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A coanalytic Menger group that is not σ-compact

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Abstract: Under $V = L$ we construct coanalytic topological subgroups of reals, demonstrating that even for *definable* groups of reals, selection principles may differ.

Key words: Coanalytic, Menger, Hurewicz, Rothberger, γ-property, σ-compact, $V = L$, productively Lindelöf, topological group

1. Introduction

All spaces are assumed to be regular. For all undefined notions we refer the reader to [9, 16, 19, 21]. $\mathbb{R}$ denotes the space of real numbers with the Euclidean topology. Consider $\mathbb{N}$ as the discrete space of all finite ordinals and $\mathbb{N}^{\mathbb{N}}$ as the Baire space with the Tychonoff product topology. $P(\mathbb{N})$, the collection of all subsets of $\mathbb{N}$, is the union of $[\mathbb{N}]^{\infty}$ and $\mathbb{N}^{<\infty}$, where $[\mathbb{N}]^{\infty}$ denotes the family of infinite subsets of $\mathbb{N}$ and $\mathbb{N}^{<\infty}$ denotes the family of finite subsets of $\mathbb{N}$. Identify $P(\mathbb{N})$ with the Cantor space $\{0,1\}^{\mathbb{N}}$, using characteristic functions.

Define the quasiorder, i.e. reflexive and transitive relation, $\leq^*$ on $\mathbb{N}^{\mathbb{N}}$ by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$. A subset D of $\mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists $f \in D$ such that $g \leq^* f$. A subset B of $\mathbb{N}^{\mathbb{N}}$ is *unbounded* if for all $g \in \mathbb{N}^{\mathbb{N}}$ there is a member $f \in B$ such that $f \not\leq^* g$; otherwise, it is called a *bounded* set.

Define the quasiorder $\subseteq^*$ on $P(\mathbb{N})$ by $A \subseteq^* B$ if $A \setminus B$ is finite. A *pseudointersection* of a family $\mathcal{F}$ is an infinite subset A such that $A \subseteq^* F$ for all $F \in \mathcal{F}$. A *tower* of cardinality κ is a set $T \subseteq [\mathbb{N}]^{\infty}$ that can be enumerated bijectively as $\{x_\alpha : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$. The tower number $\mathfrak{t}$ is the minimal cardinality of a tower that has no pseudointersection.

We denote the *cardinality of the continuum* by $\mathfrak{c}$. Recall that $\mathfrak{b}$ ($\mathfrak{d}$) is the minimal cardinality of unbounded (dominating) subsets of $\mathbb{N}^{\mathbb{N}}$. It is known that $\mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ [8].

A subset of a Polish space is *analytic* if it is a continuous image of the space $\mathbb{P}$ of irrationals. We denote by $\mathbf{\Sigma^1_1}$ the family of analytic subsets of a Polish space. For a Polish space X, a set $A \subseteq X$ is *coanalytic* if $X \setminus A$ is analytic [19]. We denote by $\mathbf{\Pi^1_1}$ the family of coanalytic subsets of X. More generally, for $n \geq 1$ the families $\mathbf{\Sigma^1_n}$, $\mathbf{\Pi^1_n}$ are known as *projective classes*; for details, see Section 37 in [19]. Since there is a connection between the projective hierarchy and the Lévy hierarchy of formulas, the family of analytic subsets is classified according to the logical complexity of the formula defining it. Let $\mathcal{A}^2$ denote the second-order arithmetic. A

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set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma}^1_1$ if it can be written as $A = \{x \in \mathbb{N}^{\mathbb{N}} : \mathcal{A}^2 \models \phi(x)\}$ where ϕ of the form $\exists^1 y\, \psi$ and ψ is an arithmetical formula, i.e. it is a formula in which all quantifiers range over $\mathbb{N}$. Then a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi}^1_1$ if it can be written as $A = \{x \in \mathbb{N}^{\mathbb{N}} : \mathcal{A}^2 \models \phi(x)\}$ where ϕ is of the form $\forall^1 y\, \psi$ and ψ is an arithmetical formula; see the section entitled "The Definability Context" in [18], and also [20].

