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On Oscillation of Two-Dimensional Time-Scale Systems with a Forcing Term

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Abstract: The oscillation and nonoscillation theories for nonlinear systems have recently received a lot of attention. We consider a two-dimensional time-scale system and find the oscillation criteria for solutions of the system by using some improper integrals and inequalities. We also give a few examples in order to highlight our main results.

Key words: Oscillation theory, time-scale systems, forcing term, nonlinear systems, nonoscillation

1. Introduction

In this paper, motivated by [22], we deal with the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t)) + c(t), \end{cases} \tag{1}$$

where $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $c \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, and f and g are nondecreasing functions such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$ and g is continuously differentiable. A time scale $\mathbb{T}$, a nonempty closed subset of real numbers, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to harmonize discrete and continuous analyses to combine them in one comprehensive theory and eliminate obscurity from both. The time-scale theory was published in a series of two books by Bohner and Peterson in 2001 and 2003; see [3, 4]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above and whenever we write $t \geq t_1$ we mean $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. Some oscillation and nonoscillation results for the nonlinear equation

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + b(t)g(x^{\sigma}(t)) = c(t) \tag{2}$$

and the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x^{\sigma}(t)) + c(t) \end{cases} \tag{3}$$

and for some variations of systems (1) and (3) are shown in [14, 15, 18, 22]. A solution (x, y) of system (1) is called oscillatory if x and y have arbitrarily large zeros. System (1) is called oscillatory if all solutions are oscillatory.

The set up of this paper is as follows: in Section 1, we give the preliminary lemmas and the time-scale calculus used in our main results. In Section 2, we give our main results by using convergence/divergence of

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some certain improper integrals. In Section 3, we give an example in order to emphasize one of our main results. Finally, we finish the last section by giving open problems and applications.

We give the following preliminaries in order to use them in our proofs. One can find details in [3, 16].

Proposition 1 (Comparison Theorem) *[16, Theorem 4.2] Suppose* $h:\mathbb{R}\to\mathbb{R}$ *is nondecreasing and* $z_1:\mathbb{T}\to\mathbb{R}$ *is such that* $h\circ z_1$ *is rd-continuous. Let* $p\geq 0$ *be rd-continuous and* $\alpha\in\mathbb{R}$*. Then*

$$z_1(t)\leq \alpha+\int_{t_0}^{t}p(\tau)h(z_1(\tau))\Delta\tau,\quad t\geq t_0$$

implies $z_1(t)\leq z_2(t)$*, where* z_2 *solves the initial value problem*

$$z_2^{\Delta}(t)=p(t)h(z_2(t)),\quad z_2(t_0)=z_{20}>\alpha.$$

Proposition 2 (Quotient Rule) *[3, Theorem 1.20 v] Assume* $h_1,h_2:\mathbb{T}\to\mathbb{R}$ *are differentiable at* $t\in\mathbb{T}^{\kappa}$ *and* $h_2(t)h_2(\sigma(t))\neq 0$*. Then* $\frac{h_1}{h_2}$ *is differentiable at* t *and*

$$\left(\frac{h_1}{h_2}\right)(t)=\frac{h_1^{\Delta}(t)h_2(t)-h_1(t)h_2^{\Delta}(t)}{h_2(t)h_2(\sigma(t))}.$$

Proposition 3 (Chain Rule) *[3, Theorem 1.90] Let* $h_1:\mathbb{R}\to\mathbb{R}$ *be continuously differentiable and suppose* $h_2:\mathbb{T}\to\mathbb{R}$ *is delta differentiable. Then* $h_1\circ h_2:\mathbb{T}\to\mathbb{R}$ *is delta differentiable and the formula*

$$(h_1\circ h_2)^{\Delta}(t)=\left\{\int_0^1 h_1'(h_2(t)+h\mu(t)h_2^{\Delta}(t))dh\right\}h_2^{\Delta}(t)$$

holds.

Proposition 4 (Integration by parts) *[3, Theorem 1.77 vi] If* $a,b\in\mathbb{T}$ *and* $h_1,h_2\in\mathrm{C}_{\mathrm{rd}}$*, then*

$$\int_a^b h_1(t)h_2^{\Delta}(t)\Delta t=(h_1h_2)(b)-(h_1h_2)(a)-\int_a^b h_1^{\Delta}(t)h_2(\sigma(t))\Delta t$$

holds.

