of real entire functions. Thus, $l_\mu(z)$ can be written as the product

$$\prod_{n\geq 1}\left(1-\frac{z}{\tau_{\mu,n}^2}\right)$$

where $\tau_{\mu,n} > 0$ for each $n \in \mathbb{N}$. Now by using the Euler–Rayleigh sum $\eta_k = \sum_{n\geq 1} \tau_{\mu,n}^{-2k}$ and the infinite sum representation of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ we get

$$\frac{l_\mu{}'(z)}{l_\mu(z)} = \sum_{n\geq 1}\frac{1}{z-\tau_{\mu,n}^2} = -\sum_{n\geq 1}\sum_{k\geq 0}\frac{1}{(\tau_{\mu,n}^2)^{k+1}}z^k = -\sum_{k\geq 0}\eta_{k+1}z^k, |z| < \tau_{\mu,1}^2, \tag{2.3}$$

$$\frac{l_\mu{}'(z)}{l_\mu(z)} = \sum_{n\geq 0}\rho_n z^n \Bigg/ \sum_{n\geq 0}\sigma_n z^n, \tag{2.4}$$

where

$$\rho_n = \frac{(-1)^{n+1}(n+1)(2n+\mu+\frac{5}{2})}{2^{2n+2}(\mu+\frac{1}{2})(\frac{\mu+2}{2})_{n+1}(\frac{\mu+3}{2})_{n+1}} \text{ and } \sigma_n = \frac{(-1)^n(2n+\mu+\frac{1}{2})}{2^{2n}(\mu+\frac{1}{2})(\frac{\mu+2}{2})_n(\frac{\mu+3}{2})_n}.$$

By equating the coefficients of (2.3) and (2.4) we obtain

$$\eta_1 = \frac{2\mu+5}{2\mu^3+11\mu^2+17\mu+6}, \eta_2 = \frac{-4\mu^4-24\mu^3+19\mu^2+295\mu+392}{(\mu+2)^2(\mu+3)^2(\mu+4)(\mu+5)(2\mu+1)^2}$$

and

$$\eta_3 = \frac{8\mu^7+44\mu^6-554\mu^5-4731\mu^4-7672\mu^3+23551\mu^2+85834\mu+72384}{(\mu+2)^3(\mu+3)^3(\mu+4)(\mu+5)(\mu+6)(\mu+7)(2\mu+1)^3}.$$

Now by considering Euler–Rayleigh inequalities $\eta_k{}^{-\frac{1}{k}} < \tau_{\mu,1}^2 < \frac{\eta_k}{\eta_{k+1}}$ for $\mu \in (-1,1), \mu \neq 0, \mu \neq -\frac{1}{2}$ and $k \in \{1,2,3\}$ we obtain the lower bounds

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)}{2\mu+5},$$

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)\sqrt{(\mu+4)(\mu+5)}}{\sqrt{-4\mu^4-24\mu^3+19\mu^2+295\mu+392}},$$

$$(\tau_{\mu,1})^2 > \frac{(\mu+2)(\mu+3)(2\mu+1)\sqrt[3]{(\mu+4)(\mu+5)(\mu+6)(\mu+7)}}{\sqrt[3]{8\mu^7+44\mu^6-554\mu^5-4731\mu^4-7672\mu^3+23551\mu^2+85834\mu+72384}}$$

and the upper bounds

$$(\tau_{\mu,1})^2 < \frac{(\mu+2)(\mu+3)(\mu+4)(\mu+5)(2\mu+1)(2\mu+5)}{-4\mu^4-24\mu^3+19\mu^2+295\mu+392},$$

$$(\tau_{\mu,1})^2 < \frac{(\mu+2)(\mu+3)(\mu+6)(\mu+7)(2\mu+1)(-4\mu^4-24\mu^3+19\mu^2+295\mu+392)}{8\mu^7+44\mu^6-554\mu^5-4731\mu^4-7672\mu^3+23551\mu^2+85834\mu+72384}.$$

□

2.3. Radii of univalence (and starlikeness) of Struve functions

Here our aim is to show that the radii of univalence of the Struve function u_ν correspond to the radii of starlikeness.