A subset of $\mathbb{R}$ is called *perfect* if it is nonempty, closed, and has no isolated points. By a set of reals, we mean a separable, metrizable space that is homeomorphic to a subset of $\mathbb{R}$. An uncountable subset of reals is *totally imperfect* if it includes no uncountable perfect set. Let κ be an infinite cardinal. $X \subseteq \mathbb{R}$ is κ*-concentrated* on a set Q if, for each open set U containing Q, $\mid X \setminus U \mid < \kappa$.

The theory of selection principles in mathematics is a study of diagonalization processes and its root goes back to Cantor. The oldest well-known selection principles are the Menger, Hurewicz, and Rothberger properties; the first two are generalizations of σ-compactness.

In 1924, Menger [23] introduced a topological property for metric spaces, which was referred to as *"property E"*. A space with property E was called *"property M"*(in honor of Menger) by Miller and Fremlin [26]. Soon thereafter, Hurewicz [15] reformulated property E as the following and nowadays it is called the *Menger property*: a topological space X satisfies the Menger property if, given any sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X, there exist finite subsets $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers X. By the following standard terminology, $S_{fin}(\mathcal{O}, \mathcal{O})$ denotes the Menger property. Menger [23] made the following conjecture:

Menger's Conjecture. A metric space X satisfies the Menger property if and only if X is σ-compact.

In 1925, Hurewicz [14] introduced a stronger property than the Menger property, which today is called the *Hurewicz property*: for any sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X one may pick finite set $\mathcal{V}_n \subset \mathcal{U}_n$ in such a way that $\{\bigcup \mathcal{V}_n : n \in \mathbb{N}\}$ is a γ-cover of X. An infinite open cover $\mathcal{U}$ is a γ*-cover* if for each $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. The collection of γ*-covers* of X is denoted by Γ. Following standard terminology let $U_{fin}(\mathcal{O}, \Gamma))$ denote the Hurewicz property. Hurewicz [14] made the following conjecture and also posed the question of whether the Menger property is strictly weaker than the Hurewicz property [14, 15].

Hurewicz's Conjecture. A metric space X satisfies the Hurewicz property if and only if X is σ-compact.

It was observed that Menger's conjecture is false, if one assumes the continuum hypothesis [15]. It was only recently that the conjecture was disproved by Miller and Fremlin in ZFC [26]. After that, many authors used different methods (topological, combinatorial) to settle Menger's conjecture (e.g., see [5, 17, 41]).

In 1938, Rothberger [31] introduced the following selection principle: a topological space X satisfies the *Rothberger property* if for every sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X, there exists a $V_n \in \mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} V_n$ covers X. It is clear that every Rothberger space is Menger. By the following standard terminology $S_1(\mathcal{O}, \mathcal{O})$ denotes the Rothberger property. There is a critical cardinal bound for the Rothberger property. $cov(\mathcal{M})$ is the minimal cardinality of a covering of the real line by meager sets. It is also known to be the minimum cardinality of a set of reals that fails to have the Rothberger property [26].

In this paper, we add a new aspect to Menger's and Hurewicz's conjectures by using the family $\mathbf{\Pi}^1_1$ of coanalytic sets. In Section 2, we construct a coanalytic unbounded tower, assuming $V = L$. In Section 3, using critical cardinalities, we present many algebraic definable examples that show the connection between Menger, Hurewicz, and Rothberger properties if $V = L$ holds.

2. Coanalytic sets with selection principles

We assume a general background about set theory. Gödel defined the class of constructible sets $L = \bigcup_{\alpha \in ON} L_\alpha$, where the sets L_α are defined by recursion on α (for details, see, e.g., [21]). The axiom of constructibility $V = L$ says that all sets in the universe are constructible, i.e. $\forall x \exists \alpha (x \in L_\alpha)$. It is well known that $V = L$ implies AC.

Now assuming $V = L$, we will employ an encoding argument that was first used by Erdös, Kunen, and Mauldin [10]. A general method was given by Miller [25]. It was also mentioned in [26].

