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Reduction formula of a double binomial sum

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Abstract: A class of double sums with binomial coefficients are evaluated by combining finite differences with partial fraction decompositions.

Key words: Binomial coefficient, finite difference, partial fraction decomposition, telescoping method

1. Introduction and motivation

There has been always a constant interest in finding closed formulae of binomial sums, including double ones (for example, Chu [8]). In the process of evaluating the quadratic moments of binomial products (cf. Chu [9] and Miana–Romero[12]), we encountered the following double sum with the closed formula being detected by *Mathematica* commands:

$$\sum_{j=0}^{n}\binom{2m-2j}{m-j}\sum_{i=0}^{j}\binom{2m}{i}\binom{2j-2m}{j-i}\frac{(m-i)^{2n+1}}{m-j}=0, \tag{1}$$

where m and n are natural numbers with $m>n$ in order to avoid the appearance of zero in denominators.

For an integer k and an indeterminate τ, define the rising and falling factorials respectively by

$$(\tau)_k=\frac{\Gamma(\tau+k)}{\Gamma(\tau)} \quad \text{and} \quad \langle\tau\rangle_k=\frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)}. \tag{2}$$

Writing the binomial coefficients in terms of shifted factorials

$$\binom{2m-2j}{m-j}\frac{1}{m-j}=\frac{(2m-2j)!}{(m-j)(m-j)!^2}=\frac{2\langle m\rangle_j^2}{\langle 2m\rangle_{2j+1}}\binom{2m}{m},$$

$$\binom{2m}{i}=\frac{\langle 2m\rangle_i}{i!},\ \binom{2j-2m}{j-i}=\frac{(-1)^{j-i}}{(j-i)!}\langle 2m-i-j-1\rangle_{j-i};$$

we can reformulate the following binomial product

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$$\begin{aligned}\frac{1}{m-j}\binom{2m-2j}{m-j}\binom{2m}{i}\binom{2j-2m}{j-i} &=2(-1)^{j-i}\binom{2m}{m}\frac{\langle m\rangle_j\langle m\rangle_j}{i!(j-i)!}\frac{\langle 2m\rangle_i\langle 2m-i-j-1\rangle_{j-i}}{\langle 2m\rangle_{2j+1}}\\ &=2(-1)^{j-i}\binom{2m}{m}\frac{\langle m\rangle_j\langle m\rangle_j}{i!(j-i)!\langle 2m-i\rangle_{j+1}}\\ &=2(-1)^{j-i}\binom{2m}{m}\binom{m}{j}\binom{j}{i}\frac{\langle m\rangle_j}{\langle 2m-i\rangle_{j+1}}\end{aligned}$$

that can be used to express the double sum in question equivalently as

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{\langle m\rangle_j}{\langle 2m-i\rangle_{j+1}}(m-i)^{2n+1}=0. \tag{3}$$

Replacing further the integer parameter m by an indeterminate $(-x)$ and introducing an extra integer parameter λ, we shall investigate the following double sum

$$\Omega(\lambda,n):=\sum_{j=0}^{n}\binom{x+j-1}{j}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{(x)_j}{(2x+i)_{j+1}}(x+i)^{2\lambda+1}. \tag{4}$$

It turns out that $\Omega(\lambda,n)$ is identical to zero for $0<\lambda\leq n$ and a polynomial in x when $\lambda>n\geq 0$. This will be accomplished by combining finite differences (cf. Boole [1, Chapter 2]) with partial fraction decompositions (cf. Chu [4]).

