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A comparative study of Gauss–Laguerre quadrature and an open type mixed quadrature by evaluating some improper integrals

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Abstract: An open type mixed quadrature rule is constructed blending the anti-Gauss 3-point rule with Steffensen's 4-point rule. The analytical convergence of the mixed rule is studied. An adaptive integration scheme is designed based on the mixed quadrature rule. A comparative study of the mixed quadrature rule and the Gauss–Laguerre quadrature rule is given by evaluating several improper integrals of the form $\int_0^\infty e^{-x} f(x)dx$. The advantage of implementing mixed quadrature rule in developing an efficient adaptive integration scheme is shown by evaluating some improper integrals.

Key words: Anti-Gaussian quadrature rule, mixed quadrature rule, adaptive integration scheme, improper integrals, Steffensen's quadrature, Gauss–Laguerre quadrature

1. Introduction

The numerical quadrature rules are broadly classified into two categories, namely closed type quadrature rules and open type quadrature rules. A quadrature rule

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k), \qquad x_1 < x_2 < \cdots < x_n$$

is said to be of closed type if the function evaluation is needed at the end points of the interval $[a, b]$. A quadrature rule is said to be of open type if both the end points are omitted from the evaluation of function. Newton–Cotes quadrature rules and Lobatto quadrature rules are examples of closed type rules whereas Steffensen's quadrature rules, Gauss–Legendre quadrature rules, and anti-Gauss quadrature rules are examples of open type rules. Open type quadrature rules are more useful for evaluation of singular integrals.

The idea of Gaussian quadrature is to give ourselves the freedom to choose not only the weighting coefficients but also the location of the abscissas at which the function is to be evaluated: they will no longer be equally spaced. Thus, we will have twice the number of degrees of freedom at our disposal; it will turn out that we can achieve Gaussian quadrature formulae whose order is, essentially, twice that of the Newton–Cotes formulae with the same number of function evaluations. High order is not the same as high accuracy. High order translates to high accuracy only when the integrand is very smooth, in the sense of being "well-approximated by a polynomial".

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There is, however, one additional feature of Gaussian quadrature formulae that adds to their usefulness. We can arrange the choice of weights and abscissas to make the integral exact for a class of integrands "polynomial times some known function $W(x)$" rather than for the usual class of integrands "polynomials". The function $W(x)$ can then be chosen to remove integrable singularities from the desired integral. For different weight functions there are different Gaussian quadrature rules, like for weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$ and $W(x) = e^{-x}$ the corresponding Gaussian quadratures are referred to as Gauss–Chebyshev and Gauss–Laguerre quadrature, respectively.

It is natural to take the challenge that if we have a task to integrate an improper integral numerically over $[0, \infty)$ then what would be our initial approach; anybody can have the intuition that certainly it would aim at Gauss–Laguerre quadrature. Let us discuss the problem follows.

Let us integrate $f(x) = \frac{1}{\sqrt{x}}$ over $[0, \infty)$. Obviously this integral is an improper integral of both first and second kind since $f(x)$ is undefined at $x = 0$ and the upper limit of the integral is infinite. Now the important thing is that if we are still looking to integrate this function then the following primary problems will arise:

(i) How could we remove the singularity?

(ii) Can we still take the continuous interval?

(iii) How can the integral be set to converge?

The answer to the first question is that if we can punch a weight on the integrand then we might be able to remove the singularities, i.e. particularly for Gauss–Laguerre quadrature over $[0, \infty)$ we choose weight function $W(x) = e^{-x}$ and then integrate $W(x)f(x) = e^{-x}f(x)$ over $[0, \infty)$.

The answer to the second question is, for a certain precision, instead of the continuous set of points we look for a discrete set of points where we can approximate the integrand exactly with the help of a polynomial, i.e.

$$\int_0^\infty W(x)f(x)dx \approx \sum_{j=1}^{n} w_j f(x_j)$$

is exact if $f(x)$ is a polynomial. As per everyone's knowledge for Laguerre quadrature, the nodes are the zeros of the Laguerre polynomial.

Now let us go back to our third question. The convergence of the integrand can only be set in force if we can have a good choice of weight function, i.e. to say the convergence of the integral now depends upon the convergence of the weight function $W(x)$. Specifically for Laguerre quadrature we choose $W(x) = e^{-x}$ because it is a rapidly decreasing function over $[0, \infty)$ and it helps $f(x)$ to converge exponentially. Let us go back to our first example, i.e. $f(x) = \frac{1}{\sqrt{x}}$. Now we draw the graphs of $f(x) = \frac{1}{\sqrt{x}}$, $W(x) = e^{-x}$, and $W(x)f(x) = e^{-x}\frac{1}{\sqrt{x}}$ over $[0, 1]$ first (shown in Figure 1).