Theorem 4 *Let $\nu \in [-\frac{1}{2}, \frac{1}{2}]$. The radius of univalence $r^*(u_\nu)$ of the normalized Struve function*

$$z \mapsto u_\nu(z) = \left(\sqrt{\pi}2^\nu \Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}}$$

corresponds to its radius of starlikeness and it is the smallest positive root $h'_{\nu,1}$ of $\mathbf{H}'_\nu$.

Proof If we consider the Maclaurin series expansion of the function

$$z \mapsto u_\nu(z) = \left(\sqrt{\pi}2^\nu \Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}}$$

we obtain

$$u(z) = z - \frac{1}{3(\nu+1)(2\nu+3)}z^3 + \frac{1}{90(\nu+1)^2(2\nu+3)^2(2\nu+5)}z^5 - \dots. \tag{2.5}$$

Therefore, the function u_ν has real coefficients. Also, we know that if the function $z \mapsto z + \alpha_2 z^2 + \dots$ has real coefficients, then its radius of starlikeness is less than or equal to its radius of univalence; see [22]. Now we should show that the radii of univalence are less than or equal to the corresponding radii of starlikeness. From the definition of $u_\nu(z)$ we can write that

$$\frac{zu'_\nu(z)}{u_\nu(z)} = \frac{1}{\nu+1}\frac{z\mathbf{H}'_\nu(z)}{\mathbf{H}_\nu(z)} = 1 - \frac{2}{\nu+1}\sum_{n\geq 1}\frac{z^2}{h^2_{\nu,n}-z^2}. \tag{2.6}$$

Thus, for $\nu \in [-\frac{1}{2}, \frac{1}{2}]$, we obtain that

$$\operatorname{Re}\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right) = 1 - \frac{2}{\nu+1}\sum_{n\geq 1}\operatorname{Re}\left(\frac{z^2}{h^2_{\nu,n}-z^2}\right) \geq 1 - \frac{2}{\nu+1}\sum_{n\geq 1}\frac{|z|^2}{h^2_{\nu,n}-|z|^2} = \frac{|z|u'_\nu(|z|)}{u_\nu(|z|)}.$$

That is,

$$\operatorname{Re}\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right) \geq \frac{ru'_\nu(r)}{u_\nu(r)}, \tag{2.7}$$

where $r = |z|$. The quantity on the right-hand side of the inequality (2.7) remains positive until the first positive zero of u'_ν. These show that indeed the radius of univalence corresponds to the radius of starlikeness of the function u_ν. □

2.4. Radii of univalence (and starlikeness) of Lommel functions

In this subsection our aim is to show that the radii of univalence of the Lommel function f_μ correspond to the radii of starlikeness.

Theorem 5 *Let $\mu \in (-\frac{1}{2}, 1), \mu \neq 0$. The radius of univalence $r^*(f_\mu)$ of the normalized Lommel function*

$$z \mapsto f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \Big(\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\Big)^{\frac{1}{\mu+\frac{1}{2}}}$$

corresponds to its radius of starlikeness and it is the smallest positive root of $s'_{\mu-\frac{1}{2},\frac{1}{2}}$.