Theorem 2.1 *$V = L$ implies there is a coanalytic unbounded tower.*

Proof Assume $V = L$. It is well known that there is a well-ordering $<_L$ on L. By using $<_L$ one can construct a $\mathbf{\Sigma^1_2}$ set of the reals ([18, Theorem 13.9]. Let X be defined by $x \in X$ if and only if $\exists z \in \mathbb{N}^{\mathbb{N}} [(M_z$ is well-founded and extensional$) \wedge (\pi_z(M_z) \models (ZF - P + V = L) \wedge (\exists n \in \mathbb{N} \; ((\pi_z(n) = x) \wedge \pi_z(M_z) \models \forall y <_L x \; \exists m(\pi_z(m) = y)))]$ where π_z denotes Mostowski's collapse by a real number z and M_z denotes the countable elementary submodel coded by a real number z. Proposition 13.8 in [18] shows that X is a $\mathbf{\Sigma^1_2}$ subset of $\mathbb{N}^{\mathbb{N}}$. Therefore, there is a coanalytic set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $p(B) = X$ where p is the projection map on the first coordinate [19, Section 37.A]. By Kondô's uniformization theorem [18, Theorem 12.3], there exists a coanalytic set $C \subset B$ that is a graph of a function f such that the domain of f is X. The importance of the uniformization is that for each $x \in X$ there exists exactly one y such that $f(x) = y$ and $(x, f(x)) \in C$. By using an arithmetical coding, we can obtain a coanalytic set of reals. For each $(x, f(x))$ in C, define

$$c_x(i) = \begin{cases} 1, & \text{if } f(x)(i) \in \operatorname{ran}(x) \\ 0, & \text{otherwise} \end{cases}$$

where $\operatorname{ran}(x)$ denotes the length of the sequence defined as $z_n = \pi_z(n) = x$. Notice that $C' = \{ c_x : x \in X \}$ can be defined as

$c_x \in C'$ if and only if $\forall x \, \psi(c_x, x)$ where ψ is the formula above in which all quantifiers are defined over $\mathbb{N}$. Therefore, C' is a coanalytic set of reals.

Since all L_α are increasing in L, we can enumerate X by using the countable levels of L. This implies that C' can be enumerated as $C' = \{ c_\alpha : x_\alpha \in X \}$. For each $\alpha < \beta < \omega_1$, $c_\beta \subseteq^* c_\alpha$, because $\operatorname{ran}(x_\beta) \setminus \operatorname{ran}(x_\alpha)$ is finite by the formula defining the set X. On the other hand, for each $g \in \mathbb{N}^{\mathbb{N}}$ there is an ordinal $\delta < \omega_1$ such that $g \in L_\delta$. Pick $x_\xi \in X$ such that $\operatorname{ran}(x_\xi) \subseteq^* \operatorname{ran}(g)$ and $\operatorname{ran}(x_\xi) \subseteq^* \operatorname{ran}(x_\delta)$. Then $g(m) \leq c_\xi(m)$ for all but finitely many $m \in \mathbb{N}$, and so $c_\xi \not\leq^* g$. □

We remark that this encoding method to construct a coanalytic set does not work for all $\mathbf{\Sigma^1_2}$ sets. Under $V = L$ there is a Luzin set, which cannot be encoded by using this method. See Miller's paper [25] for more details.

By using semifilters, Tsaban and Zdomskyy [41] introduced a general combinatorial method to disprove Menger's conjecture. Simplified versions of this method are described nicely in Tsaban's paper [39]. To investigate a definable version of Menger's conjecture, Tall and Tokgöz used a combinatorial method from [39] and obtained the following result, which was mentioned in [26]:

Theorem 2.2 ([36]) *$V = L$ implies there is a coanalytic Menger set of reals that is not σ-compact.*

However, we have a stronger result:

Gerlits and Nagy [13] introduced a covering property that satisfies all of the former selection principles mentioned above. An open cover $\mathcal{U}$ is called an *ω-cover* of X if for each finite $F \subseteq X$ there is $U \in \mathcal{U}$ such that $F \subseteq U$. A topological space X satisfies the *γ-property* if for every sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open ω-covers of X, there exists a $V_n \in \mathcal{U}_n$ such that $\{V_n\}_{n\in\mathbb{N}}$ is a γ-cover for X. Following standard terminology $S_1(\Omega, \Gamma)$ denotes the γ-property. γ-spaces that are homeomorphic to sets of reals are called *γ-sets*.

Let $\mathfrak{p}$ be the minimal cardinality of a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ that is closed under finite intersections and has no pseudointersection. It is well known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t}$ [8].

We note that any γ-set is totally imperfect [17]. By the Cantor–Bendixon theorem, every uncountable σ-compact set of reals contains a perfect set. Therefore, uncountable γ-sets are never σ-compact.

