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Reducing subspaces of Toeplitz operators on Dirichlet type spaces of the bidisk

HongZhao LIN*
College of Computer and Information Sciences, Fujian Agriculture and Forestry University, Fujian, P. R. China

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Abstract: The reducing subspaces of Toeplitz operators $T_{z_1^N}$ (or $T_{z_2^N}$), $T_{z_1^N z_2^N}$, and $T_{z_1^N z_2^M}$ on Dirichlet type spaces of the bidisk $\mathcal{D}_\alpha(\mathbb{D}^2)$ are described, which extends the results for the corresponding operators on the Bergman space of the bidisk.

Key words: Reducing subspace, Toeplitz operator, Dirichlet type spaces, bidisk

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of complex plane $\mathbb{C}$ and $\mathbb{R}$ denote the set of real numbers. $\mathbb{D}^2 = \{(z_1, z_2); z_1 \in \mathbb{D}, z_2 \in \mathbb{D}\}$ is called the bidisk. We say that a function $f : \mathbb{D}^2 \to \mathbb{C}$ is holomorphic if it is holomorphic in each variable separately. Each holomorphic function f on the bidisk can be represented as

$$f(z, w) = \sum_{i,j\in\mathbb{N}} a_{i,j} z_1^i z_2^j,$$

with $(z, w) \in \mathbb{D}^2$ and $a_{i,j} \in \mathbb{C}$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$; the Dirichlet type space of the bidisk $\mathcal{D}_\alpha(\mathbb{D}^2)$ consists of all holomorphic functions f on the bidisk satisfying

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{D}^2)} = \sum_{i,j\in\mathbb{N}} |a_{i,j}|^2 (1+i)^{\alpha_1}(1+j)^{\alpha_2} < \infty.$$

$\mathcal{D}_\alpha(\mathbb{D}^2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{i,j\in\mathbb{N}} a_{i,j}\overline{b_{i,j}}(1+i)^{\alpha_1}(1+j)^{\alpha_2},$$

where $f = \sum_{i,j\in\mathbb{N}} a_{i,j} z_1^i z_2^j$ and $g = \sum_{i,j\in\mathbb{N}} b_{i,j} z_1^i z_2^j$. Given $z = (z_1, z_2) \in \mathbb{D}^2$, each point evaluation $\lambda_z^\alpha(f) = f(z)$ is a bounded linear functional on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Hence, for each $z \in \mathbb{D}^2$, there exists a unique reproducing kernel $K_z(w) \in \mathcal{D}_\alpha(\mathbb{D}^2)$ with $w = (w_1, w_2) \in \mathbb{D}^2$ such that

$$f(z) = \langle f(w), K_z(w)\rangle, \ \forall f \in \mathcal{D}_\alpha(\mathbb{D}^2).$$

*Correspondence: hzl_076@163.com
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Actually, it can be calculated that $K_z(w) = \sum_{i,j\geq 0} \frac{w_1^i w_2^j \bar{z}_1^i \bar{z}_2^j}{(i+1)^{\alpha_1}(j+1)^{\alpha_2}}$. One can see [1] for more details about Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$. Throughout this paper, we denote $\gamma_{i,j}^\alpha = \|z_1^i z_2^j\|_{\mathcal{D}_\alpha(\mathbb{D}^2)} = \sqrt{(i+1)^{\alpha_1}(j+1)^{\alpha_2}}$. For simplicity, we denote $\|z_1^i z_2^j\|_{\mathcal{D}_\alpha(\mathbb{D}^2)}$ by $\|z_1^i z_2^j\|$.

It is easy to see that $\mathcal{D}_{(0,0)}(\mathbb{D}^2)$ is the Hardy space over the bidisk $H^2(\mathbb{D}^2)$ and $\mathcal{D}_{(-1,-1)}(\mathbb{D}^2)$ is the Bergman space over the bidisk $B^2(\mathbb{D}^2)$. In this paper, we only deal with $\mathcal{D}_\alpha(\mathbb{D}^2)$ satisfying $\alpha_1 \cdot \alpha_2 \neq 0$.

Given holomorphic function f on the bidisk $\mathbb{D}^2$, if $hf \in \mathcal{D}_\alpha(\mathbb{D}^2)$ for any $h \in \mathcal{D}_\alpha(\mathbb{D}^2)$, we define $T_f : \mathcal{D}_\alpha(\mathbb{D}^2) \to \mathcal{D}_\alpha(\mathbb{D}^2)$ by

$$T_f(h) = fh, \quad \forall\ h \in \mathcal{D}_\alpha(\mathbb{D}^2).$$

If N, M are integers larger than 1 with $N \neq M$, it is easy to check that $T_{z_1^N}$ (or $T_{z_2^N}$) is a bounded linear operator on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Note that

$$\|T_{z_1^N z_2^N}\| = \|T_{z_1^N} T_{z_2^N}\| \leq \|T_{z_1^N}\|\|T_{z_2^N}\|, \quad \|T_{z_1^N z_2^M}\| = \|T_{z_1^N} T_{z_2^M}\| \leq \|T_{z_1^N}\|\|T_{z_2^M}\|;$$

both $T_{z_1^N z_2^N}$ and $T_{z_1^N z_2^M}$ are bounded linear operators on $\mathcal{D}_\alpha(\mathbb{D}^2)$.

Suppose that $\mathfrak{M}$ is a closed subspace of Hilbert space $\mathcal{H}$. Recall that $\mathfrak{M}$ is a reducing subspace of the operator T if $T(\mathfrak{M}) \subseteq \mathfrak{M}$ and $T^*(\mathfrak{M}) \subseteq \mathfrak{M}$. A reducing subspace $\mathfrak{M}$ is said to be minimal if there are no nontrivial reducing subspaces of T contained in $\mathfrak{M}$.

Stessin and Zhu [6] completely characterized the reducing subspaces of weighted unilaterial shift operators of finite multiplicity. As a consequence, they gave the description of the reducing subspaces of T_{z^N} on the Bergman space and Dirichlet space of the unit disk. For more general symbols, the reducing subspaces of the Toeplitz operators with finite Blaschke product were well studied (see [2, 3, 8] for example). Recently, Lu et al. extended the result in [6] to Bergman space with several variables. They completely characterized the reducing subspaces of $T_{z_1^N}$ and $T_{z_1^N z_2^N}$ in [4] on the weighted Bergman space of the bidisk and on the weighted Bergman space over polydisk in [7], respectively. Moreover, they [5] solved the problems of $T_{z_1^N z_2^M}$ with $N \neq M$ on both settings.

Motivated by the above work, we will investigate the reducing subspaces of Toeplitz operators $T_{z_1^N}$ (or $T_{z_2^N}$), $T_{z_1^N z_2^N}$, and $T_{z_1^N z_2^M}$ on Dirichlet type spaces of the bidisk. The paper is organized as follows. In section 2, we give the description of the reducing subspace of Toeplitz operators $T_{z_1^N}$ (or $T_{z_2^N}$). We characterize the reducing subspaces of $T_{z_1^N z_2^N}$ in section 3 and the case of $T_{z_1^N z_2^M}$ is discussed in section 4.

2. The reducing subspace of $T_{z_1^N}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$

In this section, we will characterize the reducing subspace of $T_{z_1^N}$ (or $T_{z_2^N}$) on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$.

We observe that each $f(z_1, z_2) = \sum_{i,j\in\mathbb{N}} a_{i,j} z_1^i z_2^j \in \mathcal{D}_\alpha(\mathbb{D}^2)$ has the decomposition

$$f = \sum_{i=0}^{\infty} z_1^i f_i(z_2), \tag{2.1}$$

where $f_i(z_2) = \sum_{j=0}^{\infty} a_{i,j} z_2^j \in \mathcal{D}_{\alpha_2}(\mathbb{D})$ for each i.