Proof If we consider the Maclaurin series expansion of the function

$$z \mapsto f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \Big(\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\Big)^{\frac{1}{\mu+\frac{1}{2}}}$$

we obtain

$$f_\mu(z) = z - \frac{2}{(\mu+2)(\mu+3)(2\mu+1)}z^3 + \frac{2\mu^3+16\mu^2+39\mu-16}{2(\mu+2)(\mu+3)(\mu+4)(\mu+5)(2\mu+1)^2}z^5 - \dots.$$

Therefore, the radius of starlikeness of the function f_μ is less than or equal to its radius of univalence; see [22]. On the other hand, from the definition of f_μ we can write that

$$\frac{zf'_\mu(z)}{f_\mu(z)} = \frac{1}{1+\frac{\mu}{2}}\frac{zs'_{\mu-\frac{1}{2},\frac{1}{2}}(z)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(z)} = 1 - \frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{z^2}{l^2_{\mu,n}-z^2}. \tag{2.8}$$

Thus, for $\mu \in (-\frac{1}{2}, 1), \mu \neq 0$ we obtain that

$$\operatorname{Re}\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right) = 1 - \frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\operatorname{Re}\left(\frac{z^2}{l^2_{\mu,n}-z^2}\right) \geq 1 - \frac{2}{1+\frac{\mu}{2}}\sum_{n\geq 1}\frac{|z|^2}{l^2_{\mu,n}-|z|^2} = \frac{|z|f'_\mu(|z|)}{f_\mu(|z|)}.$$

That is,

$$\operatorname{Re}\left(\frac{zf'_\mu(z)}{f_\mu(z)}\right) \geq \frac{rf'_\mu(r)}{f_\mu(r)}, \tag{2.9}$$

where $r = |z|$. The quantity on the right-hand side of the inequality (2.9) remains positive until the first positive zero of f'_μ is reached. These show that indeed the radius of univalence corresponds to the radius of starlikeness of the function f_μ. □

2.5. Radii of convexity of Bessel functions

In this subsection we consider two different normalized forms of the Bessel functions of the first kind. Here we show that the radii of convexity of these functions are the smallest positive roots of some transcendental equations. Moreover, we will present some inequalities for the radii of convexity of the same functions.

Theorem 6 *Let $\nu > -1$. Then the radius of convexity $r^c(g_\nu)$ of the function*

$$z \mapsto g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$$

is the smallest positive root of the equation $(zg'_\nu(z))' = 0$ and satisfies the following inequalities:

$$\frac{2\sqrt{\nu+1}}{3} < r^c(g_\nu) < 6\sqrt{\frac{(\nu+1)(\nu+2)}{56\nu+137}},$$

$$2\sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{56\nu+137}} < r^c(g_\nu) < \sqrt{\frac{2(56\nu+137)(\nu+1)(\nu+3)}{208\nu^2+1172\nu+1693}},$$

$$\sqrt[6]{\frac{32(\nu+1)^3(\nu+2)(\nu+3)}{208\nu^2+1172\nu+1693}} < r^c(g_\nu) < 2\sqrt{\frac{2(\nu+1)(\nu+2)(\nu+4)(208\nu^2+1172\nu+1693)}{3104\nu^4+36768\nu^3+161424\nu^2+312197\nu+223803}}.$$

Proof By using the Alexander duality theorem for starlike and convex functions we can say that the function g_ν is convex if and only if $z \mapsto zg'_\nu(z)$ is starlike. However, the smallest positive zero of $z \mapsto (zg'_\nu(z))'$ is actually the radius of starlikeness of $z \mapsto zg'_\nu(z)$, according to [7, 8]. Therefore, the radius of convexity $r^c(g_\nu)$ is the smallest positive root of the equation $(zg'_\nu(z))' = 0$. See also [10] for more details. Now, by considering the Bessel differential equation

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0 \tag{2.10}$$

and the infinite series representations of the Bessel function and its derivative

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{2^{2n+\nu} n!\Gamma(n+\nu+1)}, \tag{2.11}$$

$$J'_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n (2n+\nu) z^{2n+\nu-1}}{2^{2n+\nu} n!\Gamma(n+\nu+1)}, \tag{2.12}$$

respectively, we obtain

$$\Delta_\nu(z) = (zg'_\nu(z))' = 1 + \sum_{n\geq 1} \frac{(-1)^n (2n+1)^2 z^{2n}}{2^{2n} n!(\nu+1)_n}. \tag{2.13}$$