This graph shows the behavior of the three functions in $[0, 1]$. Needless to say, these three graphs are decreasing, but which one is the fastest? This probably better clarified in Figure 2 by increasing the domain to $[0, 10)$ and restricting range to $[0, 1]$.

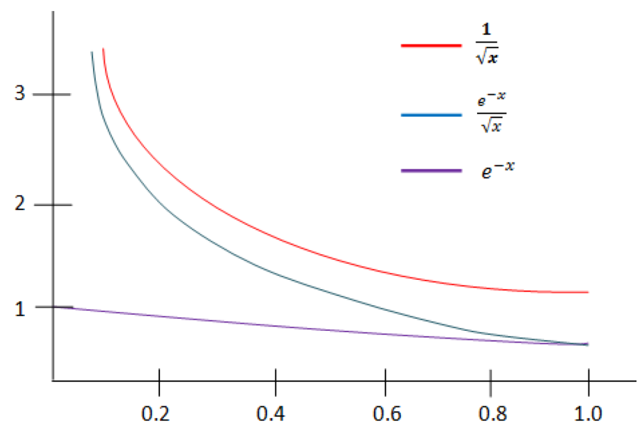

Figure 1. Plotting of $\dfrac{1}{\sqrt{x}}$, e^{-x}, and $e^{-x}\dfrac{1}{\sqrt{x}}$ over $[0,1]$.

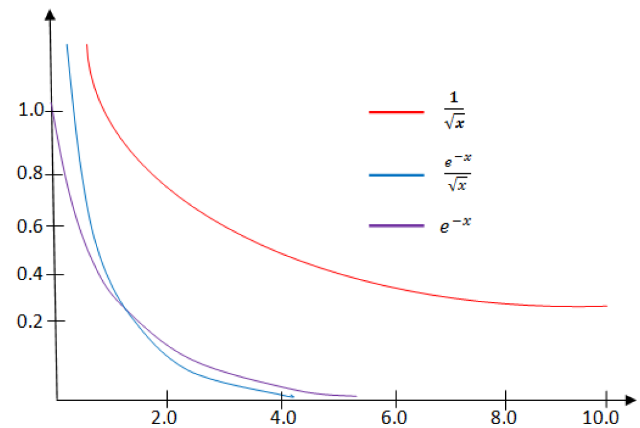

Figure 2. Plotting of $\dfrac{1}{\sqrt{x}}$, e^{-x}, and $e^{-x}\dfrac{1}{\sqrt{x}}$ over $[0,10]$.

Figure 2 gives us a clear vision that the convergence of $e^{-x}\dfrac{1}{\sqrt{x}}$ is much much faster than the other two. That is why we need a weight function that helps the integral to converge. The necessity of $W(x) = e^{-x}$ in Laguerre quadrature now becomes distinct. Though $\dfrac{1}{\sqrt{x}}$ is decaying, it is not defined at $x = 0$ and obviously the integral becomes unbounded as x approaches ∞.

Figure 3 shows the nodes and weights for several values of n of Laguerre quadrature. Since the domain of integration $[0,\infty)$ is infinite, the quadrature nodes x_j get larger and larger. As the nodes get larger, the corresponding weights decay rapidly.

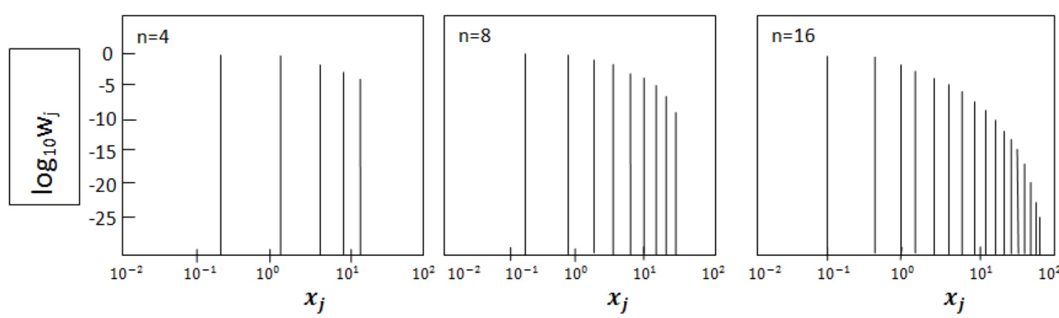

Figure 3. Nodes and weights of Gauss–Laguerre quadrature for various values of n. In each case, the location of the vertical line indicates x_j, while the height of the line shows $\log_{10} w_j$. Note that the horizontal axis is scaled logarithmically. As n increases the quadrature rule includes larger and larger nodes to account for the infinite domain of integration; however, the weights are exceptionally small for the larger nodes. For example, for $n = 16$, $w_{16} \approx 10^{-23}$.