If we denote $\gamma_i^a = \sqrt{(1+i)^a}$ for $i \in \mathbb{N}$ and $a \in \mathbb{R}$, it is easy to get the relationship between $\gamma_{i,j}^\alpha$ and $\gamma_i^{\alpha_1}$ or $\gamma_j^{\alpha_2}$, where $\alpha = (\alpha_1, \alpha_2)$. Direct computation shows the following lemma.

Lemma 2.1 *Suppose i, j, k are nonnegative integers; then* $\frac{\gamma_{i,j}^\alpha}{\gamma_{k,j}^\alpha} = \frac{\gamma_i^{\alpha_1}}{\gamma_k^{\alpha_1}}$, $\frac{\gamma_{i,j}^\alpha}{\gamma_{i,k}^\alpha} = \frac{\gamma_j^{\alpha_2}}{\gamma_k^{\alpha_2}}$.

Lemma 2.2 *Suppose i, n are nonnegative integers; then*

$$\frac{\gamma_{i+hN}^{\alpha_1}}{\gamma_i^{\alpha_1}} = \frac{\gamma_{n+hN}^{\alpha_1}}{\gamma_n^{\alpha_1}}, \quad \forall h \in \mathbb{N},$$

holds if and only if $i = n$.

Proof We only need to prove the necessity. By the assumption,

$$\frac{\gamma_{i+hN}^{\alpha_1}}{\gamma_{n+hN}^{\alpha_1}} = \frac{\gamma_i^{\alpha_1}}{\gamma_n^{\alpha_1}}, \quad \forall h \in \mathbb{N}.$$

Taking $h \to \infty$ in the above equation,

$$\lim_{h\to\infty} \frac{\gamma_{i+hN}^{\alpha_1}}{\gamma_{n+hN}^{\alpha_1}} = 1,$$

which implies that

$$\frac{\gamma_{i+hN}^{\alpha_1}}{\gamma_{n+hN}^{\alpha_1}} = 1, \quad \forall h \in \mathbb{N}.$$

By the definition of $\gamma_i^{\alpha_1}$, it is equivalent to

$$(i + hN + 1)^{\alpha_1} = (n + hN + 1)^{\alpha_1}, \quad \forall h \in \mathbb{N}. \tag{2.2}$$

Since $\alpha_1 \neq 0$, then $i = n$. □

Lemma 2.3 *Suppose $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Then the following statements hold:*

(1) If $f = \sum_{n\geq 0} z_1^n f_n(z_2) \in \mathfrak{M}$ with $f_n(z_2) \neq 0$, then $z_1^n f_n(z_2) \in \mathfrak{M}$;

(2) If $g = \sum_{n\geq 0} z_2^n g_n(z_1) \in \mathfrak{M}^\perp$ with $g_n(z_1) \neq 0$, then $z_2^n g_n(z_1) \in \mathfrak{M}^\perp$.

Proof First assume $f = z_1^p f_p(z_2)$. Let $z_1^p f_p(z_2) = u(z_1, z_2) + v(z_1, z_2)$ be the orthogonal decomposition on $\mathfrak{M}$, where $u(z_1, z_2) = \sum_{k=0}^\infty z_1^k u_k(z_2) \in \mathfrak{M}$ and $v(z_1, z_2) \in \mathfrak{M}^\perp$.

For nonnegative integer h, note that $T^*_{z_1^{hN}} T_{z_1^{hN}} z_1^k z_2^j = \frac{(\gamma_{k+hN}^{\alpha_1})^2}{(\gamma_k^{\alpha_1})^2} z_1^k z_2^j$. By Lemma 2.1, we calculate

$$\begin{aligned} T^*_{z_1^{hN}} T_{z_1^{hN}} P_{\mathfrak{M}}(z_1^p f_p(z_2)) &= T^*_{z_1^{hN}} T_{z_1^{hN}} P_{\mathfrak{M}}(u + v) \\ &= T^*_{z_1^{hN}} T_{z_1^{hN}} u \\ &= T^*_{z_1^{hN}} T_{z_1^{hN}} \sum_{k=0}^\infty z_1^k u_k(z_2) \\ &= \sum_{k=0}^\infty \frac{(\gamma_{k+hN}^{\alpha_1})^2}{(\gamma_k^{\alpha_1})^2} z_1^k u_k(z_2). \end{aligned}$$

On the other hand, direct computation shows that

$$\begin{aligned} P_{\mathfrak{M}} T^*_{z_1^{hN}} T_{z_1^{hN}}(z_1^p f_p(z_2)) &= P_{\mathfrak{M}}\left(\frac{(\gamma_{p+hN}^{\alpha_1})^2}{(\gamma_p^{\alpha_1})^2} z_1^p f_p(z_2)\right) \\ &= \frac{(\gamma_{p+hN}^{\alpha_1})^2}{(\gamma_p^{\alpha_1})^2} u \\ &= \sum_{k=0}^{\infty} \frac{(\gamma_{p+hN}^{\alpha_1})^2}{(\gamma_p^{\alpha_1})^2} z_1^k u_k(z_2). \end{aligned}$$

Since $T_{z_1^{hN}}$ and $T^*_{z_1^{hN}}$ commute with $P_{\mathfrak{M}}$, it follows that, if $u_k(z_2) \neq 0$, then for each positive integer h

$$\frac{\gamma_{k+hN}^{\alpha_1}}{\gamma_k^{\alpha_1}} = \frac{\gamma_{p+hN}^{\alpha_1}}{\gamma_p^{\alpha_1}},$$

which is equivalent to

$$\frac{\gamma_{k+hN}^2}{\gamma_{p+hN}^2} = \frac{\gamma_k^2}{\gamma_p^2}.$$

Since $\alpha_1 \neq 0$, Lemma 2.2 implies that

$$k = p.$$

Therefore, $P_{\mathfrak{M}}(z_1^p f_p(z_2)) = u = z_1^p u_p(z_2) \in \mathfrak{M}$.

Now assume $f = \sum_{n\geq 0} z_1^n f_n(z_2) \in \mathfrak{M}$ and $P_{\mathfrak{M}} z_1^n f_n(z_2) = z_1^n u_n(z_2)$. By the above discussion,

$$P_{\mathfrak{M}} f = P_{\mathfrak{M}} \sum_{n\geq 0} z_1^n f_n(z_2) = \sum_{n\geq 0} z_1^n u_n(z_2).$$

Note that $P_{\mathfrak{M}} f = f$. Comparing the expressions, it follows that

$$f_n(z_2) = u_n(z_2).$$

That is

$$P_{\mathfrak{M}} z_1^n f_n(z_2) = z_1^n f_n(z_2),$$

which means that $z_1^n f_n(z_2) \in \mathfrak{M}$. Thus statement (1) is obtained.

Note that if $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N}$, so is $\mathfrak{M}^{\perp}$. By the symmetry, statement (2) is desired. □

Let $\mathcal{M}_{n_1}^{(1)} = \overline{span}\{z_1^{n_1+hN} : h \in \mathbb{N}\}$. The next theorem characterizes the reducing subspaces of $T_{z_1^N}$ on Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$.

Theorem 2.4 *Suppose n_1 is an integer with $0 \leq n_1 \leq N-1$. The reducing subspaces*

$$f(z_2)\mathcal{M}_{n_1}^{(1)} = \overline{span}\{f(z_2) z_1^{n_1+hN} : h \in \mathbb{N}\}, \quad \text{where } f(z_2) \in \mathcal{D}_\alpha(\mathbb{D})$$

are the only minimal reducing subspaces of $T_{z_1^N}$ on Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$. Each reducing subspace $\mathfrak{M}$ contains some minimal reducing subspace as above.