Since the function g_ν belongs to the Laguerre–Pólya class of entire functions and $\mathcal{LP}$ is closed under differentiation, we can say that the function Δ_ν belongs also to the Laguerre–Pólya class. Therefore, the zeros of the function Δ_ν are all real. Suppose that $\beta_{\nu,n}$s are the zeros of the function Δ_ν. Then the function Δ_ν has the infinite product representation as follows:

$$\Delta_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\beta^2_{\nu,n}}\right). \tag{2.14}$$

By taking the logarithmic derivative of (2.14) we get

$$\frac{\Delta'_\nu(z)}{\Delta_\nu(z)} = -2\sum_{k\geq 0} \rho_{k+1} z^{2k+1}, |z| < \beta^2_{\nu,1}, \tag{2.15}$$

where $\rho_k = \sum_{n\geq 1} \beta_{\nu,n}^{-2k}$. On the other hand, by considering the infinite sum representation of $\Delta_\nu(z)$ we obtain

$$\frac{\Delta_\nu'(z)}{\Delta_\nu(z)} = \sum_{n\geq 0} \xi_n z^{2n+1} \Big/ \sum_{n\geq 0} \kappa_n z^{2n}, \tag{2.16}$$

where

$$\xi_n = \frac{(-1)^{n+1} 2(2n+3)^2}{2^{2n+2} n!(\nu+1)_{n+1}} \text{ and } \kappa_n = \frac{(-1)^n (2n+1)^2}{2^{2n} n!(\nu+1)_n}.$$

By comparing the coefficients of (2.15) and (2.16) we obtain

$$\rho_1 = \frac{9}{4(\nu+1)}, \rho_2 = \frac{56\nu+137}{16(\nu+1)^2(\nu+2)}, \rho_3 = \frac{208\nu^2+1172\nu+1693}{32(\nu+1)^3(\nu+2)(\nu+3)}$$

and

$$\rho_4 = \frac{3104\nu^4+36768\nu^3+161424\nu^2+312197\nu+223803}{216(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}.$$

Now by considering the Euler–Rayleigh inequalities $\rho_k^{-\frac{1}{k}} < \beta_{\nu,1}^2 < \frac{\rho_k}{\rho_{k+1}}$ for $\nu > -1$ and $k \in \{1,2,3\}$, we obtain the following inequalities:

$$\frac{2\sqrt{\nu+1}}{3} < r^c(g_\nu) < 6\sqrt{\frac{(\nu+1)(\nu+2)}{56\nu+137}},$$

$$2\sqrt[4]{\frac{(\nu+1)^2(\nu+2)}{56\nu+137}} < r^c(g_\nu) < \sqrt{\frac{2(56\nu+137)(\nu+1)(\nu+3)}{208\nu^2+1172\nu+1693}},$$

$$\sqrt[6]{\frac{32(\nu+1)^3(\nu+2)(\nu+3)}{208\nu^2+1172\nu+1693}} < r^c(g_\nu) < 2\sqrt{\frac{2(\nu+1)(\nu+2)(\nu+4)(208\nu^2+1172\nu+1693)}{3104\nu^4+36768\nu^3+161424\nu^2+312197\nu+223803}}.$$

□

Theorem 7 *Let $\nu > -1$. Then the radius of convexity $r^c(h_\nu)$ of the function*

$$z \mapsto h_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z})$$

is the smallest positive root of the equation $(zh_\nu'(z))' = 0$ and satisfies the following inequalities:

$$\nu+1 < r^c(h_\nu) < \frac{16(\nu+1)(\nu+2)}{7\nu+23},$$

$$\sqrt{\frac{16(\nu+1)^2(\nu+2)}{7\nu+23}} < r^c(h_\nu) < \frac{2(\nu+1)(\nu+3)(7\nu+23)}{9\nu^2+60\nu+115},$$

$$\sqrt[3]{\frac{32(\nu+1)^3(\nu+2)(\nu+3)}{9\nu^2+60\nu+115}} < r^c(h_\nu) < \frac{8(\nu+1)(\nu+2)(\nu+4)(9\nu^2+60\nu+115)}{47\nu^4+621\nu^3+3136\nu^2+7221\nu+6195}.$$