For the sake of simplicity in our proofs, let us set

$$A(t,s)=\int_t^s a(u)\Delta u,\quad B(t,s)=\int_t^s b(u)\Delta u,$$

$$C(t,s)=\int_t^s |c(u)|\Delta u,\quad D(t,s)=\int_t^s\left(b(u)-\frac{c(u)}{g(x(u))}\right)\Delta u,$$

$$I(t,s)=\int_t^s\frac{y^{\sigma}(u)x^{\Delta}(u)\int_0^1\left[g'\left(x(u)+h\mu(u)x^{\Delta}(u)\right)dh\right]}{g(x(u))g(x^{\sigma}(u))}\Delta u.$$

Since the following lemma was proved by Anderson [1] for the component functions x and y in the case $c(t)\equiv 0$, we skip the proof because they are very similar.

Lemma 1.1 *Let* (x, y) *be a nonoscillatory solution of system* (1). *Then the component function* x *is also nonoscillatory.*

Lemma 1.2 *Suppose that* (x, y) *is a nonoscillatory solution of system* (1) *and* $t_1, t_2 \in \mathbb{T}$. *If there exists a constant* $K > 0$ *such that*

$$H(t) \geq K, \quad t \geq t_2, \tag{4}$$

where H *is defined as*

$$H(t) = -\frac{y(t_1)}{g(x(t_1)} + D(t_1, t) + I(t_1, t_2), \tag{5}$$

then $y(t) \leq -Kg(x(t_2)), \quad t \geq t_2$.

Proof Suppose that (x, y) is a nonoscillatory solution of system (1). Then by Lemma 1.1 we have that x is also nonoscillatory. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$, where $t_1, t_0 \in \mathbb{T}$. Integrating the second equation of system (1) from t_1 to t and Proposition 4 give us

$$\int_{t_1}^{t} b(s)\Delta s = \frac{y(t_1)}{g(x(t_1))} - \frac{y(t)}{g(x(t))} + \int_{t_1}^{t} \left(\frac{1}{g(x(s))}\right)^{\Delta} y^{\sigma}(s)\Delta s + \int_{t_1}^{t} \frac{c(s)}{g(x(s))}\Delta s. \tag{6}$$

By applying Propositions 2 and 3 for equation (6), we have

$$\int_{t_1}^{t} b(s)\Delta s = \frac{y(t_1)}{g(x(t_1))} - \frac{y(t)}{g(x(t))} + \int_{t_1}^{t} \frac{c(s)}{g(x(s))}\Delta s - I(t_1, t), \quad t \geq t_1. \tag{7}$$

Rewriting equation (7) gives us

$$-\frac{y(t)}{g(x(t))} = D(t_1, t) - \frac{y(t_1)}{g(x(t_1))} + I(t_1, t), \quad t \geq t_1. \tag{8}$$

Now by using (4) and (5), we get

$$-\frac{y(t)}{g(x(t))} \geq K + I(t_2, t), \quad t \geq t_2 \geq t_1. \tag{9}$$

Note that $y(t) < 0$ and $x^{\Delta}(t) < 0$ for $t \geq t_2$ since $y(t)x^{\Delta}(t) = a(s)y(s)f(y(s)) > 0$. Otherwise, we would have $\frac{-y(t)}{g(x(t))} > 0$, which is a contradiction. Let

$$\frac{-v(t)}{g(x(t))} = K + I(t_2, t), \quad t \geq t_2. \tag{10}$$

Then we have

$$\left(\frac{-v(t)}{g(x(t))}\right)^{\Delta} = \frac{y^{\sigma}(t)x^{\Delta}(t)\int_0^1 \left[g'\left(x(t) + h\mu(t)x^{\Delta}(t)\right)dh\right]}{g(x(t)g(x^{\sigma}(t))} > 0, \quad t \geq t_2. \tag{11}$$

Since $x(t) > 0$ and $v(t) < 0$ for $t \geq t_2$, it follows that $\frac{-y(t)}{g(x(t))} \geq \frac{-v(t)}{g(x(t))}$, i.e. $y(t) \leq v(t) < 0$ for $t \geq t_2$. Therefore, we have by (11) that

$$\left(\frac{-v(t)}{g(x(t))}\right)^{\Delta} \geq \frac{v^{\sigma}(t)x^{\Delta}(t)\int_0^1 \left[g'\left(x(t) + h\mu(t)x^{\Delta}(t)\right)dh\right]}{g(x(t)g(x^{\sigma}(t))} > 0, \quad t \geq t_2$$