Another important thing to be stated here is that in every Gaussian quadrature we are approximating through an orthogonal polynomial and in Laguerre quadrature we are taking the Laguerre polynomial, which is also orthogonal. The reason behind this is whenever a function fails to give value at a certain point or has holes in the specified domain then this orthogonal polynomial smoothens the function by making a bridge over the holes and gives its own zeros as our nodes to evaluate our integrals. If we increase the order, the function starts merging with the orthogonal polynomial, which can be seen in Figure 4.

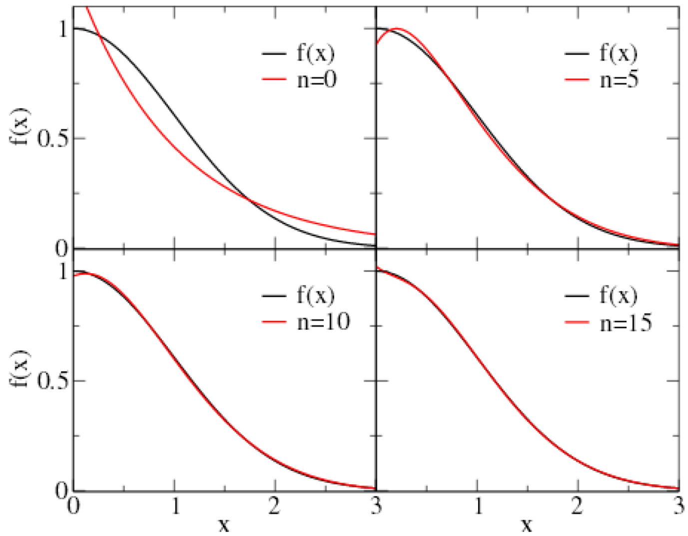

Figure 4. The function $f(x) = e^{-\frac{1}{2}x^2}$ and the representation of $f(x)$ in an expansion of Laguerre polynomials of order n.

The method of mixed quadrature is a new method of enhancing the precision of quadrature. This was first coined by Das and Pradhan [7]. In this method a quadrature rule of higher precision is formed by taking the linear/convex combination of two or more quadrature rules of equal lower precision. In the literature, precision of quadrature rules was enhanced through either Richardson extrapolation or Kronrod extension. Richardson extrapolation gives us a family of formulae of higher precision, taking into account a trapezoidal formula as the base formula (refer to [1, 2, 13, 16]). On the other hand, [11, 15] presented quadrature rules of higher precision by taking Gaussian quadrature as the base rule. These methods of precision enhancement, each having single base rule, are very much cumbersome, but the enhancement of precision by mixed quadrature approach with the aid of two rules is very simple and easy to handle. Das and Dash [4–6, 8] were the first to use the mixed quadrature rule for approximation of real definite integrals in an adaptive environment.

For evaluation of improper integrals of the form

$$\int_0^\infty e^{-x} f(x)dx, \tag{1.1}$$

usually Gauss–Laguerre type of rules are used, but the Gauss–Laguerre quadrature rules are not suitable for adaptive integration schemes since they are used only when the interval of integration is $[0, \infty)$.

In this paper our attempt is to evaluate (1.1) efficiently in the adaptive integration scheme as described in [3, 9] using an alternative mixed quadrature rule as the base rule.

As Steffensen's 4-point rule and the anti-Gauss 3-point rule are of same precision (i.e. precision-3), we have constructed a mixed quadrature rule of higher precision (i.e. precision-5) taking the convex combination of these two rules. This mixed rule is called an open type rule as the two constituent rules are of open type.

To apply the quadrature/mixed quadrature rule for evaluation of $\int_0^\infty e^{-x} f(x)dx$ in adaptive integration schemes we transform the interval of integration from $[0, \infty)$ to $[0, 1]$ as

$$\int_0^\infty e^{-x} f(x)dx = \int_0^1 f\left(\ln\left(\frac{1}{x}\right)\right). \tag{1.2}$$

Here we have designed a new adaptive integration scheme involving the above mixed quadrature rule and its constituent rules. Using the scheme some improper test integrals have been evaluated and the results are compared with the results obtained by the Gauss–Laguerre quadrature rules. For numerical computation in respect to Gauss–Laguerre quadrature we have taken the nodes and corresponding weights from Table 1. The results are reflected in Tables 2, 3, and 4. A comprehensive conclusion is given at the end.

