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Regularity and projective dimension of the edge ideal of a generalized theta graph

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Abstract: Let $k \geq 3$ and $G = \theta_{n_1,\ldots,n_k}$ be a graph consisting of k paths that have common endpoints. In this paper, we show that the projective dimension of $R/I(G)$ equals $bightI(G)$ or $bightI(G)+1$. For some special cases, we explain $depth(R/I(G))$ in terms of invariants of graphs. Moreover, we prove the regularity of $R/I(G)$ equals c_G or c_G+1, where c_G is the maximum number of 3-disjoint edges in G.

Key words: Big height, projective dimension, regularity, depth

1. Introduction

Given a simple graph G with the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$, we can associate to G the square-free monomial ideal $I(G)$ in polynomial ring $R = k[x_1, \ldots, x_n]$, which is generated by x_ix_j such that $\{x_i, x_j\} \in E(G)$. Recently, one of the most important research topics is to establish a dictionary between algebraic properties of $I(G)$, most notably, the projective dimension and the Castelnuovo–Mumford regularity of $R/I(G)$, and combinatorial invariants of G. Let I be a monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$. Then we can associate to R/I a minimal graded free resolution of the form

$$0 \to \oplus_j R(-j)^{\beta_{l,j}} \to \oplus_j R(-j)^{\beta_{l-1,j}} \to \cdots \to \oplus_j R(-j)^{\beta_{1,j}} \to R \to R/I \to 0,$$

where $l \leq n$ and $R(-j)$ is the R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}$ is called the ijth *graded Betti number* of R/I.

The *regularity* of R/I, denoted by $reg(R/I)$, is defined by

$$reg(R/I) := max\{j - i|\beta_{i,j}(R/I) \neq 0\}.$$

The *projective dimension* of R/I, denoted by $pd(R/I)$, is defined by

$$pd(R/I) := max\{i|\beta_{i,j}(R/I) \neq 0 \; for \; some \; j\}.$$

In [11], Zheng explained the regularity and projective dimension of tree graphs. He proved that if G is a tree, then $reg(R/I(G)) = c_G$, where c_G is the maximum number of pairwise 3-disjoint edges in G. In [2], Hà and Van Tuyl extended it to chordal graphs. In [4], Kimura described the projective dimension of chordal

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graphs, which actually extended Zheng' s work. Furthermore, Van Tuyl in [9] showed that if G is a sequentially Cohen–Macaulay bipartite graph, then the relation $reg(R/I(G)) = c_G$ is satisfied.

Recall that if I is a squarefree monomial ideal, then the inequality

$$ht(I) \leq bight(I) \leq pd(R/I) \leq ara(I) \leq \mu(I)$$

holds in general, where $\mu(I)$ is the minimum number of generators of the ideal I.

In [5], Khosh-Ahang and Moradi considered the class of C_5-free vertex decomposable graphs that contains forest graphs and sequentially Cohen–Macaulay bipartite graphs. For the class of graphs, they proved that $reg(R/I(G)) = c_G$ and $pd(R/I(G)) = bightI(G)$. In [6], Mohammadi and Kiani investigated the graphs consisting of some cycles and lines that have a common vertex. It is shown that the projective dimension equals the arithmetical rank for all such graphs. A graph G is called an n-*cyclic graph* with a common edge if G is a graph consisting of n cycles $C_{3r_1+1}, \ldots, C_{3r_{k_1}+1}, C_{3t_1+2}, \ldots, C_{3t_{k_2}+2}, C_{3s_1}, \ldots, C_{3s_{k_3}}$ connected through a common edge, where $k_1 + k_2 + k_3 = n$; see [12, Definition 2.4]. In [12], Zhu et al. proved that $pd(R/I(G)) = bightI(G) = ara(I(G))$ for some special n-cyclic graphs with a common edge.

Motivated by the above-mentioned works, we look for the equalities $pd(R/I(G)) = bightI(G)$ and $reg(R/I(G)) = c_G$ in the case of the graphs $G = \theta_{n_1,\ldots,n_k}$, by combining combinatorial methods with homological techniques.

Suppose that $min\{n_1, \ldots, n_k\} = n_t$. One can consider the graph $\theta_{n_1,\ldots,n_k}$ as $k-1$-cyclic graph with common path L_{n_t} consisting of $k-1$ cycles of lengths $n_i + n_t - 2$ for any $1 \leq i \neq t \leq k$, which generalizes the concept n-cyclic graphs with a common edge. For this class of graphs we describe the projective dimension and depth of $R/I(G)$ and show that $pd(R/I(G)) = bightI(G)$ unless $n_i \equiv 0 \pmod 3$ for any $1 \leq i \leq k$ or there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$ and for any $1 \leq i \neq j \leq k$ we have $n_i \equiv 2 \pmod 3$; then it yields $pd(G) = bightI(G) + 1$. Moreover, we deduce that $reg(R/I(G) = c_G$ for this class of graphs unless $n_i \equiv 2 \pmod 3$ for any $1 \leq i \leq k$ or there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$ and for any $1 \leq i \neq j \leq k$ we have $n_i \equiv 0 \pmod 3$; then it yields $reg(G) = c_G + 1$.

