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Exponential stability of periodic solutions of recurrent neural networks with functional dependence on piecewise constant argument

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Abstract: In this study, we develop a model of recurrent neural networks with functional dependence on piecewise constant argument of generalized type. Using the theoretical results obtained for functional differential equations with piecewise constant argument, we investigate conditions for existence and uniqueness of solutions, bounded solutions, and exponential stability of periodic solutions. We provide conditions based on the parameters of the model.

Key words: Recurrent neural networks, functional differential equations, piecewise constant argument, periodic solution, stability

1. Introduction and preliminaries

Differential equations with piecewise constant argument have been developed and studied by many authors during the last few decades [1, 6, 13, 16–19, 22–24]. In the literature, most of the results have been obtained by reducing these equations to discrete equations. Later, Akhmet [2–4] generalized this class of differential equations by taking any piecewise constant functions as arguments, and he recently introduced functional dependence on piecewise constant argument in [5]. Differential equations with piecewise constant argument of generalized type have been considered widely in the book [4], which develops new methods of investigation. These methods are more effective since they do not depend on the reduction to discrete equations and they enable one to consider systems that are nonlinear with respect to values of solutions at the discrete moments of time.

Differential equations with piecewise constant argument have widespread applications. One of these application areas is neural networks [7–12, 25]. Neural networks are systems comprising numerous processing units that correspond to the neurons in the brain. These units together with input, activation functions of neurons, and the connection weight produce an output. Neural networks are very important in many areas such as finance, economy, medicine, and electronics (see, for example, [14, 15, 20] and the references cited therein). Therefore, it is worthwhile to study and develop neural networks using the theory of differential equations with piecewise constant argument of generalized type.

In the literature, investigations of periodic solutions of neural networks are used practically in the learning theory. This theory indicates that certain activities and motions can be learned by repetition. In biological

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neural networks, learning takes place as follows: neural cells are interconnected with links, which have specific numerical weights. By creating new connections or adjusting repeatedly these weights representing memory, learning of the networks is provided. Mathematically, by weighting values of the input and producing output, networks learn. Later, the learning process is completed, i.e. when the information is loaded into neurons, it can be retrieved from the neurons. In this context, it is important that a neural network has a stable solution since it expresses that samples stored on learning outcomes can be called back. In the past few years, sufficient conditions have been obtained for the stability of solutions of delayed neural networks [7, 8, 25]. Thus, it is desirable to design neural networks that have bounded solutions, periodic solutions, and, in fact, exponentially stable periodic solutions.

In this paper, it is aimed to consider a model of recurrent neural networks with functional dependence on piecewise constant argument of generalized type. Existence and uniqueness of solutions, bounded solutions, and exponential stability of periodic solutions will be addressed for the proposed model.

Denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{N}$ the sets of all integers, real numbers, and natural numbers, respectively. Let $||\cdot||$ represent the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real valued sequences $\theta = \{\theta_i\}$, $\zeta = \{\zeta_i\}$, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$ with $|\theta_i| \to \infty$ as $|i| \to \infty$ and $\theta_i \leq \zeta_i \leq \theta_{i+1}$. For fixed numbers $\tau \in \mathbb{R}$ and $n \in \mathbb{N}$, let $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n)$ describe the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the uniform norm $\|\phi\|_0 = \max_{[-\tau,0]} \|\phi\|$. Consider a subset $\mathcal{D} \subset \mathbb{R} \times \mathcal{C}$ and continuous functionals $h, g : \mathcal{D} \to \mathbb{R}^n$. Let $\mathcal{C}_s = \{\phi \in \mathcal{C} | \, \|\phi\|_0 \leq s\}$ where $0 < s \in \mathbb{R}$. Moreover, let $C_0(W)$ denote the set of all bounded and continuous functions on W.

We shall consider the following recurrent neural networks with functional dependence on piecewise constant argument of generalized type:

$$x'(t) = -Ax(t) + Bx(\gamma(t)) + Ch(t, x_t) + Dg\left(t, x_{\gamma(t)}\right) + E, \tag{1}$$

where $x \in \mathbb{R}^n$ is the neuron state vector, $t \in \mathbb{R}$, and $\gamma(t) = \zeta_i$ if $\theta_i \leq t < \theta_{i+1}$. In the model, h and g stand for activation functions of neurons, and E is a constant external input vector. Besides, $A = \text{diag}(a_1, ..., a_n)$ and B is a matrix with positive entries, while C and D denote the connection weight and delayed connection weight matrices, respectively. In equation (1), x_t and $x_{\gamma(t)}$ mean $x_t(s) = x(t+s)$ and $x_{\gamma(t)} = x(\gamma(t) + s)$ for $s \in [-\tau, 0]$.

Throughout this paper, the following assumptions will be needed:

(N1) $h, g \in C_0(\mathbb{R} \times \mathcal{C}_\Omega)$ for each positive $\Omega \in \mathbb{R}$;

(N2) there exist positive Lipschitz constants L^h and L^g such that

$$\|h(t, \phi_1) - h(t, \phi_2)\| \leq L^h \|\phi_1 - \phi_2\|_0$$

and

$$\|g(t, \phi_1) - g(t, \phi_2)\| \leq L^g \|\phi_1 - \phi_2\|_0$$

for all $(t, \phi_1), (t, \phi_2) \in \mathcal{D}$;

(N3) there exist positive numbers $\overline{\theta}$ and $\overline{\zeta}$ such that $\theta_{i+1} - \theta_i \leq \overline{\theta}$ and $\zeta_{i+1} - \zeta_i \leq \overline{\zeta}$, $i \in \mathbb{Z}$.

For convenience, we adopt the following notation:

$$L = \max\left(L^h \left|C^T\right| + L^g \left|D^T\right|\right).$$

2. Existence and uniqueness of solutions

Let I denote the $n \times n$ identity matrix. Denote by $X(t,s),\ X(s,s) = \mathrm{I},\ t,s \in \mathbb{R}$, the state transition matrix of the system

$$x'(t) = -Ax(t). \tag{2}$$

It is clear that $X(t,s) = \mathrm{e}^{-A(t-s)}$. Besides, the matrix function $M_i(t)$ for system (1) is as follows [5]:

$$M_i(t) = \mathrm{e}^{-A(t-\zeta_i)} + \int_{\zeta_i}^{t} \mathrm{e}^{-A(t-s)} B ds, i \in \mathbb{Z}.$$

For a fixed $t_0 \in \mathbb{R}$, there exists a fundamental matrix $Z(t) = Z(t,t_0),\ Z(t_0) = \mathrm{I}$ of solutions of

$$x'(t) = -Ax(t) + Bx(\gamma(t))$$

such that

$$\frac{dZ}{dt} = -AZ(t) + BZ(\gamma(t)).$$

Let $\theta_i \le t_0 < \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. If $t \in [t_0, \theta_{i+1}]$, then

$$Z(t,t_0) = M_i(t) M_i^{-1}(t_0).$$

If $t \in [\theta_l, \theta_{l+1}]$ for arbitrary $l > i$,

$$Z(t) = M_l(t) \left[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k) M_{k-1}(\theta_k) \right] M_i^{-1}(t_0).$$

If $t \in [\theta_j, \theta_{j+1}],\ j < i$,

$$Z(t) = M_j(t) \left[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1}) M_{k+1}(\theta_{k+1}) \right] M_i^{-1}(t_0).$$

We suppose that the following assumptions are valid.

(N4) For every fixed $i \in \mathbb{Z}$, $\det[M_i(t)] \neq 0,\ \forall t \in [\theta_i, \theta_{i+1}]$.