Table 1. Nodes and weights for Gauss–Laguerre quadrature rules.

n	$Nodes\ x_k$	$Weights\ \lambda_k$
1	0.5857864376	0.8535533906
	3.4142135624	0.1464466094
2	0.4157745568	0.7110930099
	2.2942803603	0.2785177336
	6.2899450829	0.0103892565
3	0.3225476896	0.6031541043
	1.7457611012	0.3574186924
	4.5366202969	0.0388879085
	9.3950709123	0.0005392947
4	0.2635603197	0.5217556106
	1.4134030591	0.3986668111
	3.5964257710	0.0759424497
	7.0858100059	0.0036117587
	12.6408008443	0.0000233700
4	0.2228466042	0.4589646740
	1.1889321017	0.4170008308
	2.9927363261	0.1133733821
	5.7751435691	0.0103991975
	9.8374674184	0.0002610172
	15.9828739806	0.0000008985

2. Basic quadrature rules

The general problem of the numerical integration/quadrature rule is to find an approximate value of the integral

$$I(f) = \int_a^b w(x)f(x)dx, \tag{2.1}$$

where $w(x) > 0$ on $[a,b]$ and $w(x)f(x)$ is integrable in the Riemann sense in $[a,b]$. The quadrature rule (2.1) can be written in the form

$$I(f) = \int_a^b w(x)f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i), \tag{2.2}$$

where $x_i, i = 0(1)n$ are called the nodes distributed within the limits of integration.

$w_i, i = 0(1)n$ are called the weights of the quadrature rule. The error of approximation is given as

$$E_n(f) = I(f) - \sum_{i=0}^{n} w_i f(x_i). \tag{2.3}$$

2.1. Steffensen's quadrature rules

Steffensen's quadrature rules are of open type Newton–Cotes quadrature rules [13]. These rules may be used when the function has singularity at the end points or the values of the function are known at the end points. These rules are useful to solve differential equations numerically when the function values at the end points are not available.

2.1.1. Steffensen's 4-point rule

$$\int_a^b f(x)dx = \frac{5h}{24}[11f(a+h) + f(a+2h) + f(a+3h) + 11(a+4h)] + E_4 \tag{2.4}$$

where $h = \dfrac{b-a}{5}$

$$\text{and} \qquad E_4 = \frac{95}{144}h^5 f^{(iv)}(\xi), \qquad a < \xi < b. \tag{2.5}$$

The degree of the precision of the rule (2.4) is 3.

2.2. Gauss–Laguerre quadrature rule

The Gauss–Laguerre quadrature rule [14] is a Gaussian quadrature over the interval $[a, b]$ with the weight function $\psi(x) = e^{-x}$. The general form is

$$\int_0^\infty e^{-x} f(x)dx = \sum_{k=1}^{n} \lambda_k f(x_i). \tag{2.6}$$

The nodes $x_i's$ are the zeros of the Laguerre polynomial [1, 10, 14]

$$L_n(x) = (-1)^n e^x \frac{d^n}{dx^n}(e^{-x}x^n), \tag{2.7}$$

which satisfies the recurrence relation

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x). \tag{2.8}$$

The weights λ_is are given by

$$\lambda_k = \frac{1}{x_k[L'_n(x_i)]^2} = \frac{x_k}{(n+1)^2[L_{n+1}(x_k)]^2}. \tag{2.9}$$

The error term is

$$E = \frac{(n!)^2}{(2n)!} f^{(2n)}(\xi), \qquad 0 < \xi < \infty. \tag{2.10}$$

2.3. Anti-Gaussian quadrature rule

Laurie [12] was the first to coin the idea of the anti-Gaussian quadrature rule. An anti-Gaussian $n+1$ point quadrature rule is a rule whose degree of precision is $2n-1$. It integrates polynomials of degree up to $2n+1$ with an error equal in magnitude but of opposite sign to that of the n-point Gaussian rule.

Using this idea of Laurie, we construct an anti-Gauss 3-point rule of precision 3 from the Gauss–Legendre 2-point rule as follows.