Proof By using the same procedure as in the previous proof we can say that the radius of convexity $r^c(h_\nu)$ is the smallest positive root of the equation $(zh'_\nu(z))' = 0$. See also [10] for more details. Now, by setting $\sqrt{z}$ instead of z in (2.10), (2.11), and (2.12), respectively, we obtain

$$\theta_\nu(z) = (zh'_\nu(z))' = 1 + \sum_{n\geq 1} \frac{(-1)^n (n+1)^2 z^n}{2^{2n} n!(\nu+1)_n}. \tag{2.17}$$

In addition, we know that h_ν belongs to the Laguerre–Pólya class of entire functions $\mathcal{LP}$. Since $\mathcal{LP}$ is closed under differentiation, we can say that the function θ_ν belongs also to the Laguerre–Pólya class. That is, the zeros of the function θ_ν are all real. Suppose that $\gamma_{\nu,n}$s are the zeros of the function θ_ν. Then the function θ_ν has the infinite product representation as follows:

$$\theta_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z}{\gamma_{\nu,n}}\right). \tag{2.18}$$

By logarithmic derivation of (2.18) we get

$$\frac{\theta'_\nu(z)}{\theta_\nu(z)} = -\sum_{k\geq 0} \varrho_{k+1} z^k, |z| < \gamma_{\nu,1}, \tag{2.19}$$

where $\varrho_k = \sum_{n\geq 1} \gamma_{\nu,n}^{-k}$. Also, by using the derivative of infinite sum representation of $\theta_\nu(z)$ we get

$$\frac{\theta'_\nu(z)}{\theta_\nu(z)} = \sum_{n\geq 0} m_n z^n \Big/ \sum_{n\geq 0} s_n z^n, \tag{2.20}$$

where

$$m_n = \frac{(-1)^{n+1}(n+2)^2}{2^{2(n+1)} n!(\nu+1)_{n+1}} \text{ and } s_n = \frac{(-1)^n (n+1)^2}{2^{2n} n!(\nu+1)_n}.$$

By comparing the coefficients of (2.19) and (2.20) we have

$$\varrho_1 = \frac{1}{\nu+1}, \varrho_2 = \frac{7\nu+23}{16(\nu+1)^2(\nu+2)}, \varrho_3 = \frac{9\nu^2+60\nu+115}{32(\nu+1)^3(\nu+2)(\nu+3)}$$

and

$$\varrho_4 = \frac{47\nu^4 + 621\nu^3 + 3136\nu^2 + 7221\nu + 6195}{256(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}.$$

By applying the Euler–Rayleigh inequalities ${\varrho_k}^{-\frac{1}{k}} < \gamma_{\nu,1} < \frac{\varrho_k}{\varrho_{k+1}}$ for $\nu > -1$ and $k \in \{1,2,3\}$ we have

$$\nu + 1 < r^c(h_\nu) < \frac{16(\nu+1)(\nu+2)}{7\nu+23},$$

$$\sqrt{\frac{16(\nu+1)^2(\nu+2)}{7\nu+23}} < r^c(h_\nu) < \frac{2(\nu+1)(\nu+3)(7\nu+23)}{9\nu^2+60\nu+115},$$

$$\sqrt[3]{\frac{32(\nu+1)^3(\nu+2)(\nu+3)}{9\nu^2+60\nu+115}} < r^c(h_\nu) < \frac{8(\nu+1)(\nu+2)(\nu+4)(9\nu^2+60\nu+115)}{47\nu^4+621\nu^3+3136\nu^2+7221\nu+6195}.$$

□

2.6. Radii of convexity of Struve functions

In this subsection we consider two different normalized Struve functions of the first kind. Here we show that the radii of convexity of these functions are the smallest positive roots of some transcendental equations. We also give some lower and upper bounds for the radii of convexity of these functions.