Theorem 2.3 *$V = L$ implies there is an uncountable coanalytic γ-set.*

Proof Following Theorem 2.1, there is an unbounded coanalytic tower T of size $\aleph_1$. Note that $\mathfrak{p} = \aleph_1$ since $V = L$. Define $X = T \cup \mathbb{N}^{<\infty}$. Then X satisfies the γ-property [29]. It is known that the family of coanalytic sets $\mathbf{\Pi}^1_1$ contains all Borel sets and is closed under countable unions [19, pp. 242]. Therefore, X is a coanalytic set. □

3. Algebraic coanalytic sets of reals

Question 1 *Is the Menger (Hurewicz) conjecture true for coanalytic topological groups?*

We will show that under $V = L$ Menger's conjecture and Hurewicz's conjecture are not true for coanalytic topological groups. Tall [35] proved that the axiom of coanalytic determinacy affirmatively settles both conjectures.

Tall and Tokgöz [36] reproved Miller and Fremlin's result [26] that the axiom of coanalytic determinacy implies that Menger coanalytic sets of reals are σ-compact. After that, Tall proved:

Theorem 3.1 (**[35]**) *The axiom of coanalytic determinacy implies that every Menger coanalytic topological group is σ-compact.*

However, under $V = L$, we can disprove Menger's conjecture.

In the following observation we add a new ingredient to obtain a coanalytic set, stronger than the earlier result in [29].

Theorem 3.2 *$V = L$ implies there is a coanalytic γ-subgroup.*

Proof By Theorem 2.3, there is an uncountable coanalytic γ-set, called H. Since the γ-property is linearly σ-additive, hereditary for closed subsets, and preserved by continuous images, there is a subgroup of reals that satisfies the γ-property [29]. For the reader's convenience we reproduce the subgroup in [37].

Let $H^0 = H$, and $H^n = H^{n-1} \times H$ for $n \geq 1$. For each natural number n, let $\Psi_{\alpha^n} : H^n \to \mathbb{R}$ be defined by $\Psi_{\alpha^n}((g_1, g_2, \ldots, g_n)) = \Sigma^n_{i=1}\alpha_i g_i$ for all $(g_1, g_2, \ldots, g_n) \in H^n$, where $\alpha^n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent subset of the set $\mathbb{Z}$ of integers. Now, for each natural number n, set $G_n = \{\Sigma^n_{i=1}\alpha_i g_i : \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{Z}$ is linearly independent and $(g_1, g_2, \ldots, g_n) \in H^n\}$. Let G_H denote the subgroup $<H>$. Since $G_H = \bigcup_n G_n$, G_H satisfies the γ-property [29, Theorem 5.2].

Claim. G_H *is coanalytic.*

Recall that a map $f\colon X \to Y$ between two topological spaces X and Y is *Borel (measurable)* if the inverse image of a Borel (equivalently, open or closed) set is Borel. It is well known that the family $\mathbf{\Pi^1_1}$ is closed under Borel preimages [19, pp. 242]. Obviously, any continuous map on topological space is a Borel map. Note that H^n is coanalytic for each natural number n, since H is a coanalytic set and each H^n is the Borel preimage of H^{n-1} under the first coordinate projection. Clearly, each Ψ_{α^n} is a linear homeomorphism onto its image, so $\Psi_{\alpha^n}\colon H^n \to \Psi_{\alpha^n}(H^n)$ is a Borel isomorphism [19, pp. 71]. This implies that each image of H^n under Ψ_{α^n} is coanalytic due to $\Psi_{\alpha^n}(H^n) = (\Psi_{\alpha^n}^{-1})^{-1}(H^n)$. Since α^n is an n-tuple, we have countably many α^n for each natural number n. Then G_n is coanalytic, as the countable union of continuous images of H^n. Therefore, $G_H = \bigcup_n G_n$ is coanalytic. □

A topological space is *productively Lindelöf* if its product with every Lindelöf space is Lindelöf [4]. A *Michael space* is a Lindelöf space M such that $M \times \mathbb{P}$ is not Lindelöf. Michael spaces can be constructed from many axioms such as $\mathfrak{d} = \aleph_1$, MA (see, e.g., [2, 3, 30]). Today it is still an open problem whether they can be constructed outright in ZFC. On the other hand, there is a close connection between productively Lindelöf spaces and Michael spaces. It is known that if there is no Michael space, then there is a productively Lindelöf metrizable space that is not σ-compact, and if there is a Michael space, then productively Lindelöf spaces are Menger [30]. Recently Tall [35] showed that there is a Michael space if and only if every productively Lindelöf Čech-complete space is σ-compact.