(N3)–(N4) imply the existence of positive constants m, M, and $\overline{M}$ such that $m \le \|Z(t,s)\| \le M,\ \|X(t,s)\| \le \overline{M}$ for $t,\ s \in [\theta_i, \theta_{i+1}],\ i \in \mathbb{Z}$. Taking the fundamental matrix $Z(t),\ t \in \mathbb{R}$, for initial data $x(t_0) = x_0$, a solution $x(t)$ of equation

$$x'(t) = -Ax(t) + Bx(\gamma(t))$$

is expressed with the equality $x(t) = Z(t,t_0)x_0,\ (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$.

(N5) $\overline{M} L (1+M) \overline{\theta} < 1$.

(N6) $\|Z(t,s)\| \leq K\mathrm{e}^{-\alpha(t-s)}$, $s \leq t$, where K and α are positive numbers;

(N7) there exist positive numbers $\underline{\theta}$, $\underline{\zeta} > 0$ such that $\theta_{i+1} - \theta_i \geq \underline{\theta}$, $\zeta_{i+1} - \zeta_i \geq \underline{\zeta}$, $i \in \mathbb{Z}$;

(N8) $\overline{\theta} L \overline{M} \left(1 + 2\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right) < 1$;

(N9) $\frac{2}{\alpha}\overline{M}\mathrm{e}^{\alpha\tau} n_1 \left(K + \left(1 + \frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\mathrm{e}^{\alpha(j-i+1)\overline{\theta}}\right) < 1$, where $\theta_i \leq t_0 \leq \theta_{i+1}$, $\theta_j \leq t \leq \theta_{j+1}$, $i < j$ and $n_1 = \max\left(L^h \left|C^T\right| + L^g \left|D^T\right| \mathrm{e}^{\alpha\overline{\theta}}\right)$;

(N10) $\overline{M} L \left(1 + K\frac{1+2\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\overline{\theta} < 1$.

Consider two functions ϕ, $\psi \in \mathcal{C}$. If $\theta_i \leq t_0 < \theta_{i+1}$ for some $i \in \mathbb{Z}$, there exist two cases for the initial condition, depending on whether $t_0 \leq \zeta_i$ or $\zeta_i < t_0$:

(IC_1) If $\theta_i \leq t_0 \leq \zeta_i < \theta_{i+1}$, then a solution $x(t) = x(t, t_0, \phi)$, $t \geq t_0$, of equation (1) satisfies the initial condition $x_{t_0}(s) = \phi(s)$, $s \in [-\tau, 0]$;

(IC_2) If $\theta_i \leq \zeta_i < t_0 < \theta_{i+1}$, then a solution $x(t) = x(t, t_0, \phi, \psi)$, $t \geq t_0$, of equation (1) satisfies the initial condition $x_{t_0}(s) = \phi(s)$ and $x_{\gamma(t_0)}(s) = \psi(s)$, $s \in [-\tau, 0]$.

Definition 2.1 *A function $x(t)$ is a solution of (1) with (IC_1) or (IC_2) on an interval $[t_0, t_0 + a)$, $a > 0$, if:*

(i) it satisfies the initial condition;

(ii) $x(t)$ is continuous on $[t_0, t_0 + a)$;

(iii) the derivative $x'(t)$ exists for $t \geq t_0$ with the possible exception of the points θ_i, where one-sided derivatives exist;

(iv) equation (1) is satisfied by $x(t)$ for all $t > t_0$, except, possibly, the points of θ and it holds for the right derivative of $x(t)$ at points θ_i.

The following lemma gives necessary conditions for existence and uniqueness of solutions.

Lemma 2.2 *Assume that conditions (N1)–(N5) hold. Then for fixed $i \in \mathbb{Z}$ and for every $(t_0, \phi, \psi) \in [\theta_i, \theta_{i+1}] \times \mathcal{C} \times \mathcal{C}$ there exists a unique solution $x(t) = x(t, t_0, \phi, \psi)$ of (1) on $[t_0, \theta_{i+1}]$.*

Proof We will consider only the initial condition given by (IC_1) and thus a solution of the form $x(t) = (x_1(t), ..., x_n(t))^T = x(t, t_0, \phi)$. Proof for ($IC_2$) coincides with that for functional differential equations [21]. We fix $i \in \mathbb{Z}$ and assume that $\theta_i \leq t_0 < \theta_{i+1}$. *Existence.* Take $x^0(t) = Z(t, t_0)\phi(t_0)$ and define a sequence $\{x^k(t)\}$, $k \geq 0$ by

$$x^{k+1}(t) = \phi(t - t_0), t \in [t_0 - \tau, t_0],$$

$$x^{k+1}(t) = Z(t,t_0)\left[\phi(t_0)+\int_{t_0}^{\zeta_i} \mathrm{e}^{-A(\zeta_i-s)}\left(Ch\left(s,x_s^k\right)+Dg\left(s,x_{\zeta_i}^k\right)+E\right)ds\right]$$
$$+\int_{\zeta_i}^{t} \mathrm{e}^{-A(t-s)}\left(Ch\left(s,x_s^k\right)+Dg\left(s,x_{\zeta_i}^k\right)+E\right)ds, t\in[t_0,\theta_{i+1}].$$

Since $h,\ g\in C_0(\mathbb{R}\times\mathcal{C}_\Omega)$, we can find an $M_\Omega\in(0,\infty)$ such that $\|h(t,x_t)\|\le M_\Omega$ and $\left\|g\left(t,x_{\gamma(t)}\right)\right\|\le M_\Omega$. Thus, we have that

$$\max_{[t_0,\theta_{i+1}]}\left\|x^{k+1}(t)-x^k(t)\right\|\le\left[\overline{M}L(1+M)\overline{\theta}\right]^k\mu,$$

where $\mu=\overline{M}(1+M)\overline{\theta}\left[M_\Omega\left(\left|C^T\right|+\left|D^T\right|\right)+|E|\right]$. Thus, by $(N5)$, there exists a unique solution $x(t)=x(t,t_0,\phi)$ of the equation

$$x(t) = Z(t,t_0)\left[\phi(t_0)+\int_{t_0}^{\zeta_i} \mathrm{e}^{-A(\zeta_i-s)}\left(Ch(s,x_s)+Dg(s,x_{\zeta_i})+E\right)ds\right]$$
$$+\int_{\zeta_i}^{t} \mathrm{e}^{-A(t-s)}\left(Ch(s,x_s)+Dg(s,x_{\zeta_i})+E\right)ds, t\in[t_0,\theta_{i+1}].$$

It is clear that $x(t)=x(t,t_0,\phi)$ is also a solution of (1), proving the existence.