The Gauss–Legendre 2-point rule is

$$\int_{-1}^{1} f(x)dx \approx R_{GL_2}(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right). \tag{2.11}$$

The anti-Gauss 3-point rule $R_{aG_3}(f)$ is taken as

$$R_{aG_3}(f) = \sum_{i=0}^{2} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2), \tag{2.12}$$

where w_is are weights and x_is are the distinct points (nodes) in the interval $[-1,1]$.

The rule is so designed that the error associated with the anti-Gaussian 3-point rule is equal to the negative of the error associated with Gauss–Legendre 2-point rule. That is,

$$I(f) - R_{aG_3}(f) = -(I(f) - R_{GL_2}(f)) \tag{2.13}$$

or

$$R_{aG_3}(f) = 2I(f) - R_{GL_2}(f). \tag{2.14}$$

The evaluation of the unknown weights and nodes is based on the following preconditions:

(i) The rule is exact for all polynomials of degree ≤ 3.

(ii) The rule integrates all polynomials of degree up to 5 with an error equal in magnitude and opposite in sign to that of Gauss–Legendre 2-point rule.

Thus, we obtain a system of six equations having six unknowns, namely w_i, x_i $(i = 1, 2, 3)$.

$$\begin{aligned}
w_0 + w_1 + w_2 &= 2 \\
w_0 x_0 + w_1 x_1 + w_2 x_2 &= 0 \\
w_0 x_0^2 + w_1 x_1^2 + w_2 x_2^2 &= \frac{2}{3} \\
w_0 x_0^3 + w_1 x_1^3 + w_2 x_2^3 &= 0 \\
w_0 x_0^4 + w_1 x_1^4 + w_2 x_2^4 &= \frac{26}{45} \\
w_0 x_0^5 + w_1 x_1^5 + w_2 x_2^5 &= 0
\end{aligned}$$

Solving the system of equations we get, $w_0 = \frac{5}{13} = w_2,\ w_1 = \frac{16}{13},\ x_0 = -\sqrt{\frac{13}{15}},\ x_1 = 0,\ x_2 = \sqrt{\frac{13}{15}}$. Substituting these into equation (2.12), we get

$$\int_{-1}^{1} f(x)dx \approx R_{aG_3}(f) = \frac{1}{13}\left[5f\left(-\sqrt{\frac{13}{15}}\right) + 16f(0) + 5f\left(\sqrt{\frac{13}{15}}\right)\right]. \tag{2.15}$$

This is anti-Gauss 3-point rule. The error associated with the rule (2.15) is

$$E_{aG_3}(f) = -\frac{1}{135}f^{(iv)}(\xi), \qquad -1 < \xi < 1. \tag{2.16}$$

Hence, the degree of precision of the anti-Gauss 3-point rule is 3.

3. Construction of mixed quadrature rule of precision five

A mixed quadrature rule of precision five is constructed by using the following two quadrature rules:

(i) anti-Gauss 3-point rule $(R_{aG_3}(f))$;

(ii) Steffensen's 4-point rule $(R_{St_4}(f))$.

The anti-Gauss 3-point rule $(R_{aG_3}(f))$ is

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{aG_3}(f) = \frac{1}{13}\left[5f\left(-\sqrt{\frac{13}{15}}\right) + 16f(0) + 5f\left(\sqrt{\frac{13}{15}}\right)\right]. \tag{3.1}$$

Steffensen's 4-point rule $(R_{St_4}(f))$ is

$$\begin{aligned} I(f) &= \int_{-1}^{1} f(x)dx \approx R_{St_4}(f) \\ &= \frac{1}{12}\left[11f\left(-\frac{3}{5}\right) + f\left(-\frac{1}{5}\right) + f\left(\frac{1}{5}\right) + 11f\left(\frac{3}{5}\right)\right]. \end{aligned} \tag{3.2}$$

Each of these rules, (3.1) and (3.2), is of precision 3. Let $E_{aG_3}(f)$ and $E_{St_4}(f)$ denote the errors in approximating the integral $I(f)$ by rules (3.1) and (3.2), respectively.