It is well known that $V = L$ implies CH. Michael [24] proved that CH implies that every productively Lindelöf metrizable space is σ-compact. Therefore, a stronger statement of Theorem 3.2 can be given in the following:

Corollary 3.3 *$V = L$ implies there is a coanalytic γ-subgroup of reals that is not productively Lindelöf.*

The first uncountable ordinal in L is denoted by ${\omega_1}^L$. Since ${\omega_1}^L$ is an ordinal of the universe, in general, it satisfies ${\omega_1}^L \le \omega_1$. Clearly, $V = L$ implies ${\omega_1}^L = \omega_1$. However, in some other models of ZFC, the inequality could be strict, since the notion of cardinality is not absolute. In fact, more generally:

The Gödel constructibility was generalized by Levy and Shoenfield to relative constructibility, which gives a transitive model $L[a]$ of ZFC for any set a.

Theorem 3.4 *Suppose ${\omega_1}^{L[a]} = \omega_1$ for some $a \in \mathbb{N}^{\mathbb{N}}$. If $\mathfrak{p} > \aleph_1$, then there is a coanalytic γ-subgroup of reals that is not productively Lindelöf.*

Proof It is known that ${\omega_1}^L = \omega_1$ implies there is a coanalytic set of reals without perfect set property [18, Theorem 13.12]. In analogy with L, the inner model $L[a]$ has a well-ordering $<_{L[a]}$, and Theorem 13.12 in [18] relativizes to produce corresponding results about $L[a]$ and $\mathbf{\Pi^1_1(a)}$ [18, pp. 171]. Then there is a coanalytic totally imperfect set of reals T of size $\aleph_1$. Any set of reals of size $< \mathfrak{p}$ is a γ-set [12]. Therefore, T is a γ-set. Consider the topology on the real line generated by the base $\mathcal{B} = \{U : U \text{ is open in } \mathbb{R}\} \cup \{p : p \in \mathbb{R} \setminus T\}$, denoted by R^*. Clearly, R^* is Lindelöf and contains $\mathbb{R}$. Since $T \times R^*$ is not normal, T cannot be productively Lindelöf [24]. By using a similar argument as in Theorem 3.2, we can obtain a coanalytic γ-subgroup of reals denoted by G_T. Notice that T is a closed subset of G_T (see [37]) and not productively Lindelöf . Every closed subset of a productively Lindelöf space is productively Lindelöf. Thus, G_T cannot be productively Lindelöf. □

Therefore, even if CH fails we have a model:

Corollary 3.5 *It is consistent that CH fails and there is a coanalytic γ-subgroup of reals that is not productively Lindelöf.*

Proof Start with the constructible universe L, and force $\mathbf{MA} + 2^{\aleph_0} = \aleph_2$ via a countable chain condition iteration. By Theorem 2.1 and Theorem 2.3, in L, there is a coanalytic tower T of cardinality of $\aleph_1$, and $T \cup \mathbb{N}^{<\infty}$ is a coanalytic γ-set. It is known that $\mathbf{MA}$ implies $\mathfrak{p} = \mathfrak{t} = \mathfrak{b} = \mathfrak{c}$ [6] and countable chain condition iterations preserve cardinality [16]. Since $\mathfrak{p} > \aleph_1$ in the extension, $T \cup \mathbb{N}^{<\infty}$ remains a γ-set [12]. Then, using Theorem 3.4, one can obtain a coanalytic γ-subgroup of reals that is not productively Lindelöf. □

We can also separate the Hurewicz and the Rothberger properties under $V = L$. In the following observation we modify the argument in [39], but we obtain a stronger definable version:

Theorem 3.6 *$V = L$ implies there is a coanalytic Rothberger subgroup of reals that is not Hurewicz.*

Proof By Theorem 2.1, there is a coanalytic unbounded tower S. By using elements of S we will construct a coanalytic Rothberger set of reals that is not Hurewicz.