Uniqueness. Denote by $x^j(t)=x^j(t,t_0,\phi)$, $j=1,2$, the solutions of (1). Let $\|\phi\|_\infty=\sup_{\mathbb{R}}\|\phi(t)\|$. We find

$$\begin{aligned}\left\|x^1(t)-x^2(t)\right\| &\le \|Z(t,t_0)\|\int_{t_0}^{\zeta_i}\left\|\mathrm{e}^{-A(\zeta_i-s)}\right\|\left|C^T\right|\left\|h\left(s,x_s^1\right)-h\left(s,x_s^2\right)\right\|ds\\ &+\|Z(t,t_0)\|\int_{t_0}^{\zeta_i}\left\|\mathrm{e}^{-A(\zeta_i-s)}\right\|\left|D^T\right|\left\|g\left(s,x_{\zeta_i}^1\right)-g\left(s,x_{\zeta_i}^2\right)\right\|ds\\ &+\int_{\zeta_i}^{t}\left\|\mathrm{e}^{-A(t-s)}\right\|\left|C^T\right|\left\|h\left(s,x_s^1\right)-h\left(s,x_s^2\right)\right\|ds\\ &+\int_{\zeta_i}^{t}\left\|\mathrm{e}^{-A(t-s)}\right\|\left|D^T\right|\left\|g\left(s,x_{\zeta_i}^1\right)-g\left(s,x_{\zeta_i}^2\right)\right\|ds\\ &\le M\int_{t_0}^{\zeta_i}\overline{M}\left[\left|C^T\right|L^h\left\|x_s^1-x_s^2\right\|_0+\left|D^T\right|L^g\left\|x_{\zeta_i}^1-x_{\zeta_i}^2\right\|_0\right]ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{\zeta_i}^{t}\overline{M}\left[\left|C^T\right|L^h\left\|x_s^1-x_s^2\right\|_0+\left|D^T\right|L^g\left\|x_{\zeta_i}^1-x_{\zeta_i}^2\right\|_0\right]ds\\
=\;& M\int_{t_0}^{\zeta_i}\overline{M}\left|C^T\right|L^h\max_{\sigma\in[-\tau,0]}\left\|x_s^1(\sigma)-x_s^2(\sigma)\right\|ds\\
&+M\int_{t_0}^{\zeta_i}\overline{M}\left|D^T\right|L^g\max_{\sigma\in[-\tau,0]}\left\|x_{\zeta_i}^1(\sigma)-x_{\zeta_i}^2(\sigma)\right\|ds\\
&+\int_{\zeta_i}^{t}\overline{M}\left|C^T\right|L^h\max_{\sigma\in[-\tau,0]}\left\|x_s^1(\sigma)-x_s^2(\sigma)\right\|ds\\
&+\int_{\zeta_i}^{t}\overline{M}\left|D^T\right|L^g\max_{\sigma\in[-\tau,0]}\left\|x_{\zeta_i}^1(\sigma)-x_{\zeta_i}^2(\sigma)\right\|ds\\
\leq\;& M\int_{t_0}^{\zeta_i}\overline{M}\Big(\left|C^T\right|L^h+\left|D^T\right|L^g\Big)\sup_{\mathbb{R}}\left\|x^1(t)-x^2(t)\right\|ds\\
&+\int_{\zeta_i}^{t}\overline{M}\Big(\left|C^T\right|L^h+\left|D^T\right|L^g\Big)\sup_{\mathbb{R}}\left\|x^1(t)-x^2(t)\right\|ds\\
\leq\;& M\int_{t_0}^{\zeta_i}\overline{M}L\left\|x^1-x^2\right\|_\infty ds+\int_{\zeta_i}^{t}\overline{M}L\left\|x^1-x^2\right\|_\infty ds\\
\leq\;& M\int_{t_0}^{\zeta_i}\overline{M}L\left\|x^1-x^2\right\|ds+\int_{\zeta_i}^{t}\overline{M}L\left\|x^1-x^2\right\|ds\\
\leq\;& \overline{M}L(1+M)\,\overline{\theta}\left\|x^1-x^2\right\|\\
\leq\;& \overline{M}L(1+M)\,\overline{\theta}\max_{[t_0,\theta_{i+1}]}\left\|x^1-x^2\right\|.
\end{aligned}
$$

Condition (N5) implies that $x^1(t)=x^2(t)$. □

The following lemma is an important auxiliary result that will be used in the sequel.

Lemma 2.3 *Assume that conditions (N1)–(N5) hold and fix $i\in\mathbb{Z}$. Then for every $(t_0,\phi,\psi)\in[\theta_i,\theta_{i+1}]\times\mathcal{C}\times\mathcal{C}$ there exists a unique solution $x(t)=x(t,t_0,\phi,\psi)$, $t\geq t_0$, of (1) and it satisfies the integral equation*

$$
x(t) \;=\; Z(t,t_0)\left[\phi(t_0)+\int_{t_0}^{\zeta_i}\mathrm{e}^{-A(t_0-s)}\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right)ds\right]
$$

$$+\sum_{k=i}^{j-1} Z\left(t,\theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Ch\left(s,x_s\right)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds \tag{3}$$

$$+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} \left(Ch\left(s,x_s\right)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds,$$

where $\theta_i \leq t_0 \leq \theta_{i+1}$, $\theta_j \leq t \leq \theta_{j+1}$, $i<j$.

3. Bounded solutions

Definition 3.1 *A function* $x\left(t\right)$ *is a solution of (1) on* $\mathbb{R}$ *if:*

(i) $x\left(t\right)$ *is continuous;*

(ii) the derivative $x'\left(t\right)$ *exists for all* $t \in \mathbb{R}$ *with the possible exception of the points* θ_i, $i \in \mathbb{Z}$, *where one-sided derivatives exists;*

(iii) equation (1) is satisfied by $x\left(t\right)$ *for all* $t \in \mathbb{R}$ *except points of* θ *and it holds for the right derivative of* $x\left(t\right)$ *at points* θ_i.

Lemma 3.2 *Assume that (N1)–(N7) are fulfilled. Then a bounded on* $\mathbb{R}$ *function* $x\left(t\right)$ *is a solution of (1) if and only if it satisfies the following integral equation:*

$$\begin{aligned} x\left(t\right) &= \int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} \left(Ch\left(s,x_s\right)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds \\ &+\sum_{k=-\infty}^{j-1} Z\left(t,\theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Ch\left(s,x_s\right)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds, \end{aligned} \tag{4}$$

where $\theta_j \leq t \leq \theta_{j+1}$.

Proof *Necessity.* It can be proved by using equation (3) and assumption (N6) in a similar manner applied to ordinary differential equations.

Sufficiency. Since the solution is bounded, there exists a positive constant Ω such that $\left\|x\left(t\right)\right\| \leq \Omega$. We have $g,\ h \in C_0\left(\mathbb{R}\times\mathcal{C}_\Omega\right)$. Thus, there is a positive number M_Ω such that $\sup\limits_{\mathbb{R}} \left\|h\left(s,x_s\right)\right\| \leq M_\Omega < \infty$ and $\sup\limits_{\mathbb{R}} \left\|g\left(s,x_{\gamma(s)}\right)\right\| \leq M_\Omega < \infty$. Then:

$$\begin{aligned} \left\|x\left(t\right)\right\| &\leq \int_{\zeta_j}^{t} \left\|\mathrm{e}^{-A(t-s)}\right\| \left(\left|C^T\right| \left\|h\left(s,x_s\right)\right\| + \left|D^T\right| \left\|g\left(s,x_{\gamma(s)}\right)\right\| + \left|E\right|\right) ds \\ &+\sum_{k=-\infty}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \left\|\mathrm{e}^{-A(\theta_{k+1}-s)}\right\| \left|C^T\right| \left\|h\left(s,x_s\right)\right\| ds \end{aligned}$$

$$+\sum_{k=-\infty}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \left\|\mathrm{e}^{-A(\theta_{k+1}-s)}\right\| \left(\left|D^T\right| \left\|g\left(s, x_{\gamma(s)}\right)\right\| + |E| \right) ds$$

$$\leq \int_{\zeta_j}^{t} \overline{M} \left(\left|C^T\right| M_\Omega + \left|D^T\right| M_\Omega + |E| \right) ds$$

$$+\sum_{k=-\infty}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left(\left|C^T\right| M_\Omega + \left|D^T\right| M_\Omega + |E| \right) ds$$

$$\leq \overline{M} \left(M_\Omega \left(\left|C^T\right| + \left|D^T\right| \right) + |E| \right) \left(\overline{\theta} + \overline{\zeta} \frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}} \right)$$

$$\leq \overline{M} \left(M_\Omega \left(\left|C^T\right| + \left|D^T\right| \right) + |E| \right) \overline{\theta} \left(1 + 2\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}} \right).$$