2. Main theorem and proof

Rewriting the binomial coefficients

$$\binom{x+j-1}{j}\binom{j}{i}=\binom{x+i-1}{i}\binom{x+j-1}{j-i}$$

and interchanging the order of double sums, we can state $\Omega(\lambda,n)$ equivalently as

$$\begin{aligned}\Omega(\lambda,n)=&\sum_{i=0}^{n}(-1)^i\binom{x+i-1}{i}(x+i)^{2\lambda+1}\\ &\times\sum_{j=i}^{n}\binom{x+j-1}{j-i}\frac{(x)_j}{(2x+i)_{j+1}}.\end{aligned} \tag{5}$$

For the sequence σ_j defined below, it is trivial to check its difference

$$\sigma_j=\frac{(x+i)_{j-i}(x)_j}{\Gamma(j-i)(2x+i)_j}\quad\text{and}\quad\sigma_{j+1}-\sigma_j=(x+i)^2\binom{x+j-1}{j-i}\frac{(x)_j}{(2x+i)_{j+1}}.$$

In view of the fact that "$\frac{1}{\Gamma(0)}=0$", the inner sum with respect to j can be evaluated by telescoping (cf. [7, 13])

$$\begin{aligned}\sum_{j=i}^{n}\binom{x+j-1}{j-i}\frac{(x)_j}{(2x+i)_{j+1}} &= \sum_{j=i}^{n}\frac{\sigma_{j+1}-\sigma_j}{(x+i)^2}=\frac{\sigma_{n+1}}{(x+i)^2}\\ &= \frac{(x+i)_{1+n-i}(x)_{n+1}}{(x+i)^2(n-i)!(2x+i)_{n+1}}.\end{aligned}$$

Substituting this equality into (5), we reduce it, after some simplification, to the following single sum.

Lemma 1 *For the two natural numbers* λ *and* n, *there holds the following identity:*

$$\Omega(\lambda,n)=\frac{(x)_{n+1}^2}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(x+i)^{2\lambda-1}}{(2x+i)_{n+1}}.$$

Now we are in a position to prove the following interesting theorem, which confirms, in particular for $\lambda=n$, the double sum identities (1) and (3).

Theorem 2 *Let* n *and* λ *be two natural numbers subject to* $0<\lambda\le n$. *Then, for the double sum defined by* (4), *we have the following identity* $\Omega(\lambda,n)=0$.

Proof For the rational function in the variable i, by decomposing it into partial fractions (cf. [2, 3])

$$\frac{(x+i)^\lambda}{(2x+i)_{n+1}}=\frac{(-1)^\lambda}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(x+j)^\lambda}{2x+i+j}, \tag{6}$$

we can express the sum in Lemma 1 as the following double sums

$$\Omega(\lambda,n)=\frac{(x)_{n+1}^2}{(n!)^2}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j+\lambda}\binom{n}{i}\binom{n}{j}\frac{(x+i)^{\lambda-1}(x+j)^\lambda}{2x+i+j}, \tag{7}$$

$$\Omega(\lambda,n)=\frac{(x)_{n+1}^2}{(n!)^2}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j+\lambda}\binom{n}{i}\binom{n}{j}\frac{(x+i)^{\lambda}(x+j)^{\lambda-1}}{2x+i+j}, \tag{8}$$

where the last one is justified by interchanging the summation indices i and j. Adding these two equalities together, we derive the following symmetric expression

$$\begin{aligned}2\Omega(\lambda,n)&=\frac{(x)_{n+1}^2}{(n!)^2}\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j+\lambda}\binom{n}{i}\binom{n}{j}(x+i)^{\lambda-1}(x+j)^{\lambda-1}\\&=(-1)^\lambda\frac{(x)_{n+1}^2}{(n!)^2}\left\{\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(x+i)^{\lambda-1}\right\}^2.\end{aligned}$$

The rightmost sum vanishes because it results in the nth differences of a polynomial with degree $\lambda-1$ less than n. This completes the proof of Theorem 2. □

3. Convolution expression

When $\lambda=0$, the last sum can be evaluated by (6) as

$$\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i}{x+i}=\frac{n!}{(x)_{n+1}}.$$

From this formula, we can retrieve the respective particular case $\lambda=0$ for both Lemma 1 and Theorem 2