Notice that we can identify elements $x \in [\mathbb{N}]^{\infty}$ with increasing elements of $\mathbb{N}^{\mathbb{N}}$ by letting $x(n)$ be the nth element in the increasing enumeration of x [41, Lemma 2.4]. Then S is both dominating (under V=L) and well-ordered by $\leq^*$. Fix $S = \{ s_\alpha : \alpha < \mathfrak{d} \} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ where $\mathbb{N}^{\uparrow\mathbb{N}}$ denotes the collection of all increasing elements of $\mathbb{N}^{\mathbb{N}}$. For each $\alpha < \mathfrak{d}$, pick $a_\alpha \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that:

(1) $a_\alpha^c \in \mathbb{N}^{\uparrow\mathbb{N}}$, i.e. the complement of the image of a_α is infinite;

(2) $a_\alpha \nleq^* s_\alpha$;

(3) $a_\alpha^c \nleq^* s_\alpha$.

Now define $A = \{ a_\alpha : \alpha < \mathfrak{d} \}$.

Claim. *A is coanalytic.*

A is defined recursively in the second-order arithmetic from the set S by a coanalytic formula. Indeed, $a \in A$ if and only if $\forall s\, \psi(a, s)$ where ψ states the formula given by (1), (2), and (3). Since ψ is arithmetical, A is coanalytic. Therefore, $A \cup \mathbb{N}^{<\infty}$ is coanalytic as a union of two coanalytic subsets.

Notice that by (3) $A \cup \mathbb{N}^{<\infty}$ is unbounded, and it cannot be Hurewicz [15]. Since A is $\mathfrak{d}$-concentrated on $\mathbb{N}^{<\infty}$, $A \cup \mathbb{N}^{<\infty}$ satisfies the Rothberger property [39]. Then by a similar argument to Theorem 3.2, we can obtain a coanalytic Rothberger (Menger) non-Hurewicz subgroup of reals. □

It is well known that the additive group of $\mathbb{R}$ with the usual topology is Borel, in fact σ-compact. Then it is a coanalytic Hurewicz group of reals. Notice that every closed subset of a Rothberger space is Rothberger [17, Theorem 3.1]. Also, every uncountable closed subset of reals contains a perfect subset by the Cantor–Bendixson result [28, 2A.1]. Therefore, $\mathbb{R}$ cannot be Rothberger, since every Rothberger space is totally imperfect [22].

Theorem 3.7 *$V = L$ implies there is a coanalytic totally imperfect Hurewicz subgroup of reals that is not Rothberger.*

Proof Borel [7] introduced the notation of a strong measure zero set (or strongly null). A set of reals X has *strong measure zero property* if for each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals, there exists a cover $\{\mathcal{I}_n\}_{n\in\mathbb{N}}$ of X such that $\operatorname{diam}(\mathcal{I}_n) < \epsilon_n$ for all n. By using a modification of Theorem 2.1 in [38], we can code a coanalytic Hurewicz set of reals that is not Rothberger: an unbounded set $\{f_\alpha : \alpha < \mathfrak{b}\}$ is called a $\mathfrak{b}$-*scale* if the enumeration is increasing with respect to $\leq^*$. $V = L$ implies $\mathfrak{b} = \mathfrak{d}$. Then there is a $\mathfrak{b}$-scale [39], called $H = \{ s_\alpha : \alpha < \mathfrak{b} \}$. A set A is called *strongly unbounded* if for each $f \in \mathbb{N}^\mathbb{N}$, $|\{s_\alpha \in A : s_\alpha \leq^* f\}| < |A|$. Notice that H is strongly unbounded since it is dominating.

Let SMZ denote the collection of strong measure zero subsets of the real line, and non(SMZ) denote the minimal cardinality for a set of reals that does not have strong measure zero. Under $V = L$, non(SMZ) $= \aleph_1 = \mathfrak{b}$ [32], and then there is a set of reals $Y = \{ y_\alpha : \alpha < \mathfrak{b} \}$ that is not strong measure zero. Without loss of generality, we may assume that $Y \subseteq \{0,1\}^\mathbb{N}$ (see, e.g., [40]). Define $H' = \{ s'_\alpha : \alpha < \mathfrak{b} \}$, where $s'_\alpha(n) = 2s_\alpha(n) + y_\alpha(n)$ for all n. Then H' is also strongly unbounded and $\mathfrak{b}$-scale. The mapping $\phi\colon H' \to Y$ defined by $s'(n) \to s'(n) \pmod 2$ for all n is a continuous and surjective map [38]. We adopt the notation from [41]. Since $\overline{\mathbb{N}^{\uparrow\mathbb{N}}} = \mathbb{N}^{\uparrow\mathbb{N}} \cup \mathbb{N}^{<\infty}$ and $H' \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ [41], ϕ can be extended to a surjective continuous mapping $\phi^*\colon H' \cup \mathbb{N}^{<\infty} \to Y \cup \mathbb{N}^{<\infty}$ [9, Corollary 3.6.6].