Thus, the series and the integral in (4) converge. If we differentiate (4), then it is seen that it satisfies equation (1):

$$x'(t) = \int_{\zeta_j}^{t} A\mathrm{e}^{-A(t-s)} \left(Ch\left(s, x_s\right) + Dg\left(s, x_{\gamma(s)}\right) + E \right) ds$$

$$+\sum_{k=-\infty}^{j-1} AZ\left(t, \theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Ch\left(s, x_s\right) + Dg\left(s, x_{\gamma(s)}\right) + E \right) ds$$

$$+\sum_{k=-\infty}^{j-1} BZ\left(\zeta_j, \theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Ch\left(s, x_s\right) + Dg\left(s, x_{\gamma(s)}\right) + E \right) ds$$

$$= Ax(t) + Bx(\gamma(t)) + Ch\left(t, x_t\right) + Dg\left(t, x_{\gamma(t)}\right) + E.$$

□

Theorem 3.3 *Suppose that conditions (N1)–(N8) are fulfilled. Then (1) admits a unique bounded on* $\mathbb{R}$ *solution.*

Proof Let us consider the complete metric space $C_0(\mathbb{R})$ with the sup-norm $\|\phi\|_\infty = \sup_{\mathbb{R}} \|\phi(t)\|$ and define on $C_0(\mathbb{R})$ the operator $\prod$ such that

$$\prod H(t) \equiv \int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} \left(Ch\left(s, H_s\right) + Dg\left(s, H_{\gamma(s)}\right) + E \right) ds$$

$$+\sum_{k=-\infty}^{j-1} Z\left(t, \theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Ch\left(s, H_s\right) + Dg\left(s, H_{\gamma(s)}\right) + E \right) ds,$$

where $t \in [\theta_j, \theta_{j+1}]$. It can be shown that $\prod : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ and verified that this operator is contractive. Let us take into account the assumption (N6). If $u,\ v \in C_0(\mathbb{R})$, then

$$\begin{aligned}
\left\| \prod u(t) - \prod v(t) \right\| &\leq \int_{\zeta_j}^{t} \overline{M} \left| C^T \right| \| h(s,u_s) - h(s,v_s) \| \, ds \\
&+ \int_{\zeta_j}^{t} \overline{M} \left| D^T \right| \left\| g\left(s, u_{\gamma(s)}\right) - g\left(s, v_{\gamma(s)}\right) \right\| ds \\
&+ \sum_{k=-\infty}^{j-1} \| Z(t, \theta_{k+1}) \| \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| C^T \right| \| h(s,u_s) - h(s,v_s) \| \, ds \\
&+ \sum_{k=-\infty}^{j-1} \| Z(t, \theta_{k+1}) \| \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| D^T \right| \left\| g\left(s, u_{\gamma(s)}\right) - g\left(s, v_{\gamma(s)}\right) \right\| ds \\
&\leq \int_{\zeta_j}^{t} \overline{M} \left(\left| C^T \right| L^h \| u_s - v_s \|_0 + \left| D^T \right| L^g \left\| u_{\gamma(s)} - v_{\gamma(s)} \right\|_0 \right) ds \\
&+ \sum_{k=-\infty}^{j-1} K \mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| C^T \right| L^h \| u_s - v_s \|_0 \, ds \\
&+ \sum_{k=-\infty}^{j-1} K \mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| D^T \right| L^g \left\| u_{\gamma(s)} - v_{\gamma(s)} \right\|_0 ds \\
&\leq \int_{\zeta_j}^{t} \overline{M} \left(\left| C^T \right| L^h + \left| D^T \right| L^g \right) \sup_{\mathbb{R}} \| u - v \| \, ds \\
&+ \sum_{k=-\infty}^{j-1} K \mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| C^T \right| L^h \sup_{\mathbb{R}} \| u - v \| \, ds \\
&+ \sum_{k=-\infty}^{j-1} K \mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left| D^T \right| L^g \sup_{\mathbb{R}} \| u - v \| \, ds \\
&\leq \left(\int_{\zeta_j}^{t} \overline{M} L ds + \sum_{k=-\infty}^{j-1} K \mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} L \right) \| u - v \|_\infty \, ds \\
&\leq \overline{\theta} L \overline{M} \| u - v \|_\infty + \overline{\zeta} K \frac{\mathrm{e}^{\alpha \overline{\theta}}}{1 - \mathrm{e}^{-\alpha \underline{\theta}}} L \overline{M} \| u - v \|_\infty
\end{aligned}$$

$$\leq \overline{\theta}L\overline{M}\left\|u-v\right\|_{\infty}+2\overline{\theta}K\frac{\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}L\overline{M}\left\|u-v\right\|_{\infty}.$$

$$\leq \overline{\theta}L\overline{M}\left(1+2\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\left\|u-v\right\|_{\infty}$$

The assumption given by (N8) implies that the operator $\prod$ is contractive. Hence, (1) has a unique solution $u(t)$, which belongs to the set $C_0(\mathbb{R})$. □

4. Exponential stability of periodic solutions

In the sequel, we assume that the neural networks system (1) is ω-periodic and additionally the following periodicity property:

(N11) there exist numbers $\omega\in\mathbb{R},\ p\in\mathbb{Z}$ such that $\theta_{k+p}=\theta_k+\omega,\ \zeta_{k+p}=\zeta_k+\omega,\ k\in\mathbb{Z}$.

(N12) $h(t+\omega,\phi)=h(t,\phi)$ and $g(t+\omega,\psi)=g(t,\psi),\ t\in\mathbb{R},\ \omega\in\mathbb{R}$.

In what follows, we assume without loss of generality that $\zeta_0=0$, and consider $t_0=\zeta_0$. In the interval $t\in[\theta_j,\theta_{j+1}]$, consider the solution $x(t)=x(t,0,x_0)$ of (1) in the form

$$\begin{aligned}x(t) &= Z(t)x_0+\sum_{k=0}^{j-1}Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right)ds\\ &\quad+\int_{\zeta_j}^{t}\mathrm{e}^{-A(t-s)}\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right)ds,\end{aligned}\tag{5}$$

where $Z(t)=Z(t,0),\ t\in\mathbb{R}$. $x(t)$ is a periodic solution if and only if x_0 satisfies

$$\left[I-Z(w)\right]x_0=\sum_{k=0}^{p-1}Z(w,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right)ds.\tag{6}$$

Let $\det\left[I-Z(w)\right]\neq 0$. Thus, for (6), it is true that

$$x_0=\left[I-Z(w)\right]^{-1}\sum_{k=0}^{p-1}Z(w,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right)ds.\tag{7}$$

If we write the value of x_0 defined by (7) in equation (5), we get

$$
\begin{aligned}
x(t) &= \sum_{k=0}^{j-1} Z(t)\left[I-Z(w)\right]^{-1} Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Ch(s,x_s)\,ds \\
&+ \sum_{k=0}^{j-1} Z(t)\left[I-Z(w)\right]^{-1} Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Dg\left(s,x_{\gamma(s)}\right)+E\right) ds \\
&+ \sum_{k=j}^{p-1} Z(t)\left[I-Z(w)\right]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Ch(s,x_s)\,ds \\
&+ \sum_{k=j}^{p-1} Z(t)\left[I-Z(w)\right]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} \left(Dg\left(s,x_{\gamma(s)}\right)+E\right) ds \\
&+ \int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} \left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds.
\end{aligned}
$$