Then,

$$I(f) = R_{aG_3}(f) + E_{aG_3}(f), \tag{3.3}$$

$$I(f) = R_{St_4}(f) + E_{St_4}(f). \tag{3.4}$$

Assuming $f(x)$ to be continuously differentiable in $-1 \leq x \leq 1$, and using Maclaurin's expansion of function $f(x)$, we can express the errors associated with the quadrature rules under reference as

$$E_{aG_3}(f) = -\frac{1}{135}f^{(iv)}(0) - \frac{1016}{7! \times 675}f^{(vi)}(0) - \cdots \tag{3.5}$$

$$E_{St_4}(f) = \frac{38}{5625}f^{(iv)}(0) + \frac{13136}{7! \times 9375}f^{(vi)}(0) + \cdots \tag{3.6}$$

Now multiplying equations (3.3) and (3.4) by $\frac{38}{125}$ and $\frac{1}{3}$ respectively, and then adding the results, we obtain

$$
\begin{aligned}
I(f) &= \frac{1}{239}(114R_{aG_3}(f)+125R_{St_4}(f))+\frac{1}{239}(114E_{aG_3}(f)+125E_{St_4}(f)) \\
\text{or} \quad I(f) &= R_{aG_3St_4}(f)+E_{aG_3St_4}(f), \qquad (3.7)\\
\text{where} \quad R_{aG_3St_4}(f) &= \frac{1}{239}(114R_{aG_3}(f)+125R_{St_4}(f)), \qquad (3.8)\\
E_{aG_3St_4}(f) &= \frac{1}{239}(114E_{aG_3}(f)+125E_{St_4}(f)). \qquad (3.9)
\end{aligned}
$$

Equation (3.8) expresses the desired mixed quadrature rule for the approximate evaluation of $I(f)$ and equation (3.9) expresses the error generated in this approximation.

Substituting equations (3.5) and (3.6) into equation (3.9), we obtain

$$E_{aG_3St_4}(f)=\frac{32}{7!\times 2151}f^{(vi)}(0)+\cdots \qquad (3.10)$$

As the first term of $E_{aG_3St_4}(f)$ contains the sixth-order derivative of the integrand, the degree of precision of the mixed quadrature rule is 5. It is called a mixed type rule as it is constructed from two different types of rules of equal lower precision.

4. Error analysis of the mixed quadrature rule

An asymptotic error estimate and an error bound of the rule (3.8) are given in Theorems 4.1 and 4.2, respectively.

Theorem 4.1 *Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $E_{aG_3St_4}(f)$ associated with the mixed quadrature rule is given by*

$$|E_{aG_3St_4}(f)|\approx\frac{32}{7!\times 2151}|f^{(vi)}(0)|.$$

Proof The proof follows from (3.10). □

Theorem 4.2 *The bound for the truncation error*

$$E_{aG_3St_4}(f)=I(f)-R_{aG_3St_4}(f)$$

is given by

$$|E_{aG_3St_4}(f)|\le\frac{38M}{10755}|\eta_2-\eta_1|, \qquad \eta_1,\eta_2\in[-1,1],$$

where $M=\max_{-1\le x\le 1}|f^{(v)}(x)|$.

Proof We have

$$E_{aG_3}(f) \approx -\frac{1}{135} f^{(iv)}(\eta_1), \qquad \eta_1 \in [-1,1];$$

$$E_{St_4}(f) \approx \frac{38}{5625} f^{(iv)}(\eta_2), \qquad \eta_2 \in [-1,1].$$

Hence, $$E_{aG_3St_4}(f) = \frac{1}{239}\left[114E_{aG_3}(f) + 125E_{St_4}(f)\right]$$

$$\approx \frac{38}{10755}\left[f^{(iv)}(\eta_2) - f^{(iv)}(\eta_1)\right]$$

$$= \frac{38}{10755}\int_{\eta_1}^{\eta_2} f^{(v)}(x)dx, \quad \text{(assuming } \eta_1 < \eta_2\text{)}.$$

From this we obtain $$|E_{aG_3St_4}(f)| \approx \left|\frac{38}{10755}\int_{\eta_1}^{\eta_2} f^{(v)}(x)dx\right|$$

$$\le \frac{38}{10755}\int_{\eta_1}^{\eta_2} |f^{(v)}(x)|dx$$

or $$|E_{aG_3St_4}(f)| \le \frac{38M}{10755}\left|(\eta_2 - \eta_1)\right|,$$

which gives only a theoretical error bound, as η_1, η_2 are unknown points in the interval $[-1,1]$. It shows that the error in the approximation will be less if the points η_1, η_2 are close to each other. □