Since the collection of all infinite sets of natural numbers $[\mathbb{N}]^{\aleph_0}$ is a semifilter, $\phi^*(H' \cup \mathbb{N}^{<\infty})$ satisfies the Hurewicz property [41, Theorem 2.14]. On the other hand, since the property of having strong measure zero is hereditary [38] and $\phi^*(H') = \phi(H')$ does not have strong measure zero, $\phi^*(H' \cup \mathbb{N}^{<\infty})$ does not have strong measure zero, and then it does not satisfy Rothberger property [26].

For each $y \in Y$ is defined by the arithmetical formula $\forall n (y(n) = s'(n) \pmod 2)$, and so Y is coanalytic. Thus, $Y \cup \mathbb{N}^{<\infty}$ is co-analytic. By following a similar argument as in Theorem 3.2, one can obtain a coanalytic totally imperfect Hurewicz subgroup of reals that is not Rothberger. □

It is not obvious that there is a coanalytic Menger subgroup of reals that is neither Hurewicz nor productively Lindelöf in ZFC. Tall [35] proved that, assuming there is a Michael space and CH holds, there is no such space.

We also have:

Corollary 3.8 *Suppose* $\omega_1{}^{L[a]} = \omega_1$ *for some* $a \in \mathbb{N}^\mathbb{N}$. *If* $\mathfrak{d} > \mathfrak{b} = \aleph_1$, *then there is a coanalytic Menger subgroup of reals that is neither Hurewicz nor productively Lindelöf.*

Proof By the discussion in Theorem 3.4, there is a coanalytic set of reals S of size $\aleph_1$ that does not contain a perfect subset. The assumption $\mathfrak{d} > \mathfrak{b} = \aleph_1$ implies S is Menger but not Hurewicz [15]. Moreover, using the same argument as in Theorem 3.4, S cannot be productively Lindelöf, since $S \times R^*$ is not normal [24]. Thus, we can construct a coanalytic Menger subgroup of reals that is neither Hurewicz nor productively Lindelöf. □

Corollary 3.9 *It is consistent that* CH *fails and there is a coanalytic Menger subgroup of reals that is neither Hurewicz nor productively Lindelöf.*

Proof There is a model of set theory satisfying these two hypotheses in Corollary 3.8. Start with the constructible universe L. Take any regular cardinal $\kappa > \aleph_1$ such that $\kappa^{\aleph_0} = \kappa$. Then, in the Cohen extension $L[G]$ via Cohen forcing $\mathbb{C}(\kappa)$, we have $\mathfrak{d} > \mathfrak{b} = \aleph_1$ [11]. Also, notice that Cohen forcing preserves the cardinality $\aleph_1$, since forcings with countable chain condition (abbreviated c.c.c.) preserve cardinalities [33]. □

4. Comments on productivity

Let P be a property (or class) of spaces. A space X is called *productively* P if $X \times Y$ has the property P for each space Y satisfying P. Productively P properties have been studied by many authors (see, e.g., [3, 27, 34]).

It is known that $\mathfrak{b} = \aleph_1$ implies every productively Lindelöf space is Menger [1], but this implication is not reversible:

Following Theorem 2.3, under the assumption $V = L$, there is an uncountable coanalytic γ-set X. Thus, X is Menger. On the other hand, X is not σ-compact, and so X is not productively Lindelöf.

Note also that one can obtain a productively Menger set by using a nonproductively Menger set in the constructible universe L: clearly, every unbounded tower of size $\mathfrak{b}$ is a scale (see, e.g., [39]) under $V = L$. Theorem 2.1 and [27, Theorem 6.2] imply that there is a coanalytic productively Menger but nonproductively Lindelöf set of reals, but $\mathfrak{d}$-concentrated sets satisfy the Menger property [39] and then any unbounded tower (under V=L) is not productively Menger by Theorem 4.8 in [34].

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