We are ready to construct the Green function $G_p(t,s)$, $t,s\in[0,\omega]$ for our model. If $t\in[\theta_j,\theta_{j+1}]$, $j=0,1,...,p-1$,

$$
G_p(t,s)=\begin{cases}
Z(t)[I-Z(\omega)]^{-1}Z^{-1}(\theta_{k+1})\mathrm{e}^{-A(\theta_{k+1}-s)}, s\in[\zeta_k,\zeta_{k+1}), k<j, \\
Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})\mathrm{e}^{-A(\theta_{k+1}-s)}, s\in[\zeta_k,\zeta_{k+1})\setminus[\zeta_j\hat{},t], k\geq j, \\
Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})\mathrm{e}^{-A(\theta_{k+1}-s)}+\mathrm{e}^{-A(t-s)}, s\in[\zeta_j\hat{},t].
\end{cases}
$$

Using the Green function, we can express the periodic solution by the integral equation

$$
x(t)=\int_0^{\omega} G_p(t,s)\left(Ch(s,x_s)+Dg\left(s,x_{\gamma(s)}\right)+E\right) ds. \tag{8}
$$

Denote by $\widetilde{H}=\max\limits_{t,s\in[0,\omega]}\|G_p(t,s)\|<\infty$. Then we can state the following theorems, which give necessary conditions for existence, uniqueness, and exponential stability of the periodic solution (8).

Theorem 4.1 *Let conditions (N1)–(N5), (N11), and (N12) and the inequality $\widetilde{H}L\omega<1$ be satisfied and the matrix $[I-Z(w)]$ be nonsingular. Then (1) admits a unique ω-periodic solution.*

Proof Let the complete metric space $C_\omega(\mathbb{R})$ denote the set of all continuous and ω-periodic functions on $\mathbb{R}$. Define on $C_\omega(\mathbb{R})$ an operator such that

$$
\prod \widetilde{S}(t)=\int_0^{\omega} G_p(t,s)\left(Ch\left(s,\widetilde{S}_s\right)+Dg\left(s,\widetilde{S}_{\gamma(s)}\right)+E\right) ds, \tag{9}
$$

where $t \in [\theta_j, \theta_{j+1}), j = 0, 1, 2, ..., p-1$ and

$$G_p(t,s) = \begin{cases} Z(t)[I - Z(\omega)]^{-1} Z^{-1}(\theta_{k+1}) \mathrm{e}^{-A(\theta_{k+1}-s)}, s \in [\zeta_k, \zeta_{k+1}), k < j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \mathrm{e}^{-A(\theta_{k+1}-s)}, s \in [\zeta_k, \zeta_{k+1}) \setminus [\hat{\zeta_j}, t], k \geq j, \\ Z(t)[I - Z(\omega)]^{-1} Z(\omega) Z^{-1}(\theta_{k+1}) \mathrm{e}^{-A(\theta_{k+1}-s)} + \mathrm{e}^{-A(t-s)}, s \in [\hat{\zeta_j}, t]. \end{cases}$$

It can be seen that $\prod : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$. Let $u,\ v \in C_\omega(\mathbb{R})$, and then we have that

$$\begin{aligned} \left\| \prod u(t) - \prod v(t) \right\| &\leq \int_0^\omega \| G_p(t,s) \| \left| C^T \right| \| h(s, u_s) - h(s, v_s) \| \, ds \\ &\quad + \int_0^\omega \| G_p(t,s) \| \left| D^T \right| \left\| g\left(s, u_{\gamma(s)}\right) - g\left(s, v_{\gamma(s)}\right) \right\| ds \\ &\leq \int_0^\omega \widetilde{H} \left(\left| C^T \right| L^h \| u_s - v_s \|_0 + \left| D^T \right| L^g \left\| u_{\gamma(s)} - v_{\gamma(s)} \right\|_0 \right) ds \\ &\leq \int_0^\omega \widetilde{H} L \| u - v \| \, ds \\ &\leq \widetilde{H} L \omega \| u - v \|, \end{aligned}$$

which completes the proof. □

Theorem 4.2 *In addition to the conditions of Theorem 4.1, assume that conditions (N6)–(N10) are true. Then (1) admits a unique exponentially stable ω-periodic solution.*

Proof Let u and v be the solutions of equation (3) with initial data (t_0, ϕ, ψ) and (t_0, η, π), respectively. Thus, we have

$$\begin{aligned} u(t) - v(t) &= Z(t, t_0)(\phi(t_0) - \eta(t_0)) \\ &\quad + Z(t, t_0) \int_{t_0}^{\zeta_i} \mathrm{e}^{-A(t_0 - s)} C(h(s, u_s) - h(s, v_s)) \, ds \\ &\quad + Z(t, t_0) \int_{t_0}^{\zeta_i} \mathrm{e}^{-A(t_0 - s)} D\left(g\left(s, u_{\gamma(s)}\right) - g\left(s, v_{\gamma(s)}\right)\right) ds \\ &\quad + \sum_{k=i}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1} - s)} C(h(s, u_s) - h(s, v_s)) \, ds \end{aligned}$$

$$+\sum_{k=i}^{j-1} Z\left(t,\theta_{k+1}\right) \int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,v_{\gamma(s)}\right)\right) ds$$

$$+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} C\left(h\left(s,u_s\right)-h\left(s,v_s\right)\right) ds$$

$$+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,v_{\gamma(s)}\right)\right) ds.$$

Let $w\left(t\right)=u\left(t\right)-v\left(t\right)$, and then w satisfies the following equation:

$$\begin{aligned} w\left(t\right) &= Z\left(t,t_0\right)\left(\phi\left(t_0\right)-\eta\left(t_0\right)\right) \\ &+Z\left(t,t_0\right)\int_{t_0}^{\zeta_i} \mathrm{e}^{-A(t_0-s)} C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right) ds \\ &+Z\left(t,t_0\right)\int_{t_0}^{\zeta_i} \mathrm{e}^{-A(t_0-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right) ds \\ &+\sum_{k=i}^{j-1} Z\left(t,\theta_{k+1}\right)\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right) ds \\ &+\sum_{k=i}^{j-1} Z\left(t,\theta_{k+1}\right)\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right) ds \\ &+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right) ds \\ &+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right) ds. \end{aligned} \tag{10}$$

Now consider equation (10) for $t_0=0$ and take initial function $w_{t_0}=\phi\left(s\right)-\eta\left(s\right)$ where $t_0=0$ assuming that $\gamma\left(0\right)\leq 0$. Let $\left\|\phi\left(s\right)-\eta\left(s\right)\right\|<\delta$, $s\in\left[-\tau,0\right]$, where $\delta>0$. Fix $\epsilon>0$ and denote

$$\mathcal{L}_{ij}(\delta)=\frac{K\delta}{1-\frac{2}{\alpha}\overline{M}\mathrm{e}^{\alpha\tau}n_1\left(K+\left(1+\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\overline{\theta}}}\right)\mathrm{e}^{\alpha(j-i+1)\overline{\theta}}\right)}.$$

Take δ so small that $\mathcal{L}_{ij}\left(\delta\right)<\epsilon$. Let ψ_δ be the set of all continuous functions that are defined on $[-\tau,\infty)$ such that:

1. $w_{t_0}=\phi\left(s\right)-\eta\left(s\right)$, $s\in\left[-\tau,0\right]$;
2. $\left\|w\left(t\right)\right\|\leq\mathcal{L}_{ij}\left(\delta\right)\mathrm{e}^{-\frac{\alpha}{2}t}$ if $t\geq 0$;