since $v(t) < 0$ and $x^{\Delta}(t) < 0$ for $t \geq t_2$. By setting

$$\frac{w(t)}{g(x(t))} = K - \int_{t_2}^{t} \frac{w^{\sigma}(s)x^{\Delta}(s)\int_0^1 \left[g'\left(x(s) + h\mu(s)x^{\Delta}(s)\right)dh\right]}{g(x(s)g(x^{\sigma}(s))}\Delta s \tag{12}$$

and using (10), we have $\frac{-v(t_2)}{g(x(t_2))} = K = \frac{w(t_2)}{g(x(t_2))}$. Then setting $z_1 = \frac{v(t)}{g(x(t))}, z_2 = \frac{-w(t)}{g(x(t))}, h(u) = \frac{u^{\sigma}(t)}{g(x(t))}$ in Proposition 1, we have $v(t) \leq -w(t)$ and therefore $y(t) \leq -w(t)$ for $t \geq t_2$. We also have by Propositions 2 and 3 that

$$\left(\frac{w(t)}{g(x(t))}\right)^{\Delta} = \frac{w^{\Delta}(t)}{g(x^{\sigma}(t))} - \frac{w^{\sigma}(t)x^{\Delta}(t)\int_0^1 g'(x(t) + h\mu(t)x^{\Delta}(t))dh}{g(x(t))g(x^{\sigma}(t))}, \quad t \geq t_2. \tag{13}$$

Taking the derivative of (12) and comparing the resulting equation with (13) yield us

$$\frac{w^{\Delta}(t)}{g(x^{\sigma}(t))} = 0, \quad i.e.\ w^{\Delta}(t) = 0, \quad t \geq t_2.$$

Therefore, we have

$$w(t_2) = K \cdot g(x(t_2)) = w(t), \quad i.e.\ y(t) \leq -w(t) = -K \cdot g(x(t_2)).$$

This proves the assertion. □

2. Oscillation results

In this section, we give the oscillation criteria of system (1) by using our convergence/divergence of $A(t_0, \infty)$, $B(t_0, \infty)$, and $C(t_0, \infty)$.

Theorem 2.1 *Suppose that* $A(t_0, \infty) = \infty, B(t_0, \infty) < \infty, C(t_0, \infty) < \infty$. *Suppose also that*

$$f(u)f(v) \leq f(uv) \leq -f(u)f(-v) \quad \text{and} \tag{14}$$

$$\int_{t_0}^{\infty} \frac{x^{\Delta}(s)}{f(g(x(s)))}\Delta s < \infty. \tag{15}$$

Then system (1) *is oscillatory if*

$$\int_{t_0}^{\infty} a(t) f\left(B(t, \infty) - k \cdot C(t, \infty)\right)\Delta t = \infty \tag{16}$$

for $k \neq 0$.

Proof Suppose that system (1) has a nonoscillatory solution (x, y) such that $x > 0$ eventually. Then there exist $t_1 \geq t_0$ and a constant k_1 such that $g(x(t)) \geq k_1$ for $t \geq t_1$ by the monotonicity of g. Then by equation (8), we have

$$\frac{y(t)}{g(x(t))} = \frac{y(t_1)}{g(x(t_1))} - D(t_1, t) - I(t_1, t), \quad t \geq t_1. \tag{17}$$

Note that $I(t_1, t) < \infty$. Otherwise, we have a contradiction to $x(t) > 0$ for $t \geq t_1$ by Lemma 1.1 since $A(t_0, \infty) = \infty$. Equality (17) can be rewritten as

$$\frac{y(t)}{g(x(t))} = \gamma + D(t, \infty) + I(t, \infty), \tag{18}$$

where $\gamma = \frac{y(t_1)}{g(x(t_1))} - D(t_1, \infty) - I(t_1, \infty), \quad t \geq t_1$. It can be shown that $\gamma \geq 0$. Otherwise, we can choose a large t_2 such that $B(t, \infty) \leq -\gamma$, $I(t_2, \infty) \leq \frac{-\gamma}{4}$, and $\left| \int_t^\infty \frac{c(s)}{g(x(s))} \Delta s \right| \leq \frac{-\gamma}{4}$ for $t \geq t_2$. Then $H(t) \geq \frac{-\gamma}{4} > 0$ for $t \geq t_2$. Then by setting $K = \frac{-\gamma}{4}$ in Lemma 1.1, we have $y(t) \leq -Kg(x(t_2))$ for $t \geq t_2$. Integrating the first equation of system (1) from t_2 to ∞ and the monotonicity of f yield us

$$x(t) \leq x(t_2) + f(-Kg(x(t_2))) \int_{t_2}^t a(s) \Delta s, \quad t \geq t_2.$$