Theorem 8 *Let $|\nu| \leq \frac{1}{2}$. Then the radius of convexity $r^c(u_\nu)$ of the function*

$$z \mapsto u_\nu(z) = \sqrt{\pi} 2^\nu z^{-\nu} \Gamma\left(\nu + \frac{3}{2}\right) \mathbf{H}_\nu(z)$$

is the smallest positive root of the equation $(zu_\nu'(z))' = 0$ and satisfies the following inequalities:

$$\sqrt{\frac{2\nu+3}{3}} < r^c(u_\nu) < \sqrt{\frac{36\nu^2+144\nu+135}{34\nu+105}},$$

$$\sqrt[4]{\frac{3(2\nu+3)^2(2\nu+5)}{34\nu+105}} < r^c(u_\nu) < \sqrt{\frac{5(2\nu+3)(2\nu+7)(34\nu+105)}{3(268\nu^2+1824\nu+3213)}},$$

$$\sqrt[6]{\frac{5(2\nu+3)^3(2\nu+5)(2\nu+7)}{268\nu^2+1824\nu+3213}} < r^c(u_\nu) < 3\sqrt{\frac{7(2\nu+3)(2\nu+5)(2\nu+9)(268\nu^2+1824\nu+3213)}{\nu^*}},$$

where $\nu^* = 160336\nu^4 + 2256464\nu^3 + 11855904\nu^2 + 27626796\nu + 24017715$.

Proof Similarly as in the proof of Theorem 6 we observe that the radius of convexity $r^c(u_\nu)$ is the smallest positive root of the equation $(zu_\nu'(z))' = 0$. See also [12] for more details. Now, by considering the Struve differential equation

$$z^2\mathbf{H}_\nu''(z) + z\mathbf{H}_\nu'(z) + (z^2 - \nu^2)\mathbf{H}_\nu(z) = \frac{4(\frac{z}{2})^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \tag{2.21}$$

and the infinite series representations of the Struve function and its derivative

$$\mathbf{H}_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\nu+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu+1}, \tag{2.22}$$

$$\mathbf{H}_\nu'(z) = \sum_{n\geq 0} \frac{(-1)^n(2n+\nu+1)}{2\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(\nu+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu}, \tag{2.23}$$

respectively, we get

$$\Omega_\nu(z) = (zu_\nu'(z))' = 1 + \sum_{n\geq 1} \frac{(-1)^n(2n+1)}{2^{2n}(\frac{1}{2})_n(\nu+\frac{3}{2})_n} z^{2n}. \tag{2.24}$$

Since the function u_ν belongs to the Laguerre–Pólya class of entire functions $\mathcal{LP}$ and this class is closed under differentiation we obtain that the function Ω_ν belongs also to the Laguerre–Pólya class. Therefore, the zeros of the function Ω_ν are all real. Suppose that $\vartheta_{\nu,n}$s are the zeros of the function Ω_ν. Then the function Ω_ν has infinite product representation as follows:

$$\Omega_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\vartheta_{\nu,n}^2}\right). \tag{2.25}$$

By taking the logarithmic derivative of (2.25) we have

$$\frac{\Omega_\nu'(z)}{\Omega_\nu(z)} = -2\sum_{k\geq 0}\chi_{k+1}z^{2k+1}, |z| < \vartheta_{\nu,1}^2, \tag{2.26}$$

where $\chi_k = \sum_{n\geq 1}\vartheta_{\nu,n}^{-2k}$. On the other hand, by considering infinite sum representation of $\Omega_\nu(z)$ we get

$$\frac{\Omega_\nu'(z)}{\Omega_\nu(z)} = \sum_{n\geq 0}\tau_n z^{2n+1}\Bigg/\sum_{n\geq 0}\varsigma_n z^{2n}, \tag{2.27}$$

where

$$\tau_n = \frac{(-1)^{n+1}(2n+3)(n+1)}{2^{2n+1}(\frac{1}{2})_{n+1}(\nu+\frac{3}{2})_{n+1}} \text{ and } \varsigma_n = \frac{(-1)^n(2n+1)}{2^{2n}(\frac{1}{2})_n(\nu+\frac{3}{2})_n}.$$