3. $w(t)$ is uniformly continuous on $[0,\infty)$,

for all $w\in\psi_\delta$. Define on ψ_δ an operator $\widetilde{\prod}$ such that

$$\widetilde{\prod}w\left(t\right)=\begin{cases}\phi\left(t\right)-\eta\left(t\right),t\in\left[-\tau,0\right],\\ \\ Z\left(t,0\right)\left(\phi\left(0\right)-\eta\left(0\right)\right)+\\ +Z\left(t,0\right)\int\limits_0^{\zeta_i}\mathrm{e}^{As}C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right)ds\\ +Z\left(t,0\right)\int\limits_0^{\zeta_i}\mathrm{e}^{As}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right)ds\\ +\sum\limits_{k=i}^{j-1}Z\left(t,\theta_{k+1}\right)\int\limits_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right)ds\\ +\sum\limits_{k=i}^{j-1}Z\left(t,\theta_{k+1}\right)\int\limits_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right)ds\\ +\int\limits_{\zeta_j}^{t}\mathrm{e}^{-A(t-s)}C\left(h\left(s,u_s\right)-h\left(s,u_s-w_s\right)\right)ds\\ +\int\limits_{\zeta_j}^{t}\mathrm{e}^{-A(t-s)}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}\right)\right)ds,t\geq0.\end{cases}$$

We shall verify that $\widetilde{\prod}:\psi_\delta\to\psi_\delta$. Let us take into account the condition (N6). Denote $\left\|\phi\right\|_1=\sup\limits_{[0,\infty)}\left\|\phi\left(t\right)\right\|$. Indeed, for $t\geq0$ it is true that

$$\begin{aligned}\left\|\widetilde{\prod}w\left(t\right)\right\| &\leq K\mathrm{e}^{-\alpha t}\left[\delta+\int\limits_0^{\zeta_i}\overline{M}\left(\left|C^T\right|L^h\left\|w_s\right\|_0+\left|D^T\right|L^g\left\|w_{\gamma(s)}\right\|_0\right)ds\right]\\ &\quad+\sum_{k=i}^{j-1}K\mathrm{e}^{-\alpha(t-\theta_{k+1})}\int\limits_{\zeta_k}^{\zeta_{k+1}}\overline{M}\left(\left|C^T\right|L^h\left\|w_s\right\|_0+\left|D^T\right|L^g\left\|w_{\gamma(s)}\right\|_0\right)ds\\ &\quad+\int\limits_{\zeta_j}^{t}\overline{M}\left(\left|C^T\right|L^h\left\|w_s\right\|_0+\left|D^T\right|L^g\left\|w_{\gamma(s)}\right\|_0\right)ds\\ &= K\mathrm{e}^{-\alpha t}\left[\delta+\int\limits_0^{\zeta_i}\overline{M}\left|C^T\right|L^h\max_{\sigma\in[-\tau,0]}\left\|w_s\left(\sigma\right)\right\|ds\right]\\ &\quad+K\mathrm{e}^{-\alpha t}\int\limits_0^{\zeta_i}\overline{M}\left|D^T\right|L^g\max_{\sigma\in[-\tau,0]}\left\|w_{\gamma(s)}\left(\sigma\right)\right\|ds\\ &\quad+\sum_{k=i}^{j-1}K\mathrm{e}^{-\alpha(t-\theta_{k+1})}\int\limits_{\zeta_k}^{\zeta_{k+1}}\overline{M}\left|C^T\right|L^h\max_{\sigma\in[-\tau,0]}\left\|w_s\left(\sigma\right)\right\|ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M}\left|D^T\right| L^g \max_{\sigma\in[-\tau,0]} \left\|w_{\gamma(s)}(\sigma)\right\| ds \\
&+\int_{\zeta_j}^{t} \overline{M}\left(\left|C^T\right| L^h \max_{\sigma\in[-\tau,0]} \|w_s(\sigma)\| + \left|D^T\right| L^g \max_{\sigma\in[-\tau,0]} \left\|w_{\gamma(s)}(\sigma)\right\|\right) ds \\
\leq\ & K\mathrm{e}^{-\alpha t}\left[\delta + \int_0^{\zeta_i} \overline{M}\left|C^T\right| L^h \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau} ds\right] \\
&+K\mathrm{e}^{-\alpha t}\int_0^{\zeta_i} \overline{M}\left|D^T\right| L^g \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau}\mathrm{e}^{\alpha\overline{\theta}} ds \\
&+\sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M}\left|C^T\right| L^h \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau} ds \\
&+\sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M}\left|D^T\right| L^g \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau}\mathrm{e}^{\alpha\overline{\theta}} ds \\
&+\int_{\zeta_j}^{t} \overline{M}\left(\left|C^T\right| L^h \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau} + \left|D^T\right| L^g \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{e}^{\alpha\tau}\mathrm{e}^{\alpha\overline{\theta}}\right) ds \\
\leq\ & K\mathrm{e}^{-\alpha t}\left[\delta + \left(-\frac{2}{\alpha}\right)\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left(\left|C^T\right| L^h + \left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\right)\mathrm{e}^{-\frac{\alpha}{2}(\zeta_i-0)}\right] \\
&+\sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})}\left(-\frac{2}{\alpha}\right)\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left|C^T\right| L^h \mathrm{e}^{-\frac{\alpha}{2}(\zeta_{k+1}-\zeta_k)} \\
&+\sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})}\left(-\frac{2}{\alpha}\right)\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\mathrm{e}^{-\frac{\alpha}{2}(\zeta_{k+1}-\zeta_k)} \\
&+\left(-\frac{2}{\alpha}\right)\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left(\left|C^T\right| L^h + \left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\right)\mathrm{e}^{-\frac{\alpha}{2}(t-\zeta_j)} \\
\leq\ & K\mathrm{e}^{-\frac{\alpha}{2}t}\left[\delta + \frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left(\left|C^T\right| L^h + \left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\right)\right] \\
&+\mathrm{e}^{-\frac{\alpha}{2}t}\sum_{k=i}^{j-1} K\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha\theta_{k+1}}\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left(\left|C^T\right| L^h + \left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\right)\mathrm{e}^{\frac{-\alpha}{2}\zeta} \\
&+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}\left(\left|C^T\right| L^h + \left|D^T\right| L^g \mathrm{e}^{\alpha\overline{\theta}}\right)\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\frac{\alpha}{2}\zeta_j} \\
\leq\ & K\mathrm{e}^{-\frac{\alpha}{2}t}\left[\delta + \frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau} n_1\right] \\
&+\mathrm{e}^{-\frac{\alpha}{2}t}\sum_{k=i}^{j-1} K\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha\theta_{k+1}}\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau} n_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha\theta_{j+1}}\\
\leq\ & K\mathrm{e}^{-\frac{\alpha}{2}t}\left[\delta+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\right]\\
&+\mathrm{e}^{-\frac{\alpha}{2}t}\sum_{k=i}^{j-1}K\mathrm{e}^{-\frac{\alpha}{2}(t-\theta_{j+1})}\mathrm{e}^{-\alpha(\theta_{j+1}-\theta_{k+1})}\mathrm{e}^{\frac{\alpha}{2}\theta_{j+1}}\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\\
&+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha\theta_{j+1}}\\
\leq\ & K\mathrm{e}^{-\frac{\alpha}{2}t}\left[\delta+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\right]+\mathrm{e}^{-\frac{\alpha}{2}t}\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\mathrm{e}^{\frac{\alpha}{2}\theta_{j+1}}\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\\
&+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha\theta_{j+1}}\\
\leq\ & K\mathrm{e}^{-\frac{\alpha}{2}t}\left[\delta+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\right]\\
&+\mathrm{e}^{-\frac{\alpha}{2}t}\frac{K\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\mathrm{e}^{\alpha(j-i+1)\overline{\theta}}\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\\
&+\frac{2}{\alpha}\overline{M}\mathcal{L}_{ij}(\delta)\mathrm{e}^{\alpha\tau}n_1\mathrm{e}^{-\frac{\alpha}{2}t}\mathrm{e}^{\alpha(j-i+1)\overline{\theta}}\\
\leq\ & \mathcal{L}_{ij}(\delta)\mathrm{e}^{-\frac{\alpha}{2}t}.
\end{aligned}
$$