Proof For each $h \in \mathbb{N}$, we calculate

$$T_{z_1^N} f(z_2) z_1^{n_1+hN} = f(z_2) z_1^{n_1+(h+1)N} \in f(z_2)\mathcal{M}_{n_1}^{(1)}$$

and

$$T_{z_1^N}^* f(z_2) z_1^{n_1+hN} = \begin{cases} \frac{(\gamma_{n_1+hN}^{\alpha_1})^2}{(\gamma_{n_1+(h-1)N}^{\alpha_1})^2} f(z_2) z_1^{n_1+(h-1)N} \in f(z_2)\mathcal{M}_{n_1}^{(1)}, & h \geq 1 \\ 0, & h = 0 \end{cases},$$

which means $f(z_2)\mathcal{M}_{n_1}^{(1)}$ is the reducing subspace of $T_{z_1^N}$. Next we will show that such type of reducing subspace is minimal.

If $\mathfrak{M} \subseteq \mathcal{D}_\alpha(\mathbb{D}^2)$ is the reducing subspace of $T_{z_1^N}$, since $\mathfrak{M}$ is nonempty, there is nonzero $f = \sum_{n\geq 0} z_1^n f_n(z_2) \in \mathfrak{M}$. By Lemma 2.3, there exists some $n_0 \in \mathbb{N}$ such that $z_1^{n_0} f_{n_0}(z_2) \in \mathfrak{M}$ and $f_{n_0}(z_2) \neq 0$. Let $n_1 = n_0 \mod N$; then $f_{n_0}(z_2)\mathcal{M}_{n_1}^{(1)} \subseteq \mathfrak{M}$ is the minimal reducing subspace.

Furthermore, if $\mathfrak{M}$ is minimal, then $f_{n_0}(z_2)\mathcal{M}_{n_1}^{(1)} = \mathfrak{M}$. The result is desired. □

Theorem 2.5 *Suppose n_2 is an integer with $0 \leq n_2 \leq N-1$. The reducing subspaces*

$$g(z_1)\mathcal{M}_{n_2}^{(2)} = \overline{span}\{g(z_1) z_2^{n_2+kN}\}, \quad \text{where} \;\; g(z_1) \in \mathcal{D}_\alpha(\mathbb{D})$$

are the only minimal reducing subspaces of $T_{z_2^N}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Each reducing subspace $\mathfrak{M}$ contains some minimal reducing subspace as above.

Proof The result is immediately followed by Theorem 2.4. □

Theorem 2.6 *Suppose N_1, N_2 are positive integers larger than 1 and $\mathfrak{M}$ is a closed subspace of $\mathcal{D}_\alpha(\mathbb{D}^2)$. If $\mathfrak{M}$ is the reducing subspaces of both $T_{z_1^{N_1}}$ and $T_{z_2^{N_2}}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$, then there exist $\mathcal{M}_{n_1}^{(1)}$ and $\mathcal{M}_{n_2}^{(2)}$ such that*

$$\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)} := \overline{span}\{z_1^{n_1+hN_1} z_2^{n_2+kN_2} : h, k \in \mathbb{N}\} \subseteq \mathfrak{M},$$

where n_1 and n_2 are integers with $0 \leq n_1 \leq N_1 - 1, 0 \leq n_2 \leq N_2 - 1$. In particular, $\mathfrak{M}$ is the minimal reducing subspace if and only if $\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)} = \mathfrak{M}$. There are totally $N_1 N_2$ such minimal reducing subspaces on $\mathcal{D}_\alpha(\mathbb{D}^2)$.

Proof Since $\mathfrak{M}$ is the reducing subspaces of $T_{z_1^{N_1}}$, then by Theorem 2.4 there exists

$$f(z_2) z_1^{n_1} \in \mathfrak{M},$$

where nonzero function $f(z_2) = \sum_{i=1}^\infty a_i z_2^i \in \mathcal{D}_\alpha(\mathbb{D})$ and $n_1 \in \mathbb{N}$ with $0 \leq n_1 \leq N_1 - 1$. Note that $f(z_2) z_1^{n_1} = (\sum_{i=1}^\infty a_i z_2^i) z_1^{n_1} = \sum_{i=1}^\infty z_2^i a_i z_1^{n_1} \in \mathfrak{M}$. Since $\mathfrak{M}$ is also the reducing subspaces of $T_{z_2^{N_2}}$, by Lemma 2.3, there exist $i_0 \in \mathbb{N}$ such that $z_2^{i_0} z_1^{n_1} \in \mathfrak{M}$. Let $n_2 = i_0 \mod N_2$, and it follows that

$$\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)} \subseteq \mathfrak{M}$$

is the reducing subspace of both $T_{z_1^{N_1}}$ and $T_{z_2^{N_2}}$.

If reducing subspace $\mathfrak{M}$ is minimal, it certainly follows that $\mathfrak{M} = \mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)}$.

Examining the proof above, it is easy to see that all the minimal reducing subspaces of both $T_{z_1^{N_1}}$ and $T_{z_2^{N_2}}$ can be expressed as $\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)}$. The total number of the minimal reducing subspaces is N_1N_2 since there are N_1 different spaces $\mathcal{M}_{n_1}^{(1)}$ and N_2 different spaces $\mathcal{M}_{n_2}^{(2)}$. This completes the proof. □

Corollary 2.7 *Suppose N is a positive integer larger than 1 and $\mathfrak{M}$ is a closed subspace of $\mathcal{D}_\alpha(\mathbb{D}^2)$. If $\mathfrak{M}$ is the reducing subspaces of both $T_{z_1^N}$ and $T_{z_2^N}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$, then there exist $\mathcal{M}_{n_1}^{(1)}$ and $\mathcal{M}_{n_2}^{(2)}$ such that*

$$\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)} \subseteq \mathfrak{M},$$

where n_1 and n_2 are integers with $0 \le n_1 \le N-1, 0 \le n_2 \le N-1$. In particular, $\mathfrak{M}$ is the minimal reducing subspace if and only if $\mathcal{M}_{n_1}^{(1)} \otimes \mathcal{M}_{n_2}^{(2)} = \mathfrak{M}$. There are totally N^2 such minimal reducing subspaces on $\mathcal{D}_\alpha(\mathbb{D}^2)$.

Proof It follows from Theorem 2.6. □

Zhou and Lu [4] describe all the reducing subspaces of $T_{z_1^N}$ (or $T_{z_2^N}$) on the Bergman space of the bidisk. Observe that $\mathcal{D}_{(-1,-1)}(\mathbb{D}^2)$ is the common Bergman space of the bidisk, and we extend the result in [4]. Comparing with the results in [4], Theorem 2.4 and Theorem 2.5 imply that $T_{z_1^N}$ (or $T_{z_2^N}$) shares the same structure of reducing subspaces on each Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$. In other words, if $\alpha \neq 0$, the structure of reducing subspaces of $T_{z_1^N}$ (or $T_{z_2^N}$) on $\mathcal{D}_\alpha(\mathbb{D}^2)$ is independent of the weight α whenever $\alpha_1\alpha_2 \neq 0$.

3. The reducing subspace of $T_{z_1^N z_2^N}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$

In this section, we will study reducing subspace of $T_{z_1^N z_2^N}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$. It shows that the structure of reducing subspaces of $T_{z_1^N z_2^N}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$ is dependent on α.

Firstly we make further study of $\gamma_{i,j}^\alpha$.

Lemma 3.1 *Suppose k, m, i, j, N are nonnegative integers and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. If*

$$\frac{(\gamma_{i+hN,j+hN}^\alpha)^2}{(\gamma_{i,j}^\alpha)^2} = \frac{(\gamma_{k+m+hN,m+hN}^\alpha)^2}{(\gamma_{k+m,m}^\alpha)^2}, \quad \forall h \in \mathbb{N},$$

then the following statements hold:

(1) If $\alpha_1 = \alpha_2$, then $(i,j) = (k+m, m)$ or $(i,j) = (m, k+m)$.

(2) If $\alpha_1 = -\alpha_2$, then $(i,j) = (k+m, m)$ or $\frac{m}{k+m} = \frac{j}{i} = 1$.