Now, by comparing the coefficients of (2.26) and (2.27), we obtain

$$\chi_1 = \frac{3}{2\nu+3}, \chi_2 = \frac{34\nu+105}{3(2\nu+3)^2(2\nu+5)}, \chi_3 = \frac{268\nu^2+1824\nu+3213}{5(2\nu+3)^3(2\nu+5)(2\nu+7)}$$

and

$$\chi_4 = \frac{160336\nu^4+2256464\nu^3+11855904\nu^2+27626796\nu+24017715}{315(2\nu+3)^4(2\nu+5)^2(2\nu+7)(2\nu+9)}.$$

By using the Euler–Rayleigh inequalities ${\chi_k}^{-\frac{1}{k}} < \vartheta_{\nu,1}^2 < \frac{\chi_k}{\chi_{k+1}}$ for $|\nu| \leq \frac{1}{2}$ and $k \in \{1,2,3\}$ we obtain

$$\sqrt{\frac{2\nu+3}{3}} < r^c(u_\nu) < \sqrt{\frac{36\nu^2+144\nu+135}{34\nu+105}},$$

$$\sqrt[4]{\frac{3(2\nu+3)^2(2\nu+5)}{34\nu+105}} < r^c(u_\nu) < \sqrt{\frac{5(2\nu+3)(2\nu+7)(34\nu+105)}{3(268\nu^2+1824\nu+3213)}},$$

$$\sqrt[6]{\frac{5(2\nu+3)^3(2\nu+5)(2\nu+7)}{268\nu^2+1824\nu+3213}} < r^c(u_\nu) < 3\sqrt{\frac{7(2\nu+3)(2\nu+5)(2\nu+9)(268\nu^2+1824\nu+3213)}{\nu^*}},$$

where $\nu^* = 160336\nu^4+2256464\nu^3+11855904\nu^2+27626796\nu+24017715$. □

Theorem 9 *Let $|\nu| \leq \frac{1}{2}$. Then the radius of convexity $r^c(w_\nu)$ of the function*

$$z \mapsto w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(\sqrt{z})$$

is the smallest positive root of the equation $(zw_\nu'(z))' = 0$ and satisfies the following inequalities:

$$\frac{3(2\nu+3)}{4} < r^c(w_\nu) < \frac{30(2\nu+3)(2\nu+5)}{26\nu+119},$$

$$\sqrt{\frac{45(2\nu+3)^2(2\nu+5)}{2(26\nu+119)}} < r^c(w_\nu) < \frac{21(2\nu+3)(2\nu+7)(26\nu+119)}{2(404\nu^2+3396\nu+8665)},$$

$$\sqrt[3]{\frac{945(2\nu+3)^3(2\nu+5)(2\nu+7)}{4(404\nu^2+3396\nu+8665)}} < r^c(w_\nu) < \frac{30(2\nu+3)(2\nu+5)(2\nu+9)(404\nu^2+3396\nu+8665)}{\nu^{**}},$$

where $\nu^{**} = 36368\nu^4 + 588848\nu^3 + 3695776\nu^2 + 10793332\nu + 11828151$.