Corollary 4.1 *The error bound for the truncation error $E_{aG_3St_4}(f)$ is given by*

$$|E_{aG_3St_4}(f)| \le \frac{76M}{10755}.$$

Proof We know from Theorem 4.2 that

$$|E_{aG_3St_4}(f)| \le \frac{38M}{10755}|\eta_2 - \eta_1|, \qquad \eta_1, \eta_2 \in [-1,1],$$

where $$M = \max_{-1 \le x \le 1} |f^{(v)}(x)|.$$

Choosing $|(\eta_2 - \eta_1)| \le 2$, we have

$$|E_{aG_3St_4}(f)| \le \frac{76M}{10755}.$$

□

5. Algorithm for adaptive quadrature routine

Applying the constituent rules $R_{aG_3}(f)$, $R_{St_4}(f)$ and the mixed quadrature rule $(R_{aG_3St_4}(f))$, we have formed a new adaptive integration scheme to evaluate improper integrals of the type (1.2). In this adaptive integration scheme, the desired accuracy is sought by progressively subdividing the interval of integration according to the computed behavior of the integrand and applying the same formula over each subinterval. The algorithm for the adaptive integration scheme is outlined using the mixed quadrature rule $(R_{aG_3St_4}(f))$ in the following four steps.

Input: Function $f:[a,b]\to\mathbb{R}$ and the prescribed tolerance ε.

Output: An approximation $Q(f)$ to the integral $I(f)=\int_a^b f(x)dx$ such that $|Q(f)-I(f)|\le\varepsilon$.

Step 1: The mixed quadrature rule $(R_{aG_3St_4}(f))$ is applied to approximate the integral $I(f)=\int_a^b f(x)dx$. The approximated value is denoted by $(R_{aG_3St_4}[a,b])$.

Step 2: The interval of integration $[a,b]$ is divided into two equal pieces, $[a,c]$ and $[c,b]$. The mixed quadrature rule $(R_{aG_3St_4}(f))$ is applied to approximate the integral $I_1(f)=\int_a^c f(x)dx$ and the approximated value is denoted by $(R_{aG_3St_4}[a,c])$. Similarly, the mixed quadrature rule $(R_{aG_3St_4}(f))$ is applied to approximate the integral $I_2(f)=\int_c^b f(x)dx$ and the approximated value is denoted by $(R_{aG_3St_4}[c,b])$.

Step 3: $(R_{aG_3St_4}[a,c])+(R_{aG_3St_4}[c,b])$ is compared with $(R_{aG_3St_4}[a,b])$ to estimate the error in $(R_{aG_3St_4}[a,c])+(R_{aG_3St_4}[c,b])$.

Step 4: If $|\ \textit{estimated error}\ |\le\frac{\varepsilon}{2}$ (termination criterion) then $(R_{aG_3St_4}[a,c])+(R_{aG_3St_4}[c,b])$ is accepted

Table 2. Numerical approximation of some improper integrals using Gauss–Laguerre 2-point, 3-point, and 4-point quadrature rules $(R_{GLag_2}(f))$, $(R_{GLag_3}(f))$, $(R_{GLag_4}(f))$.

Integrals	Exact value $I(f)$	Approximate value $(Q(f))$		
		$R_{GLag_2}(f)$	$R_{GLag_3}(f)$	$R_{GLag_4}(f)$
$I_1(f)=\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$	1.77245385	1.94477	1.290822	1.350961
$I_2(f)=\int_0^\infty e^{-x}\cos x dx$	0.5	0.570208	0.476520	0.502493
$I_3(f)=\int_0^\infty e^{-x}\frac{\sin x}{x}dx$	0.7853981634	0.794019	0.781780	0.785921
$I_4(f)=\int_0^\infty \frac{e^{-x}}{1+x^2}dx$	0.62144962	0.647058	0.651006	0.636426
$I_5(f)=\int_0^\infty e^{-x}\sqrt{x}dx$	0.8862269254	0.923879	0.906440	0.8992802
$I_6(f)=\int_0^\infty e^{-x}\sin x dx$	0.5	0.432459	0.496029	0.504879
$I_7(f)=\int_0^\infty e^{-x-\frac{1}{x}}dx$	0.2797317636	0.264089	0.253158	0.260403
$I_8(f)=\int_0^\infty e^{-x}\log x dx$	–0.5772156649	–0.276651	–0.373671	–0.423307
$I_9(f)=\int_1^\infty \frac{e^{-x}}{x}dx$	0.21938393439	0.210216	0.216399	0.218263
$I_{10}(f)=\int_0^\infty e^{-x}\log(1+x)dx$	0.5963473623	0.611005	0.5999141	0.597446

as an approximation to $I(f) = \int_a^b f(x)dx$. Otherwise, the same procedure is applied to $[a, c]$ and $[c, b]$, allowing each piece a tolerance of $\frac{\varepsilon}{2}$. If the termination criterion is not satisfied on one or more of the subintervals, then those subintervals must be further subdivided and the entire process is repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q(f)$ of the integral $I(f)$ such that $|Q(f) - I(f)| \leq \varepsilon$.