(3) If $|\alpha_1| \neq |\alpha_2|$, then $(i,j) = (k+m, m)$.

Proof By the assumption,

$$\frac{(\gamma_{i+hN,j+hN}^\alpha)^2}{(\gamma_{k+m+hN,m+hN}^\alpha)^2} = \frac{(\gamma_{i,j}^\alpha)^2}{(\gamma_{k+m,m}^\alpha)^2}, \quad \forall h \in \mathbb{N},$$

Taking $h \to \infty$ in the above equation, it follows that for any positive integer h

$$\frac{(\gamma^{\alpha}_{i+hN,j+hN})^2}{(\gamma^{\alpha}_{k+m+hN,m+hN})^2} = 1.$$

By definition of $\gamma^{\alpha}_{i,j}$, it turns into

$$(i+hN+1)^{\alpha_1}(j+hN+1)^{\alpha_2} = (k+m+hN+1)^{\alpha_1}(m+hN+1)^{\alpha_2}. \tag{3.1}$$

If $\alpha_1 = \alpha_2 \neq 0$, the above equation is equivalent to

$$(i+hN+1)(j+hN+1) = (k+m+hN+1)(m+hN+1),$$

which implies statement (1).

If $\alpha_1 = -\alpha_2 \neq 0$, (3.1) is equivalent to

$$(i+hN+1)(m+hN+1) = (k+m+hN+1)(j+hN+1),$$

which implies statement (2).

Now suppose $|\alpha_1| \neq |\alpha_2|$. Without loss of generality, we may assume $\alpha_1 > 0$ and $\alpha_2 > 0$. If $\lambda = hN$, (3.1) is equivalent to

$$(i+\lambda+1)^{\alpha_1}(j+\lambda+1)^{\alpha_2} = (k+m+\lambda+1)^{\alpha_1}(m+\lambda+1)^{\alpha_2}, \quad \forall \lambda \in \mathbb{C}.$$

Note that $\alpha_1 \neq \alpha_2$; by comparing the multiplicity of zeros on both sides of the equation, statement (3) follows. $\square$

Next we describe the projection of monomial on reducing subspace $\mathfrak{M}$.

Theorem 3.2 *Suppose that $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$ on Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$ and $P_{\mathfrak{M}}$ is the projection onto $\mathfrak{M}$. Then the following statements hold:*

(1) If $\alpha_1 = \alpha_2$, then

$$P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) = (az_1^k + bz_2^k)(z_1z_2)^m, \quad (a-|a|^2)\|z_1^k(z_1z_2)^m\|^2 = |b|^2\|z_2^k(z_1z_2)^m\|^2.$$

(2) If $\alpha_1 = -\alpha_2$, then

$$P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) = az_1^k(z_1z_2)^m, \quad a = |a|^2, \quad \textit{for } \ k > 0;$$

$$P_{\mathfrak{M}}((z_1z_2)^m) = \sum_{i=0}^{\infty} a_i(z_1z_2)^i, \quad a_m = \sum_{i=0}^{\infty}|a_i|^2, \quad \textit{for } \ k = 0.$$

(3) If $|\alpha_1| \neq |\alpha_2|$, then

$$P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) = az_1^k(z_1z_2)^m, \quad a = |a|^2.$$

Proof Let $z_1^k(z_1z_2)^m = f(z_1,z_2) + g(z_1,z_2)$ be the orthogonal decomposition on $\mathfrak{M}$, where $f(z_1,z_2) = \sum_{i,j=0}^{\infty} f_{i,j}z_1^iz_2^j \in \mathfrak{M}$ and $g(z_1,z_2) = \sum_{i,j=0}^{\infty} g_{i,j}z_1^iz_2^j \in \mathfrak{M}^{\perp}$.

For nonnegative integer h, we calculate

$$\begin{aligned} T^*_{z_1^{hN}z_2^{hN}}T_{z_1^{hN}z_2^{hN}}P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) &= T^*_{z_1^{hN}z_2^{hN}}T_{z_1^{hN}z_2^{hN}}P_{\mathfrak{M}}(f+g) \\ &= T^*_{z_1^{hN}z_2^{hN}}T_{z_1^{hN}z_2^{hN}}f \\ &= T^*_{z_1^{hN}z_2^{hN}}\sum_{i,j=0}^{\infty} f_{i,j}z_1^{i+hN}z_2^{j+hN} \\ &= \sum_{i,j=0}^{\infty} f_{i,j}\frac{(\gamma^{\alpha}_{i+hN,j+hN})^2}{(\gamma^{\alpha}_{i,j})^2}z_1^i z_2^j. \end{aligned}$$

On the other hand, a direct computation shows that

$$\begin{aligned} P_{\mathfrak{M}}T^*_{z_1^{hN}z_2^{hN}}T_{z_1^{hN}z_2^{hN}}(z_1^k(z_1z_2)^m) &= P_{\mathfrak{M}}T^*_{z_1^{hN}z_2^{hN}}(z_1^k(z_1z_2)^{m+hN}) \\ &= P_{\mathfrak{M}}((\frac{\gamma^{\alpha}_{k+m+hN,m+hN})^2}{(\gamma^{\alpha}_{k+m,m})^2}z_1^k(z_1z_2)^m) \\ &= \sum_{i,j=0}^{\infty} f_{i,j}\frac{(\gamma^{\alpha}_{k+m+hN,m+hN})^2}{(\gamma^{\alpha}_{k+m,m})^2}z_1^i z_2^j. \end{aligned}$$

Since $T_{z_1^{hN}z_2^{hN}}$ and $T^*_{z_1^{hN}z_2^{hN}}$ commute with $P_{\mathfrak{M}}$, if $f_{i,j}\neq 0$, it follows that

$$\frac{(\gamma^{\alpha}_{i+hN,j+hN})^2}{(\gamma^{\alpha}_{i,j})^2} = \frac{(\gamma^{\alpha}_{k+m+hN,m+hN})^2}{(\gamma^{\alpha}_{k+m,m})^2}, \quad \forall \ h\in\mathbb{N}.$$

If $\alpha_1=\alpha_2$, by Lemma 3.1, $(i,j)=(m+k,m)$ or $(i,j)=(m,m+k)$, which implies that

$$P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) = (az_1^k+bz_2^k)(z_1z_2)^m, \quad \text{for constants } \ a,b.$$

Since $\langle z_1^k(z_1z_2)^m - P_{\mathfrak{M}}(z_1^k(z_1z_2)^m), P_{\mathfrak{M}}(z_1^k(z_1z_2)^m)\rangle = 0$, it follows that

$$\langle z_1^k(z_1z_2)^m, P_{\mathfrak{M}}(z_1^k(z_1z_2)^m)\rangle = \|P_{\mathfrak{M}}(z_1^k(z_1z_2)^m)\|^2. \tag{3.2}$$

A direct computation shows that $a\|z_1^k(z_1z_2)^m\|^2 = |a|^2\|z_1^k(z_1z_2)^m\|^2+|b|^2\|z_2^k(z_1z_2)^m\|^2$. Thus statement (1) holds.

If $\alpha_1=-\alpha_2$, by Lemma 3.1, $(i,j)=(m+k,m)$ or $\frac{m}{k+m}=\frac{j}{i}=1$, which implies that for $k>0$

$$P_{\mathfrak{M}}(z_1^k(z_1z_2)^m) = az_1^k(z_1z_2)^m, \quad \text{for constant } \ a,$$

and for $k=0$

$$P_{\mathfrak{M}}((z_1z_2)^m) = \sum_{i=0}^{\infty} a_i(z_1z_2)^i, \quad \text{for constants } \ a_i.$$

Observe that (3.2) holds for any α, and statement (2) follows from the direct computation with (3.2).