Thus as $t \to \infty$, we have a contradiction to $x > 0$ eventually. Therefore, $\gamma \geq 0$. Then, by equation (18), we have

$$y(t) \geq g(x(t)) \left[\int_t^\infty b(s) \Delta s - \frac{1}{k_1} \int_t^\infty |c(s)| \Delta s \right], \quad t \geq t_2.$$

By the first equation of system (1), the monotonicity of f and equation (14), we have

$$x^\Delta(t) \geq a(t) f\left(g(x(t))\right) f\left(\int_t^\infty b(s) \Delta s - \frac{1}{k_1} \int_t^\infty |c(s)| \Delta s \right), \quad t \geq t_2. \tag{19}$$

Then, by (19) and (15), we have

$$\int_{t_2}^t a(s) f\left(\int_s^\infty b(u) \Delta u - k \int_s^\infty |c(u)| \Delta u \right) \leq \int_{t_2}^t \frac{x^\Delta(s)}{f(g(x(s))} \Delta s < \infty,$$

where $k = \frac{1}{k_1}$. However, this is contradiction to (16) as $t \to \infty$. This completes the proof. (For $x < 0$ eventually, k can be considered a negative number and the proof can be shown similarly.) □

Theorem 2.2 *Suppose that $A(t_0, \infty) = \infty, B(t_0, \infty) = \infty$, and $C(t_0, \infty) < \infty$. Then system (1) is oscillatory.*

Proof Proof is by a contradiction. Hence assume that there exists a nonoscillatory solution (x, y) of system (1) such that $x > 0$ eventually. The case $x < 0$ eventually can be shown similarly. By the monotonicity of g, there exist $t_1 \geq t_0$ and $k_1 > 0$ such that $g(x(t)) \geq k_1$ for $t \geq t_1$. Then, since $C(t_0, \infty) < \infty$, we have that there exists $0 < k_2 < \infty$ such that

$$\left| \int_{t_1}^t \frac{c(s)}{g(x(s))} \Delta s \right| \leq \frac{1}{k_1} \int_{t_1}^t |c(s)| \Delta s < k_2, \quad t \geq t_1. \tag{20}$$

The first equation of system (1), Lemma 1.1 and the monotonicity of g give us that there exist $K > 0$ and $t_2 \geq t_1$ so large that

$$x^\Delta(t) \leq a(t) f(-Kg(x(t_2))), \quad t \geq t_2. \tag{21}$$

Integrating (21) from t_2 to t yields

$$x(t) \le x(t_2) + k_3 \int_{t_2}^{t} a(s)\Delta s, \quad \text{where} \quad k_3 = f(-Kg(x(t_2))) < 0, \quad t \ge t_2.$$

As $t \to \infty$, we have a contradiction to $x(t) > 0$ for $t \ge t_2$. This proves the assertion. □

3. Examples

In this section, we give an example in one of the best-known time scales for Theorem 2.2. We not only focus on showing the result of Theorem 2.2 but w ealso solve our dynamical system explicitly.

Example 1 *Let* $\mathbb{T} = 5\mathbb{Z}^+$, $a(t) = \frac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)}$, $b(t) = \frac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)}$, $f(z) = z^{\frac{1}{3}}$, $g(z) = z^3$, $c(t) = \frac{(-1)^{3t}(-3t^5-27t^4-125t^3-237t^2-195t-59)}{5(t+1)^4(t+4)(t+6)(t+9)}$, *and* $t = 5n$, *where* $n \in \mathbb{N}$. *We show that* $A(t_0,\infty) = \infty, B(t_0,\infty) = \infty$, *and* $C(t_0,\infty) < \infty$. *Indeed,*

$$A(5,T) = \int_5^T \frac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)}\Delta t = \sum_{t\in[5,T)_{5\mathbb{Z}+}} \frac{(t+4)^{\frac{1}{3}}(2t+7)}{(t+1)^{\frac{2}{3}}(t+6)}.$$

Thus, as $T \to \infty$, *we have*

$$\sum_{n=1}^{\infty} \frac{(5n+4)^{\frac{1}{3}}(10n+7)}{(5n+1)^{\frac{2}{3}}(5n+6)} = \infty \quad \textit{by the limit comparison test. Therefore, } A(5,\infty) = \infty.$$