Proof By using the same idea as in the proof of Theorem 6 we have that the radius of convexity $r^c(w_\nu)$ is the smallest positive root of the equation $(zw'_\nu(z))' = 0$. See also [12] for more details. Now, if we put $\sqrt{z}$ instead of z in (2.21), (2.22), and (2.23), respectively, after some calculations we obtain

$$\psi_\nu(z) = (zw'_\nu(z))' = 1 + \sum_{n\geq 1} \frac{(-1)^n(n+1)^2}{2^{2n}(2n+1)(\frac{1}{2})_n(\nu+\frac{3}{2})_n} z^n. \tag{2.28}$$

On the other hand, we know that the function w_ν belongs to the Laguerre–Pólya class of entire functions $\mathcal{LP}$ and the Laguerre–Pólya class of entire functions is closed under differentiation. Therefore, we get that the function ψ_ν belongs also to the Laguerre–Pólya class. Hence, the zeros of the function ψ_ν are all real. Suppose that $\epsilon_{\nu,n}$s are the zeros of the function ψ_ν. Then the function ψ_ν has the infinite product representation as follows:

$$\psi_\nu(z) = \prod_{n\geq 1}\left(1 - \frac{z}{\epsilon_{\nu,n}}\right). \tag{2.29}$$

If we take the derivative of (2.29) logarithmically, then we get

$$\frac{\psi'_\nu(z)}{\psi_\nu(z)} = -\sum_{k\geq 0} \varphi_{k+1} z^k, |z| < \epsilon_{\nu,1}, \tag{2.30}$$

where $\varphi_k = \sum_{n\geq 1} \epsilon_{\nu,n}^{-k}$. Also, by taking the derivative of (2.28), we have

$$\frac{\psi'_\nu(z)}{\psi_\nu(z)} = \sum_{n\geq 0} t_n z^n \Big/ \sum_{n\geq 0} r_n z^n, \tag{2.31}$$

where

$$t_n = \frac{(-1)^{n+1}(n+2)^2(n+1)}{2^{2n+2}(2n+3)(\frac{1}{2})_{n+1}(\nu+\frac{3}{2})_{n+1}} \text{ and } r_n = \frac{(-1)^n(n+1)^2}{2^{2n}(2n+1)(\frac{1}{2})_n(\nu+\frac{3}{2})_n}.$$

Now, by comparing the coefficients of (2.30) and (2.31) we get

$$\varphi_1 = \frac{4}{3(2\nu+3)}, \varphi_2 = \frac{2(26\nu+119)}{45(2\nu+3)^2(2\nu+5)}, \varphi_3 = \frac{4(404\nu^2+3396\nu+8665)}{945(2\nu+3)^3(2\nu+5)(2\nu+7)}$$

and

$$\varphi_4 = \frac{2(36368\nu^4+588848\nu^3+3695776\nu^2+10793332\nu+11828151)}{14175(2\nu+3)^4(2\nu+5)^2(2\nu+7)(2\nu+9)}.$$

When we use the Euler–Rayleigh inequalities $\varphi_k^{-\frac{1}{k}} < \epsilon_{\nu,1} < \frac{\varphi_k}{\varphi_{k+1}}$ for $|\nu| \leq \frac{1}{2}$ and $k \in \{1,2,3\}$ we obtain the following inequalities:

$$\frac{3(2\nu+3)}{4} < r^c(w_\nu) < \frac{30(2\nu+3)(2\nu+5)}{26\nu+119},$$

$$\sqrt{\frac{45(2\nu+3)^2(2\nu+5)}{2(26\nu+119)}} < r^c(w_\nu) < \frac{21(2\nu+3)(2\nu+7)(26\nu+119)}{2(404\nu^2+3396\nu+8665)},$$

$$\sqrt[3]{\frac{945(2\nu+3)^3(2\nu+5)(2\nu+7)}{4(404\nu^2+3396\nu+8665)}} < r^c(w_\nu) < \frac{30(2\nu+3)(2\nu+5)(2\nu+9)(404\nu^2+3396\nu+8665)}{\nu^{**}},$$

where $\nu^{**} = 36368\nu^4 + 588848\nu^3 + 3695776\nu^2 + 10793332\nu + 11828151$. □

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