Let $w^1,\ w^2\in\psi_\delta$. Then we get

$$
\begin{aligned}
\widetilde{\prod}w^1(t) &= Z(t,0)\left(\phi(0)-\eta(0)\right)\\
&\quad+Z(t,0)\int_0^{\zeta_i}\mathrm{e}^{As}C\left(h(s,u_s)-h\left(s,u_s-w_s^1\right)\right)ds\\
&\quad+Z(t,0)\int_0^{\zeta_i}\mathrm{e}^{As}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^1\right)\right)ds\\
&\quad+\sum_{k=i}^{j-1}Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}C\left(h(s,u_s)-h\left(s,u_s-w_s^1\right)\right)ds\\
&\quad+\sum_{k=i}^{j-1}Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}\mathrm{e}^{-A(\theta_{k+1}-s)}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^1\right)\right)ds\\
&\quad+\int_{\zeta_j}^{t}\mathrm{e}^{-A(t-s)}C\left(h(s,u_s)-h\left(s,u_s-w_s^1\right)\right)ds\\
&\quad+\int_{\zeta_j}^{t}\mathrm{e}^{-A(t-s)}D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^1\right)\right)ds,
\end{aligned}
$$

$$\begin{aligned}
\widetilde{\prod} w^2(t) &= Z(t,0)(\phi(0)-\eta(0)) \\
&+Z(t,0)\int_0^{\zeta_i} \mathrm{e}^{As} C\left(h(s,u_s)-h\left(s,u_s-w_s^2\right)\right) ds \\
&+Z(t,0)\int_0^{\zeta_i} \mathrm{e}^{As} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^2\right)\right) ds \\
&+\sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} C\left(h(s,u_s)-h\left(s,u_s-w_s^2\right)\right) ds \\
&+\sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^2\right)\right) ds \\
&+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} C\left(h(s,u_s)-h\left(s,u_s-w_s^2\right)\right) ds \\
&+\int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} D\left(g\left(s,u_{\gamma(s)}\right)-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^2\right)\right) ds,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\prod} w^1(t)-\widetilde{\prod} w^2(t) &= Z(t,0)\int_0^{\zeta_i} \mathrm{e}^{As} C\left(-h\left(s,u_s-w_s^1\right)+h\left(s,u_s-w_s^2\right)\right) ds \\
&+ Z(t,0)\int_0^{\zeta_i} \mathrm{e}^{As} D\left(-g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^1\right)+g\left(s,u_{\gamma(s)}-w_{\gamma(s)}^2\right)\right) ds \\
&- \sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Ch\left(s,u_s-w_s^1\right) ds \\
&+ \sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Ch\left(s,u_s-w_s^2\right) ds \\
&- \sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Dg\left(s,u_{\gamma(s)}-w_{\gamma(s)}^1\right) ds \\
&+ \sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} \mathrm{e}^{-A(\theta_{k+1}-s)} Dg\left(s,u_{\gamma(s)}-w_{\gamma(s)}^2\right) ds
\end{aligned}$$

$$+ \int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} C \left(-h\left(s, u_s - w_s^1\right) + h\left(s, u_s - w_s^2\right)\right) ds$$

$$+ \int_{\zeta_j}^{t} \mathrm{e}^{-A(t-s)} D \left(-g\left(s, u_{\gamma(s)} - w_{\gamma(s)}^1\right) + g\left(s, u_{\gamma(s)} - w_{\gamma(s)}^2\right)\right) ds.$$

Let us take into account condition (N6). Thus, we attain that

$$\begin{aligned}
\left\| \widetilde{\prod} w^1(t) - \widetilde{\prod} w^1(t) \right\|_1 \leq\ & K\mathrm{e}^{-\alpha t} \int_0^{\zeta_i} \overline{M} \left|C^T\right| L^h \sup_{[0,\infty)} \left\|w_s^1 - w_s^2\right\|_0 ds \\
& + K\mathrm{e}^{-\alpha t} \int_0^{\zeta_i} \overline{M} \left|D^T\right| L^g \sup_{[0,\infty)} \left\|w_{\gamma(s)}^1 - w_{\gamma(s)}^2\right\|_0 ds \\
& + \sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left|C^T\right| L^h \sup_{[0,\infty)} \left\|w_s^1 - w_s^2\right\|_0 ds \\
& + \sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left|D^T\right| L^g \sup_{[0,\infty)} \left\|w_{\gamma(s)}^1 - w_{\gamma(s)}^2\right\|_0 ds \\
& + \int_{\zeta_j}^{t} \overline{M} \left|C^T\right| L^h \sup_{[0,\infty)} \left\|w_s^1 - w_s^2\right\|_0 ds \\
& + \int_{\zeta_j}^{t} \overline{M} \left|D^T\right| L^g \sup_{[0,\infty)} \left\|w_{\gamma(s)}^1 - w_{\gamma(s)}^2\right\|_0 ds \\
\leq\ & K \int_0^{\zeta_i} \overline{M} \left(\left|C^T\right| L^h + \left|D^T\right| L^g\right) \sup_{[0,\infty)} \left\|w^1(t) - w^2(t)\right\| ds \\
& + \sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left|C^T\right| L^h \sup_{[0,\infty)} \left\|w^1(t) - w^2(t)\right\| ds \\
& + \sum_{k=i}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \overline{M} \left|D^T\right| L^g \sup_{[0,\infty)} \left\|w^1(t) - w^2(t)\right\| ds \\
& + \int_{\zeta_j}^{t} \overline{M} \left(\left|C^T\right| L^h + \left|D^T\right| L^g\right) \sup_{[0,\infty)} \left\|w^1(t) - w^2(t)\right\| ds
\end{aligned}$$

$$\begin{aligned}
&\leq K\int_0^{\zeta_i}\overline{M}L\left\|w^1-w^2\right\|_1 ds\\
&\quad+\sum_{k=i}^{j-1}K\mathrm{e}^{-\alpha(t-\theta_{k+1})}\int_{\zeta_k}^{\zeta_{k+1}}\overline{M}L\left\|w^1-w^2\right\|_1 ds\\
&\quad+\int_{\zeta_j}^{t}\overline{M}L\left\|w^1-w^2\right\|_1 ds\\
&\leq \overline{M}L\left(\overline{\theta}(K+1)+K\frac{\overline{\zeta}\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\left\|w^1-w^2\right\|_1\\
&\leq \overline{M}L\left(K+1+K\frac{2\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\overline{\theta}\left\|w^1-w^2\right\|_1\\
&\leq \overline{M}L\left(1+K\frac{1+2\mathrm{e}^{\alpha\overline{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}\right)\overline{\theta}\left\|w^1-w^2\right\|_1.
\end{aligned}$$