Similarly,

$$B(5,T) = \int_5^T \frac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)}\Delta t = \sum_{t\in[5,T)_{5\mathbb{Z}+}} \frac{t^5+t^4+t^3+t^2+t+1}{(t+1)(t+4)(t+6)(t+9)}$$
$$\ge \sum_{t\in[5,T)_{5\mathbb{Z}+}} \frac{t^5}{(t+1)(t+4)(t+6)(t+9)}.$$

Taking the limit as $T \to \infty$ *gives us*

$$B(5,\infty) \ge 625 \cdot \sum_{n=1}^{\infty} \frac{n^5}{(5n+1)(5n+4)(5n+6)(5n+9)} = \infty$$

by the limit divergence test. Therefore, $B(5,\infty) = \infty$ *by the comparison test. Finally, we show* $C(t_0,\infty) < \infty$.

$$C(5,T) = \sum_{t\in[5,T)_{5\mathbb{Z}+}} \frac{3t^5+27t^4+125t^3+237t^2+195t+59}{(t+1)^4(t+4)(t+6)(t+9)}$$
$$\le \sum_{t\in[5,T)_{5\mathbb{Z}+}} \frac{3}{t^2} + \frac{27}{t^3} + \frac{125}{t^5} + \frac{195}{t^6} + \frac{59}{t^7}.$$

Hence, as $T \to \infty$*, we have*

$$C(5,\infty) \leq \sum_{n=1}^{\infty} \frac{3}{n^2} + \frac{27}{n^3} + \frac{125}{n^5} + \frac{195}{n^6} + \frac{59}{n^7} < \infty$$

by the geometric series. One can also show that $\left(\frac{(-1)^{t+1}}{t+1}, \frac{(-1)^{3t}}{(t+1)(t+4)}\right)$ *is an oscillatory solution of system*

$$\begin{cases} x^{\Delta}(t) = \frac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)} y^{\frac{1}{3}}(t) \\ y^{\Delta}(t) = -\frac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)} x^3(t) + \frac{(-1)^{3t}(-3t^5-27t^4-125t^3-237t^2-195t-59)}{5(t+1)^4(t+4)(t+6)(t+9)}, \end{cases}$$

where we define $h^{\Delta}(t) = \dfrac{h(\sigma(t)) - h(t)}{\mu(t)}$ *for* $\sigma(t) = t+5$ *and* $\mu(t) = 5$; *see [3].*

4. Open problems and applications

This paper deals with a very general nonlinear system and investigates the oscillation criteria. One can also consider

$$\begin{cases} x^{\Delta}(t) = a(t)^{\alpha} \left|y(t)\right|^{\frac{1}{\alpha}} sgny(t) \\ y^{\Delta}(t) = -b(t) \left|x^{\sigma}(t)\right|^{\beta} sgnx^{\sigma}(t), \end{cases} \tag{22}$$

where $\alpha, \beta > 0$ and $a, b \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and find the oscillation criteria. Note that system (22) is a special case of system (3). By intuition, we can relax the monotonicity conditions on f and g. Since we waive the strict assumptions on f and g, the results might be very interesting. System (22) is referred to as an Emden–Fowler dynamic system and it has several applications such as in astrophysics, gas dynamics, and fluid mechanics (see [19]), relativistic mechanics, nuclear physics, and chemically reacting systems (see [2, 7, 13, 20]). For example, the fundamental problem in studying the stellar structure for gaseous dynamics in astrophysics was to look into the equilibrium formation of the mass of spherical clouds of gas for the continuous case, proposed by Kelvin and Lane; see [12, 21]. They considered the equation

$$\frac{1}{t^2}\frac{d}{dt}\left(t^2\frac{du}{dt}\right) + u^n = 0 \tag{23}$$

for $n = 1.5$ and $n = 2.5$. This equation is referred to as the Lane–Emden equation; see [5, 6]. Note that it is a very special case of equation (2) for $\mathbb{T} = \mathbb{R}$. At that time, astrophysicists were interested in equation (23) for initial conditions $u(0) = 1$ and $u'(0) = 0$. Special cases of (23) have explicit solutions when $n = 0, 1, 5$, namely

$$u_1(t) = \frac{sint}{t} \quad \text{and} \quad u_5(t) = \frac{1}{\sqrt{1+\frac{1}{3t^2}}} \quad \text{for} \quad n = 1 \text{ and } 5 \quad \text{respectively.}$$

Note that u_1 is an oscillatory solution while u_5 is a nonoscillatory one. Much information about the solutions of equation (23) was provided by Ritter (see [17]), in a series of eighteen papers published during 1878–1889. The mathematical foundation for the study of such an equation was made by Fowler in a series of four papers during 1914–1931; see [8–11].

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