If $|\alpha_1|\neq|\alpha_2|$, by Lemma 3.1, $(i,j)=(m+k,m)$, which together with (3.2) implies statement (3) holds. $\square$

Lemma 3.3 *Let Λ be an index set and let $\mathcal{H}$ be a Hilbert space. Suppose that $\mathcal{H}$ is the direct sum of its closed subspace X_i $(i \in \Lambda)$, that is $\mathcal{H} = \bigoplus_{i\in\Lambda} X_i$, $\mathfrak{M}$ is a reducing subspace of bounded linear operator T on $\mathcal{H}$ and $P_{\mathfrak{M}} X_i \subseteq X_i$. If $f = \sum_{i\in\Lambda} f_i \in \mathfrak{M}$ with $f_i \in X_i$, then $f_i \in \mathfrak{M}$ for each $i \in \Lambda$.*

Proof Note that

$$\sum_{i\in\Lambda} f_i = f = P_{\mathfrak{M}} f = \sum_{i\in\Lambda} P_{\mathfrak{M}} f_i.$$

The result follows from $f_i = P_{\mathfrak{M}} f_i$ since $f_i \in X_i$ and $P_{\mathfrak{M}} f_i \in X_i$. □

Theorem 3.4 *Suppose that $\mathfrak{M}$ is a nontrivial reducing subspace of $T_{z_1^N z_2^N}$ in $\mathcal{D}_\alpha(\mathbb{D}^2)$ and $f = \sum_{i,j=0}^{\infty} f_{i,j} z_1^i z_2^j \in \mathfrak{M}$. Then the following statements hold:*

(1) If $\alpha_1 = \alpha_2$, then $f_{i,i} z_1^i z_2^i \in \mathfrak{M}$ and $f_{i,j} z_1^i z_2^j + f_{j,i} z_1^j z_2^i \in \mathfrak{M}$ with $i \neq j$.

(2) If $\alpha_1 = -\alpha_2$, then $\sum_{i=0}^{\infty} f_{i,i} z_1^i z_2^i \in \mathfrak{M}$ and $f_{i,j} z_1^i z_2^j \in \mathfrak{M}$ with $i \neq j$.

(3) If $|\alpha_1| \neq |\alpha_2|$, then $z_1^i z_2^j \in \mathfrak{M}$.

Proof If $\alpha_1 = \alpha_2$, rewrite $f = \sum_{i\geq 0} f_{i,i} z_1^i z_2^i + \sum_{i>j} f_{i,j} z_1^i z_2^j + f_{j,i} z_1^j z_2^i$. Theorem 3.2 implies that

$$P_{\mathfrak{M}} z_1^i z_2^i \subseteq \overline{span}\{z_1^i z_2^i\} \quad \text{and} \quad P_{\mathfrak{M}} \overline{span}\{z_1^i z_2^j, z_1^j z_2^i\} \subseteq \overline{span}\{z_1^i z_2^j, z_1^j z_2^i\}.$$

By Lemma 3.3, it follows that

$$f_{i,i} z_1^i z_2^i \in \mathfrak{M} \quad \text{and} \quad f_{i,j} z_1^i z_2^j + f_{j,i} z_1^j z_2^i \in \mathfrak{M}.$$

Thus statement (1) holds.

If $\alpha_1 = -\alpha_2$, rewrite $f = \sum_{i\geq 0} f_{i,i} z_1^i z_2^i + \sum_{i\neq j} f_{i,j} z_1^i z_2^j$. Theorem 3.2 implies that

$$P_{\mathfrak{M}} \overline{span}\{z_1^i z_2^i; i \in \mathbb{N}\} \subseteq \overline{span}\{z_1^i z_2^i; i \in \mathbb{N}\} \quad \text{and} \quad P_{\mathfrak{M}} z_1^i z_2^j \subseteq \overline{span}\{z_1^i z_2^j\}.$$

By Lemma 3.3, it follows that

$$\sum_{i\geq 0} f_{i,i} z_1^i z_2^i \in \mathfrak{M} \quad \text{and} \quad f_{i,j} z_1^i z_2^j \in \mathfrak{M}.$$

Thus statement (2) holds.

If $|\alpha_1| \neq |\alpha_2|$, Theorem 3.2 implies that $P_{\mathfrak{M}}(z_1^i z_2^j) \subseteq \overline{span}\{z_1^i z_2^j\}$. Statement (3) is also achieved from Lemma 3.3 . □

Theorem 3.5 *Suppose that M is the minimal reducing subspaces of $T_{z_1^N z_2^N}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Then the following statements hold:*

(1) If $\alpha_1 = \alpha_2$, then

$$M = \overline{span}\{(a z_1^k + b z_2^k)(z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}, \quad a, b \in \mathbb{C}, m, k \in \mathbb{N}, 0 \leq m \leq N-1.$$

(2) If $\alpha_1 = -\alpha_2$*, then*

$$
\begin{aligned}
&M = \overline{span}\{(\sum_{i=1}^{\infty} a_i z_1^i z_2^i)(z_1^{lN} z_2^{lN}) : l \in \mathbb{N}\}, \quad a_i \in \mathbb{C}, m \in \mathbb{N} \\
or \quad &M = \overline{span}\{a z_1^k (z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}, \quad a \in \mathbb{C}, m, k \in \mathbb{N}, 0 \le m \le N-1, \\
or \quad &M = \overline{span}\{b z_2^k (z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}, \quad b \in \mathbb{C}, m, k \in \mathbb{N}, 0 \le m \le N-1.
\end{aligned}
$$

(3) If $|\alpha_1| \neq |\alpha_2|$*, then*

$$
\begin{aligned}
&M = \overline{span}\{a z_1^k (z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}, \quad a \in \mathbb{C}, m, k \in \mathbb{N}, 0 \le m \le N-1, \\
or \quad &M = \overline{span}\{b z_2^k (z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}, \quad b \in \mathbb{C}, m, k \in \mathbb{N}, 0 \le m \le N-1.
\end{aligned}
$$

Proof Suppose that $\mathfrak{M}$ is a nontrivial reducing subspace. Then there exists nonzero function $f = \sum_{i,j\ge 0} f_{i,j} z_1^i z_2^j \in \mathfrak{M}$.

For $\alpha_1 = \alpha_2$. We consider the following two cases. If $f_{i,i} \neq 0$ for some i, by Theorem 3.4, $z_1^i z_2^i \in \mathfrak{M}$. Note that if $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$, then $T_{z_1^{lN} z_2^{lN}} \mathfrak{M} \subseteq \mathfrak{M}$ and $T^*_{z_1^{lN} z_2^{lN}} \mathfrak{M} \subseteq \mathfrak{M}$. Let $a+b = 1, k = 0, m = i$ mod N; then $M \subseteq \mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$.

If $f_{i,j} \neq 0$ for some multi-index (i,j) with $i > j$, by Theorem 3.4, $f_{i,j} z_1^i z_2^j + f_{j,i} z_1^j z_2^i \in \mathfrak{M}$. Let $a = f_{i,j}, b = f_{j,i}, k = i - j$ and $m = j$ mod N; then $M \subseteq \mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$.

For $\alpha_1 = -\alpha_2$. If $\sum_{i=1}^{\infty} f_{i,i} z_1^i z_2^i \neq 0$, by Theorem 3.4, $\sum_{i=1}^{\infty} f_{i,i} z_1^i z_2^i \in \mathfrak{M}$. Therefore, $M = \overline{span}\{(\sum_{i=1}^{\infty} f_{i,i} z_1^i z_2^i)(z_1^{lN} z_2^{lN}) : h \in \mathbb{N}\} \subseteq \mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$.

If $f_{i,j} \neq 0$ for some multi-index (i,j) with $i \neq j$, without loss of generality we assume $i > j$. By Theorem 3.4, $z_1^i z_2^j \in \mathfrak{M}$. Let $a = f_{i,j}, k = i - j$ and $m = j$ mod N; then $M = \overline{span}\{a z_1^k (z_1^{m+lN} z_2^{m+lN}) : h \in \mathbb{N}\} \subseteq \mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$.