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{(x+i)(2x+i)_{n+1}}=\frac{n!}{2(x)_{n+1}^2}, \tag{9}$$

$$\sum_{j=0}^{n}\binom{x+j-1}{j}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{(x+i)(x)_j}{(2x+i)_{j+1}}=\frac{1}{2}. \tag{10}$$

When $\lambda > n$, we need the following equality, instead of (6)

$$\frac{(x+i)^n}{(2x+i)_{n+1}} = \frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(x+j)^n}{2x+i+j}.$$

Substituting this into the equation displayed in Lemma 1, we have

$$\begin{aligned} 2\Omega(\lambda,n) &= (-1)^n\frac{(x)_{n+1}^2}{(n!)^2}\sum_{i,j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j} \\ &\times \frac{(x+i)^{2\lambda-n-1}(x+j)^n+(x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j}. \end{aligned}$$

Rewriting the last fraction by

$$\begin{aligned} &\frac{(x+i)^{2\lambda-n-1}(x+j)^n+(x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j} \\ =&(x+i)^n(x+j)^n\frac{(x+i)^{2\lambda-2n-1}+(x+j)^{2\lambda-2n-1}}{(x+i)+(x+j)} \\ =&\sum_{k=1+n-\lambda}^{\lambda-n-1}(-1)^{\lambda-n-k-1}(x+i)^{\lambda+k-1}(x+j)^{\lambda-k-1}, \end{aligned}$$

we derive the following polynomial expression.

Proposition 3 *Let n and λ be two natural numbers subject to $\lambda > n$. Then for the double sum defined by (4), we have the following convolution formula*

$$\Omega(\lambda,n) = \frac{(x)_{n+1}^2}{2(n!)^2}\sum_{k=1+n-\lambda}^{\lambda-n-1}(-1)^{\lambda-k-1}P_{\lambda+k-1,n}(x)P_{\lambda-k-1,n}(x),$$

where $P_{m,n}(x)$ is a polynomial defined by

$$P_{m,n}(x) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(x+i)^m \quad \textit{for} \quad m,n\in\mathbb{N} \quad \textit{with} \quad m>n.$$

Observing that $P_{m,n}(x)$ is essentially the nth differences of a monomial with the degree m such that $m > n$, we assert (cf. [5, 6]) that $P_{m,n}(x)$ results in a polynomial of degree $m-n$ in x. In particular, we have $P_{m,n}(0) = n!\,S(m,n)$, where $S(m,n)$ is the Stirling number of the second kind (cf. [10, §5.1] and [11, §6.1]).

According to Proposition 3, we can evaluate the double sum $\Omega(\lambda,n)$ for $\lambda > n$. The first three examples are displayed below:

$$\Omega(1+n,n) = (-1)^n \frac{(x)_{n+1}^2}{2}, \tag{11}$$

$$\Omega(2+n,n) = (-1)^n \frac{(n+1)(x)_{n+1}^2}{12}\big\{2n^2+6xn+n+6x^2\big\}, \tag{12}$$

$$\Omega(3+n,n) = (-1)^n \frac{(n+1)(n+2)(x)_{n+1}^2}{720}\Big\{180x^2(n+x)^2 \tag{13}$$

$$+120nx(n+1)(n+x)+n(2n+1)(2n+3)(5n-1)\Big\}.$$

In view of Lemma 1, they correspond to the following binomial identities:

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(x+i)^{2n+1}}{(2x+i)_{n+1}} = (-1)^n\frac{n!}{2}, \tag{14}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(x+i)^{2n+3}}{(2x+i)_{n+1}} = (-1)^n\frac{(n+1)!}{12}\big\{2n^2+6xn+n+6x^2\big\}, \tag{15}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(x+i)^{2n+5}}{(2x+i)_{n+1}} = (-1)^n\frac{(n+2)!}{720}\Big\{120nx(n+1)(n+x) \tag{16}$$

$$+\,180x^2(n+x)^2+n(2n+1)(2n+3)(5n-1)\Big\}. \tag{17}$$

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