Now we show that there is no other solution of the initial value problem. Consider first the interval $[\theta_0,\theta_1]$, $\theta_0\leq 0\leq\theta_1$. Assume on this interval that u, v are two different solutions of the problem. Denote $w=u-v$, $\overline{m}=\max\limits_{[0,\theta_1]}\|w(t)\|$, $\overline{m}>0$. Then we have

$$\begin{aligned}
u(t) &= Z(t,0)\phi(0)+Z(t,0)\int_0^{\zeta_0}\mathrm{e}^{-A(\zeta_0-s)}\left(Ch(s,u_s)+Dg(s,u_{\zeta_i})+E\right)ds\\
&\quad+\int_{\zeta_0}^{t}\mathrm{e}^{-A(t-s)}\left(Ch(s,u_s)+Dg(s,u_{\zeta_i})+E\right)ds,\\
v(t) &= Z(t,0)\phi(0)+Z(t,0)\int_0^{\zeta_0}\mathrm{e}^{-A(\zeta_0-s)}\left(Ch(s,v_s)+Dg(s,v_{\zeta_i})+E\right)ds\\
&\quad+\int_{\zeta_0}^{t}\mathrm{e}^{-A(t-s)}\left(Ch(s,v_s)+Dg(s,v_{\zeta_i})+E\right)ds,\\
w(t) &= Z(t,0)\int_0^{\zeta_0}\mathrm{e}^{-A(\zeta_0-s)}C\left(h(s,u_s)-h(s,u_s-w_s)\right)ds\\
&\quad+Z(t,0)\int_0^{\zeta_0}\mathrm{e}^{-A(\zeta_0-s)}D\left(g(s,u_{\zeta_i})-g(s,u_{\zeta_i}-w_{\zeta_i})\right)ds\\
&\quad+\int_{\zeta_0}^{t}\mathrm{e}^{-A(t-s)}\left(C\left(h(s,u_s)-h(s,u_s-w_s)\right)+D\left(g(s,u_{\zeta_i})-g(s,u_{\zeta_i}-w_{\zeta_i})\right)\right)ds.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\|w(t)\| &= M\int_0^{\zeta_0}\overline{M}\left|C^T\right|\|h(s,u_s)-h(s,u_s-w_s)\|\,ds \\
&\quad +M\int_0^{\zeta_0}\overline{M}\left|D^T\right|\|g(s,u_{\zeta_i})-g(s,u_{\zeta_i}-w_{\zeta_i})\|\,ds \\
&\quad +\int_{\zeta_0}^{t}\overline{M}\left|C^T\right|\|h(s,u_s)-h(s,u_s-w_s)\|\,ds \\
&\quad +\int_{\zeta_0}^{t}\overline{M}\left|D^T\right|\|g(s,u_{\zeta_i})-g(s,u_{\zeta_i}-w_{\zeta_i})\|\,ds \\
&\leq M\int_0^{\zeta_0}\overline{M}\left(\left|C^T\right|L^h\|w_s\|_0+\left|D^T\right|L^g\|w_{\zeta_i}\|_0\right)ds \\
&\quad +\int_{\zeta_0}^{t}\overline{M}\left(\left|C^T\right|L^h\|w_s\|_0+\left|D^T\right|L^g\|w_{\zeta_i}\|_0\right)ds \\
&\leq \left(M\int_0^{\zeta_0}\overline{M}Lds+\int_{\zeta_0}^{t}\overline{M}L\right)\|w\|\,ds \\
&\leq \overline{M}L(1+M)\,\overline{\theta}\max_{[0,\theta_1]}\|w\|\,ds \\
&\leq \overline{M}L(1+M)\,\overline{\theta}\overline{m}. \qquad (11)
\end{aligned}$$

(11) contradicts condition $(N5)$. Now, using induction, one can easily prove the uniqueness for all $t\geq 0$. □

References

[1] Aftabizadeh AR, Wiener J, Xu JM. Oscillatory and periodic solutions of delay differential equations with piecewise constant argument. P Am Math Soc 1987; 99: 673-679.

[2] Akhmet MU. On the integral manifolds of the differential equations with piecewise constant argument of generalized type. In: Agarval RP, Perera K, editors. Proceedings of the Conference on Differential and Difference Equations at the Florida Institute of Technology; 1–5 August 2005; Melbourne, Florida. Cairo, Egypt: Hindawi Publishing Corp., 2006, pp. 11-20.

[3] Akhmet MU. Stability of differential equations with piecewise constant arguments of generalized type. Nonlinear Anal 2008; 68: 794-803.

[4] Akhmet MU. Nonlinear Hybrid Continuous Discrete Time Models. Amsterdam, the Netherlands: Atlantis Press, 2011.

[5] Akhmet MU. Quasilinear retarded differential equations with functional dependence on piecewise constant argument. Commun Pure Appl Ana 2014; 13: 929-947.

[6] Akhmet MU, Aruğaslan D. Lyapunov-Razumikhin method for differential equations with piecewise constant argument. Discret Contin Dyn-A 2009; 25: 457-466.

[7] Akhmet MU, Aruğaslan D, Yılmaz E. Stability analysis of recurrent neural networks with piecewise constant argument of generalized type. Neural Networks 2010; 23: 805-811.

[8] Akhmet MU, Aruğaslan D, Yılmaz E. Stability in cellular neural networks with a piecewise constant argument. J Comput Appl Math 2010; 233: 2365-2373.

[9] Akhmet MU, Yılmaz E. Hopfield-type neural networks systems equations with piecewise constant argument. Int J Qual Theory Differ Equat Appl 2009; 3: 8-14.

[10] Akhmet MU, Yılmaz E. Global attractivity in impulsive neural networks with piecewise constant delay. In: Proceedings of Neural, Parallel, and Scientific Computations. Atlanta, GA, USA: Dynamic Publishers, Inc., 2010, pp. 11-18.

[11] Akhmet MU, Yılmaz E. Global exponential stability of neural networks with non-smooth and impact activations. Neural Networks 2012; 34: 18-27.

[12] Akhmet MU, Yılmaz E. Neural Networks with Discontinuous/Impact Activations. New York, NY, USA: Springer, 2014.

[13] Aruğaslan D. Differential equations with discontinuities and population dynamics. PhD, Middle East Technical University, Ankara, Turkey, 2009.

[14] Belair J, Campbell SA, Driessche PVD. Frustration, stability, and delay-induced oscillations in a neural network model. SIAM J Appl Math 1996; 56: 245-255.

[15] Chen TP. Global exponential stability of delayed Hopfield neural networks. Neural Networks 2001; 14: 977-980.

[16] Cooke KL, Wiener J. Retarded differential equations with piecewise constant delays. J Math Anal Appl 1984; 99: 265-297.

[17] Cooke KL, Wiener J. Stability for linear equations with piecewise continuous delay. Comput Math Appl-A 1986; 12: 695-701.

[18] Cooke KL, Wiener J. An equation alternately of retarded and advanced type. P Am Math Soc 1987; 99: 726-732.

[19] Cooke KL, Wiener J. Neutral differential equations with piecewise constant argument. Boll Un Mat Ital 1987; 7: 321-346.

[20] Driessche PVD, Zou X. Global attractivity in delayed Hopfield neural network models. SIAM J Appl Math 1998; 58: 1878-1890.

[21] Hale J. Functional Differential Equations. New York, NY, USA: Springer, 1971.

[22] Papaschinopoulos G. On the integral manifold for a system of differential equations with piecewise constant argument. J Math Anal Appl 1996; 201: 75-90.

[23] Wiener J. Generalized Solutions of Functional Differential Equations. Singapore: World Scientific, 1993.

[24] Wiener J, Lakshmikantham V. A damped oscillator with piecewise constant time delay. Nonlinear Stud 2000; 7: 78-84.

[25] Yang X. Existence and exponential stability of almost periodic solution for cellular neural networks with piecewise constant argument. Acta Math Appl Sin 2006; 29: 789-800.