A similar discussion implies the case of $|\alpha_1| \neq |\alpha_2|$.

From the above proof, we deduce that each reducing subspace $\mathfrak{M}$ contains a reducing subspace M, which means that M consists of all the minimal reducing subspaces.

Therefore, if $\mathfrak{M}$ is a minimal reducing subspace then $\mathfrak{M} = M$.

This complete the proof. □

Theorem 2.5 in [4] showed that if $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^N}$ on the Bergman space over the bidisk, then there exist nonnegative integers a, b, k, m with $0 \le m \le N-1$ and $a, b \in \mathbb{C}$ (Carefully examining the original proof, we find that it should be $a, b \in \mathbb{C}$ instead of $b \in \{0, 1\}$ in [4] Theorem 2.5.) such that

$$
\overline{span}\{(a z_1^k + b z_2^k)(z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\} \subseteq \mathfrak{M}. \tag{3.3}
$$

In particular, $\mathfrak{M}$ is minimal if and only if $\mathfrak{M} = \overline{span}\{(a z_1^k + b z_2^k)(z_1^{m+lN} z_2^{m+lN}) : l \in \mathbb{N}\}$. Note that since $\mathcal{D}_{(-1,-1)}(\mathbb{D}^2)$ is the bidisk Bergman space, Theorem 3.5 extends the result of Theorem 2.5 in [4] to more general spaces.

4. The reducing subspace of $T_{z_1^N z_2^M}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$

In this section, we will study reducing subspaces of $T_{z_1^N z_2^M}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $\alpha \neq 0$. The result shows that the structure of reducing subspaces of $T_{z_1^N z_2^M}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$ is dependent on α. Here we define $\rho_1(n) = \frac{M(n+1)}{N} - 1$ and $\rho_2(m) = \frac{N(m+1)}{M} - 1$.

Lemma 4.1 *Suppose n, m, i, j, N are nonnegative integers. If*

$$\frac{\gamma^\alpha_{i+hN,j+hM}}{\gamma^\alpha_{i,j}} = \frac{\gamma^\alpha_{n+hN,m+hM}}{\gamma^\alpha_{n,m}}, \quad \forall h \in \mathbb{N},$$

holds, then the following statements hold.

(1) If $\alpha_1 = \alpha_2$, then $(i,j) = (n,m)$ or $(i,j) = (\rho_2(m), \rho_1(n))$ if $\rho_2(m), \rho_1(n) \in \mathbb{N}$.

(2) If $\alpha_1 = -\alpha_2$, then $(i,j) = (n,m)$ or $\frac{i+1}{j+1} = \frac{n+1}{m+1} = \frac{N}{M}$.

(3) If $|\alpha_1| \neq |\alpha_2|$, then $(i,j) = (n,m)$.

Proof By the assumption,

$$\frac{\gamma^\alpha_{i+hN,j+hM}}{\gamma^\alpha_{n+hN,m+hM}} = \frac{\gamma^\alpha_{i,j}}{\gamma^\alpha_{n,m}}, \quad \forall h \in \mathbb{N},$$

Taking $h \to \infty$ in the above equation, it follows that for any $h \in \mathbb{N}$

$$\frac{\gamma^\alpha_{i+hN,j+hM}}{\gamma^\alpha_{n+hN,m+hM}} = 1,$$

By the definition of $\gamma^\alpha_{i,j}$, it converts to

$$(i + hN + 1)^{\alpha_1}(j + hM + 1)^{\alpha_2} = (n + hN + 1)^{\alpha_1}(m + hM + 1)^{\alpha_2}. \tag{4.1}$$

If $\alpha_1 = \alpha_2$, then (4.1) is equivalent to

$$(i + hN + 1)(j + hM + 1) = (n + hN + 1)(m + hM + 1).$$

It is easy to see

$$\frac{i+1}{N} = \frac{n+1}{N}, \frac{j+1}{M} = \frac{m+1}{M}$$

or

$$\frac{i+1}{N} = \frac{m+1}{M}, \frac{j+1}{M} = \frac{n+1}{N},$$

which implies statement (1).

If $\alpha_1 = -\alpha_2$, then (4.1) is equivalent to

$$(i + hN + 1)(m + hM + 1) = (n + hN + 1)(j + hM + 1).$$

It is easy to see

$$\frac{i+1}{N} = \frac{n+1}{N}, \quad \frac{m+1}{M} = \frac{j+1}{M}$$

or

$$\frac{i+1}{N} = \frac{j+1}{M}, \quad \frac{m+1}{M} = \frac{n+1}{N},$$

which implies statement (2).

If $|\alpha_1| \neq |\alpha_2|$. Firstly, we consider the case of $\alpha_1 \cdot \alpha_2 > 0$. Without loss of generality, we assume $\alpha_1 > 0$ and $\alpha_2 > 0$. Observe that the left of (4.1) has zeros $-\frac{i+1}{N}$, $-\frac{j+1}{M}$ with order α_1, α_2, respectively, while the right of (4.1) has zeros $-\frac{n+1}{N}$, $-\frac{m+1}{M}$ with order α_1, α_2, respectively. For $-\frac{i+1}{N} \neq -\frac{j+1}{M}$, note that $\alpha_1 \neq \alpha_2$, and it follows that

$$-\frac{i+1}{N} = -\frac{n+1}{N}, \quad -\frac{j+1}{M} = -\frac{m+1}{M}.$$

Thus statement (3) holds.

For $-\frac{i+1}{N} = -\frac{j+1}{M}$, it follows that

$$-\frac{i+1}{N} = -\frac{n+1}{N} = -\frac{j+1}{M} = -\frac{m+1}{M},$$

which also implies statement (3).

Secondly, as to the case of $\alpha_1 \cdot \alpha_2 < 0$, without loss of generality, we assume $\alpha_1 > 0$ and $\alpha_2 < 0$. Therefore, (4.1) turns into

$$(i + hN + 1)^{\alpha_1}(m + hM + 1)^{-\alpha_2} = (n + hN + 1)^{\alpha_1}(j + hM + 1)^{-\alpha_2}.$$

A similar discussion shows that $i = n$ and $m = j$. This complete the proof. $\square$

Next we describe the projection of monomial on reducing subspace of $T_{z_1^N z_2^M}$.

Theorem 4.2 *Suppose that $\mathfrak{M}$ is a reducing subspace of $T_{z_1^N z_2^M}$ on Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$ and $P_\mathfrak{M}$ is the projection onto $\mathfrak{M}$. Then the following statements hold:*

(1) If $\alpha_1 = \alpha_2$, then

$$P_\mathfrak{M}(z_1^n z_2^m) = a z_1^n z_2^m + b\delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)},$$

where $a = |a|^2 + |b\delta_{n,m}|^2$ and $\delta_{n,m} = \begin{cases} 1 & , \quad if \quad \rho_2(m), \rho_1(n) \in \mathbb{N} \\ 0 & , \quad if \quad others \end{cases}.$

(2) If $\alpha_1 = -\alpha_2$, let $S_{N,M,n,m} = \{(i,j); \frac{i+1}{j+1} = \frac{n+1}{m+1} = \frac{N}{M}, i \in \mathbb{N}, j \in \mathbb{N}\}$, $S'_{N,M,n,m} = S_{N,M,n,m} - (n,m)$, then

$$P_\mathfrak{M}(z_1^n z_2^m) = a z_1^n z_2^m + \sum_{(i,j) \in S'_{N,M,n,m}} a_{i,j} z_1^i z_2^j, \qquad a = |a|^2 + \sum_{(i,j) \in S'_{N,M,n,m}} |a_{i,j}|^2.$$

(3) If $|\alpha_1| \neq |\alpha_2|$, then

$$P_\mathfrak{M}(z_1^k (z_1 z_2)^m) = a z_1^k (z_1 z_2)^m, \qquad a = |a|^2.$$

Proof

Let $z_1^n z_2^m = f(z_1,z_2)+g(z_1,z_2)$ be the orthogonal decomposition on $\mathfrak{M}$, where $f(z_1,z_2)=\sum_{i,j=0}^{\infty} f_{i,j} z_1^i z_2^j \in \mathfrak{M}$ and $g(z_1,z_2)=\sum_{i,j=0}^{\infty} g_{i,j} z_1^i z_2^j \in \mathfrak{M}^{\perp}$.