N.B. In this algorithm we can use any quadrature rule to evaluate real definite integrals in the adaptive integration scheme.

6. Conclusion

(1) When results of the test integrals appearing in Tables 2 and 3 are compared with those of Table 4, one can smartly derive the conclusion that the adaptive integration scheme of mixed quadrature in evaluation of improper integrals using the mixed rule as the base rule is significantly a much better numerical quadrature tool than Gauss–Laguerre quadrature rules.

(2) When results of Table 4 are analyzed, one can derive the conclusion that the adaptive integration scheme having the mixed rule as the base rule is much better than the scheme having constituent rules (of mixed

Table 3. Comparative study of the quadrature/mixed quadrature rules $R_{St_4}(f)$, $R_{aG_3}(f)$, and $R_{aG_3St_4}(f)$ for approximating some improper integrals (as given in Table 2) in an adaptive environment.

Integrals	Approximate value ($Q(f)$) by					
	$R_{St_4}(f)$	#steps	$R_{aG_3}(f)$	# steps	$R_{aG_3St_4}(f)$	#steps
$I_1(f) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$	1.772452765	155	1.772455116	151	1.7724531660	81
$I_2(f) = \int_0^\infty e^{-x}\cos x dx$	0.499999881	71	0.5000001008	71	0.499999841	31
$I_3(f) = \int_0^\infty e^{-x}\frac{\sin x}{x}dx$	0.785397034	49	0.78539947	49	0.78539671	19
$I_4(f) = \int_0^\infty \frac{e^{-x}}{1+x^2}dx$	0.621449770	29	0.621449289	27	0.621449798	13
$I_5(f) = \int_0^\infty e^{-x}\sqrt{x}dx$	0.886226632	63	0.886227119	65	0.886226698	33
$I_6(f) = \int_0^\infty e^{-x}\sin x dx$	0.4999999	103	0.5000000116	111	0.4999999701	41
$I_7(f) = \int_0^\infty e^{-x-\frac{1}{x}}dx$	0.2797313121	37	0.2797321516	39	0.2797313572	17
$I_8(f) = \int_0^\infty e^{-x}\log x dx$	–0.577215394	97	–0.577215656	101	–0.577215475	49
$I_9(f) = \int_1^\infty \frac{e^{-x}}{x}dx$	0.219384251	23	0.219383642	23	0.2193842303	13
$I_{10}(f) = \int_0^\infty e^{-x}\log(1+x)dx$	0.596346775	43	0.596347895	43	0.596347132	23

N.B. The prescribed tolerance is ε=0.000001. # steps: No. of steps.
All the computations are done using the 'C' program [5]. It is important to note that the results are corrected up to five decimal places.

quadrature $R_{aG_3St_4}(f)$), Steffensen's-4-point rule ($R_{St_4}(f)$, and the anti-Gauss 3-point rule ($R_{aG_3}(f)$) as the base rules as far as the number of steps is concerned.

Table 4. Numerical approximation of some improper integrals (as given in Table 2) using Gauss–Laguerre 5-point and 6-point quadrature rules ($R_{GLag_5}(f)$), ($R_{GLag_6}(f)$).

Integrals	Approximate value ($Q(f)$)	
	$R_{GLag_5}(f)$	$R_{GLag_6}(f)$
$I_1(f) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$	1.393054	1.424628
$I_2(f) = \int_0^\infty e^{-x}\cos x dx$	0.500538	0.499737
$I_3(f) = \int_0^\infty e^{-x}\frac{\sin x}{x}dx$	0.785401	0.785379
$I_4(f) = \int_0^\infty \frac{e^{-x}}{1+x^2}dx$	0.626377	0.621717
$I_5(f) = \int_0^\infty e^{-x}\sqrt{x}dx$	0.895537	0.893295
$I_6(f) = \int_0^\infty e^{-x}\sin x dx$	0.498903	0.5000494
$I_7(f) = \int_0^\infty e^{-x-\frac{1}{x}}dx$	0.268896	0.275142
$I_8(f) = \int_0^\infty e^{-x}\log x dx$	–0.453474	–0.473752
$I_9(f) = \int_1^\infty \frac{e^{-x}}{x}dx$	0.218919	0.219176
$I_{10}(f) = \int_0^\infty e^{-x}\log(1+x)dx$	0.59674009	0.5965031

Finally, we conclude that the mixed rule adaptive integration scheme is a highly efficient scheme both theoretically and practically.

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