For nonnegative integer h, we calculate

$$\begin{aligned} T^*_{z_1^{hN} z_2^{hM}} T_{z_1^{hN} z_2^{hM}} P_{\mathfrak{M}}(z_1^n z_2^m) &= T^*_{z_1^{hN} z_2^{hM}} T_{z_1^{hN} z_2^{hM}} P_{\mathfrak{M}}(f+g) \\ &= T^*_{z_1^{hN} z_2^{hM}} T_{z_1^{hN} z_2^{hM}} f \\ &= T^*_{z_1^{hN} z_2^{hM}} \sum_{i,j=0}^{\infty} f_{i,j} z_1^{i+hN} z_2^{j+hM} \\ &= \sum_{i,j=0}^{\infty} f_{i,j} \frac{(\gamma^{\alpha}_{i+hN,j+hM})^2}{(\gamma^{\alpha}_{i,j})^2} z_1^i z_2^j. \end{aligned}$$

On the other hand, direct computation shows that

$$\begin{aligned} P_{\mathfrak{M}} T^*_{z_1^{hN} z_2^{hM}} T_{z_1^{hN} z_2^{hM}} (z_1^n z_2^m) &= P_{\mathfrak{M}} T^*_{z_1^{hN} z_2^{hM}} (z_1^{n+hN} z_2^{m+hM}) \\ &= P_{\mathfrak{M}} \Big(\frac{(\gamma^{\alpha}_{n+hN,m+hM})^2}{(\gamma^{\alpha}_{n,m})^2} z_1^n z_2^m \Big) \\ &= \sum_{i,j=0}^{\infty} f_{i,j} \frac{(\gamma^{\alpha}_{n+hN,m+hM})^2}{(\gamma^{\alpha}_{n,m})^2} z_1^i z_2^j. \end{aligned}$$

Since $T_{z_1^{hN} z_2^{hM}}$ and $T^*_{z_1^{hN} z_2^{hM}}$ commute with $P_{\mathfrak{M}}$, if $f_{i,j} \neq 0$, it follows that

$$\frac{\gamma^{\alpha}_{i+hN,j+hM}}{\gamma^{\alpha}_{i,j}} = \frac{\gamma^{\alpha}_{n+hN,m+hM}}{\gamma^{\alpha}_{n,m}}, \quad \forall h \in \mathbb{N}.$$

If $\alpha_1 = \alpha_2$, Lemma 4.1 indicates that $(i,j)=(n,m)$ or $(i,j)=(\rho_2(m),\rho_1(n))$ if $\rho_2(m),\rho_1(n) \in \mathbb{N}$, which implies for constants a, b

$$P_{\mathfrak{M}}(z_1^n z_2^m) = a z_1^n z_2^m + b\delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)}.$$

Note that

$$\|P_{\mathfrak{M}}(z_1^n z_2^m)\|^2_{\mathcal{D}_\alpha(\mathbb{D}^2)} = \langle P_{\mathfrak{M}}(z_1^n z_2^m), z_1^n z_2^m \rangle \tag{4.2}$$

and

$$\|z_1^n z_2^m\|^2 = \|z_1^{\rho_2(m)} z_2^{\rho_1(n)}\|^2,$$

direct computation shows that $|a|^2 + |b\delta_{n,m}|^2 = a$.

If $\alpha_1 = -\alpha_2$, Lemma 4.1 indicates that $(i,j)=(n,m)$ or $\frac{i+1}{j+1} = \frac{n+1}{m+1} = \frac{N}{M}$. It follows that $P_{\mathfrak{M}}(z_1^n z_2^m) = b z_1^n z_2^m + \sum_{(i,j)\in S_{N,M,n,m}} a_{i,j} z_1^i z_2^j$. Combining like terms, we can write $P_{\mathfrak{M}}(z_1^n z_2^m) = a z_1^n z_2^m + \sum_{(i,j)\in S'_{N,M,n,m}} a_{i,j} z_1^i z_2^j$. By (4.2) and the fact that $\|z_1^i z_2^j\|^2 = \|z_1^n z_2^m\|^2$ whenever $(i,j) \in S'_{N,M,n,m}$, direct computation shows that $a = |a|^2 + \sum_{(i,j)\in S'_{N,M,n,m}} |a_{i,j}|^2$.

If $|\alpha_1| \neq |\alpha_2|$, Lemma 4.1 indicates that $(i,j)=(n,m)$. It follows that $P_{\mathfrak{M}}(z_1^n z_2^m) = a z_1^n z_2^m$. By (4.2), direct computation shows that $a=|a|^2$. □

Note that if $\frac{n+1}{m+1} \neq \frac{N}{M}$, then $S_{N,M,n,m}=\emptyset$. Consequently, $S'_{N,M,n,m}=\emptyset$ for most of $(n,m)\in\mathbb{N}^2$. That is, $P_{\mathfrak{M}}(z_1^n z_2^m)$ always equals $a z_1^n z_2^m$ in statement (2) of Theorem 4.2.

Theorem 4.3 *Suppose $\mathfrak{M}$ is a nontrivial reducing subspace of $T_{z_1^N z_2^M}$ in $\mathcal{D}_\alpha(\mathbb{D}^2)$ and $f=\sum_{i,j=0}^{\infty} f_{i,j} z_1^i z_2^j \in \mathfrak{M}$. Then the following statements hold:*

(1) If $\alpha_1=\alpha_2$, then

$$f_{n,m} z_1^n z_2^m + f_{\rho_2(m),\rho_1(n)} \delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)} \in \mathfrak{M}.$$

(2) If $\alpha_1=-\alpha_2$, then

$$f_{n,m} z_1^n z_2^m + \sum_{(i,j)\in S'_{N,M,n,m}} f_{i,j} z_1^i z_2^j \in \mathfrak{M}.$$

(3) If $|\alpha_1| \neq |\alpha_2|$, then

$$f_{n,m} z_1^n z_2^m \in \mathfrak{M}.$$

Proof For $\alpha_1=\alpha_2$, if $\rho_2(m),\rho_1(n)\in\mathbb{N}$, then $\delta_{n,m}=1$. Note that $\rho_2(\rho_1(n))=n$, $\rho_1(\rho_2(m))=m$; by Theorem 4.2 it follows that

$$P_{\mathfrak{M}} \overline{span}\{z_1^n z_2^m, \delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)}\} \subseteq \overline{span}\{z_1^n z_2^m, \delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)}\}. \tag{4.3}$$

If $\rho_2(m)) \notin \mathbb{N}$ or $\rho_1(n)\notin\mathbb{N}$, then $\delta_{n,m}=0$. It is easy to see that (4.3) holds either. That is, (4.3) holds for any (n,m). Using Lemma 3.3, statement (1) holds.

For $\alpha_1=-\alpha_2$, if $\frac{n+1}{m+1}\neq\frac{N}{M}$, then $S'_{N,M,n,m}=\emptyset$. By Theorem 4.2, it is easy to see

$$P_{\mathfrak{M}} \overline{span}\{z_1^h z_2^k; (h,k)\in (n,m)\bigcup S'_{N,M,n,m}\} \subseteq \overline{span}\{z_1^h z_2^k; (h,k)\in (n,m)\bigcup S'_{N,M,n,m}\}. \tag{4.4}$$

If $\frac{n+1}{m+1}=\frac{N}{M}$, note that $S_{N,M,i,j}=S_{N,M,n,m}$ if $(i,j)\in S_{N,M,n,m}$; then

$$P_{\mathfrak{M}}(z_1^i z_2^j) \in \overline{span}\{z_1^h z_2^k; (h,k)\in (i,j)\bigcup S'_{N,M,i,j}\} = \overline{span}\{z_1^h z_2^k; (h,k)\in S_{N,M,n,m}\}.$$

Thus (4.4) holds either. Therefore, (4.4) holds for any (n,m). Statement (2) follows from Lemma 3.3.

By Theorem 4.2 and Lemma 3.3, a similar discussion comes to statement (3). The proof is complete. □

Theorem 4.4 *Suppose that M is the minimal reducing subspaces of $T_{z_1^N z_2^M}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Then the following statements hold:*

(1) If $\alpha_1=\alpha_2$, then

$$M=\overline{span}\{(a z_1^n z_2^m + b\delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)}) z_1^{hN} z_2^{hM} : h\in\mathbb{N}\}.$$

where $a,b\in\mathbb{C}$ and $m,n\in\mathbb{N}$ such that $0\le n\le N-1$ or $0\le m\le M-1$.

(2) If $\alpha_1 = -\alpha_2$, *then*

$$M = \overline{span}\{az_1^n z_2^m (z_1^{hN} z_2^{hM}) : h \in \mathbb{N}\}, \tag{4.5}$$

where $a \in \mathbb{C}$ *and* $m, n \in \mathbb{N}$ *such that* $0 \le n \le N-1$ *or* $0 \le m \le M-1$, *or*

$$M = \overline{span}\{z_1^{(h+1)N-1} z_2^{(h+1)M-1} : h \in \mathbb{N}\}. \tag{4.6}$$

(3) If $|\alpha_1| \ne |\alpha_2|$, *then*

$$M = \overline{span}\{az_1^n z_2^m (z_1^{hN} z_2^{hM}) : h \in \mathbb{N}\},$$

where $a \in \mathbb{C}$ *and* $m, n \in \mathbb{N}$ *such that* $0 \le n \le N-1$ *or* $0 \le m \le M-1$.

Proof Suppose that $\mathfrak{M}$ is a nontrivial reducing subspace. Then there exists nonzero $f = \sum_{i,j\ge 0} f_{i,j} z_1^i z_2^j \in \mathfrak{M}$. If $\alpha_1 = \alpha_2$, then by Theorem 4.3

$$g_{k,l} \triangleq az_1^k z_2^l + b\delta_{k,l} z_1^{\rho_2(l)} z_2^{\rho_1(k)} \in \mathfrak{M} \text{ for any } (k,l),$$

where $a = f_{k,l}$ and $b = f_{\rho_2(l),\rho_1(k)}$. Note that there exists $h_0 \in \mathbb{N}$ such that $(T^*_{z_1^N z_2^M})^{h_0} g_{k,l} \ne 0$ and $(T^*_{z_1^N z_2^M})^{h_0+1} g_{k,l} = 0$. Let $n = k - h_0 N$, $m = l - h_0 M$; then $0 \le n \le N-1$ or $0 \le m \le M-1$. Since $\mathfrak{M}$ is the reducing subspace of $T_{z_1^N z_2^M}$ and $\delta_{k,l} = \delta_{n,m}$, then $az_1^n z_2^m + b\delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)} \in \mathfrak{M}$. Thus statement (1) holds.

If $|\alpha_1| \ne |\alpha_2|$, then by Theorem 4.3

$$r_{k,l} \triangleq az_1^k z_2^l \in \mathfrak{M} \text{ for any } (k,l),$$

where $a = f_{k,l}$. Note that there exists $h_0 \in \mathbb{N}$ such that $(T^*_{z_1^N z_2^M})^{h_0} r_{k,l} \ne 0$ and $(T^*_{z_1^N z_2^M})^{h_0+1} r_{k,l} = 0$. Let $n = k - h_0 N$, $m = l - h_0 M$; then $0 \le n \le N-1$ or $0 \le m \le M-1$. Since $\mathfrak{M}$ is the reducing subspace of $T_{z_1^N z_2^M}$, then $az_1^n z_2^m \in \mathfrak{M}$. Thus statement (3) holds.

If $\alpha_1 = -\alpha_2$, then by Theorem 4.3

$$q_{k,l} \triangleq az_1^k z_2^l + \sum_{(i,j)\in S'_{N,M,k,l}} b_{i,j} z_1^i z_2^j \in \mathfrak{M} \text{ for any } (k,l),$$

where $a = f_{k,l}$ and $b_{i,j} = f_{i,j}$.

If $\frac{k+1}{l+1} \ne \frac{N}{M}$, then $S'_{N,M,k,l} = \emptyset$. Therefore, $q_{k,l} = az_1^k z_2^l \in \mathfrak{M}$. A similar discussion as the case of $|\alpha_1| \ne |\alpha_2|$ shows $\mathfrak{M}$ has a reducing subspace as (4.5).

If $\frac{k+1}{l+1} = \frac{N}{M}$, then $(k,l) \in S_{N,M,k,l}$. Therefore

$$q_{k,l} = \sum_{(i,j)\in S_{N,M,k,l}} b_{i,j} z_1^i z_2^j \in \mathfrak{M}.$$

Since $(i,j) \in S_{N,M,k,l}$, then $i = (h+1)N - 1$ and $j = (h+1)M - 1$ for some $h \in \mathbb{N}$. Let $h_0 = min\{h \in \mathbb{N} : b_{i,j} \ne 0\}$ and write

$$q_{k,l} = b_{i_0,j_0} z_1^{i_0} z_2^{j_0} + q'_{k,l}$$

where $(i_0, j_0) = ((h_0+1)N-1, (h_0+1)M-1)$, and $q'_{k,l} = q_{k,l} - b_{i_0,j_0} z_1^{i_0} z_2^{j_0}$. Note that $T_{z_1^{h_0N} z_2^{h_0M}} T^*_{z_1^{(h_0+1)N} z_2^{(h_0+1)M}} q_{k,l} = q'_{k,l} \in \mathfrak{M}$, and it follows that

$$z_1^{i_0} z_2^{j_0} \in \mathfrak{M}.$$

Since $\mathfrak{M}$ is the reducing subspace of $T_{z_1^N z_2^M}$, then $z_1^{N-1} z_2^{M-1} \in \mathfrak{M}$.

Consequently, the reducing subspace developed by $z_1^{N-1} z_2^{M-1}$ has the form of (4.6). Thus statement (2) holds.

From the above proof, we deduce that each reducing subspace $\mathfrak{M}$ contains a reducing subspace M, which means that M consists of all the minimal reducing subspaces.

Since each M is a reducing subspace of $T_{z_1^N z_2^M}$, if $\mathfrak{M}$ is minimal then $\mathfrak{M} = M$.

The proof is complete. □

Theorem 2.4 in [5] by Shi and Lu showed that on the Bergman space over the bidisk the minimal reducing subspaces of $T_{z_1^N z_2^M}$ has the form

$$M_{n,m,a,b} = \overline{span}\{a z_1^{n+hN} z_2^{m+hM} + b\delta_{n,m} z_1^{\rho_2(m+hM)} z_2^{\rho_1(n+hN)}\}.$$

Note that since $(a z_1^n z_2^m + b\delta_{n,m} z_1^{\rho_2(m)} z_2^{\rho_1(n)}) z_1^{hN} z_2^{hM} = a z_1^{n+hN} z_2^{m+hM} + b\delta_{n,m} z_1^{\rho_2(m+hM)} z_2^{\rho_1(n+hN)}$ and $\mathcal{D}_{(-1,-1)}(\mathbb{D}^2)$ is the Bergman space of the bidisk, Theorem 4.4 extends the result of Shi and Lu.

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