2. Projective dimension and depth

Let k be an integer number and $n_1, \ldots, n_k$ be a sequence of positive integers. Let $\theta_{n_1,\ldots,n_k}$ be the graph constructed by k paths with $n_1, \ldots, n_k$ vertices such that only their endpoints are in common. Since the graphs are assumed simple, then at most one of $n_1, \ldots, n_k$ can be equal to 2. If $k = 1$ or 2, then $\theta_{n_1,\ldots,n_k}$ would be a path or a cycle and homological properties of these graphs are completely studied in [3]; hence in this paper we suppose that $k \geq 3$.

We present the following theorem of Terai that plays a fundamental role in the study of the projective dimension and regularity of the graph $\theta_{n_1,\ldots,n_k}$.

Theorem 2.1 (see [8]) *Let I be a square-free monomial ideal. Then $pd(I^\vee) = reg(R/I)$.*

The following lemma is frequently needed in the sequel:

Lemma 2.2 ([5], Corollary 2.2) *Suppose that G is a graph, $x \in V(G)$ and $|N_G(x)| = t$. Let $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$. Then*

1. $pd(I(G)^{\vee}) \leq max\{pd(I(G')^{\vee}), pd(I(G'')^{\vee}) + 1\}$;
2. $reg(I(G)^{\vee}) \leq max\{reg(I(G')^{\vee}) + 1, reg(I(G'')^{\vee}) + t\}$.

The *big height* of $I(G)$, denoted by $bightI(G)$, is the maximum size of a minimal vertex cover of G.

Lemma 2.3 *For any graph G, the following relations are satisfied:*

1. *(See [2, Theorem 6.5].)* $c_G \leq reg(R/I(G))$.
2. *(See [7, Corollary 3.33].)* $bightI(G) \leq pd(R/I(G))$.

Let G be a finite simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Let e and e' be two distinct edges of G. The distance between e and e' in G, denoted by $dist_G(e, e')$, is defined by the minimum length l among sequences $e_0 = e, e_1, \dots, e_l = e'$ with $e_{i-1} \cap e_i \neq \phi$, where $e_i \in E_G$. If there is no such sequence, we define $dist_G(e, e') = \infty$. We say that e and e' are 3-disjoint in G if $dist_G(e, e') \geq 3$. A subset $E \subset E_G$ is said to be *pairwise 3-disjoint* if every pair of distinct edges $e, e' \in E$ are 3-disjoint in G; see [2, Definitions 2.2 and 6.3].

The graph B with $V(B) = \{w, z_1, \dots, z_d\}$ and $E(B) = \{\{w, z_i\} : i = 1, \dots, d\}$ $(d \geq 1)$ is called a bouquet. Then the vertex w is called the root of B, the vertices z_i flowers of B, and the edges $\{w, z_i\}$ stems of B; see [11, Definition 1.7]. Let $\mathbf{B} = \{B_1, B_2, \dots, B_j\}$ be a set of bouquets of G. We set

$$F(\mathbf{B}) := \{z \in V_G : z \; is \; a \; flower \; of \; some \; bouquet \; in \; \mathbf{B}\},$$

$$R(\mathbf{B}) := \{w \in V_G : w \; is \; a \; root \; of \; some \; bouquet \; in \; \mathbf{B}\},$$

$$S(\mathbf{B}) := \{s \in E_G : s \; is \; a \; stem \; of \; some \; bouquet \; in \; \mathbf{B}\}.$$

The type of $\mathbf{B}$ is defined by $(|F(\mathbf{B})|, |R(\mathbf{B})|)$; see [4].

Definition 2.4 ([4], Definition 2.1) *A set $\mathbf{B} = \{B_1, B_2, \dots, B_j\}$ of bouquets of G is said to be strongly disjoint in G if the following conditions are satisfied:*

1. *$V(B_k) \cap V(B_l) = \phi$ for all $k \neq l$.*
2. *For any $1 \leq k \leq j$, there exists a stem s_k in B_k such that $\{s_1, s_2, \dots, s_j\}$ are pairwise 3-disjoint in G.*

Definition 2.5 ([4], Definition 5.1) *A set $\mathbf{B} = \{B_1, B_2, \dots, B_j\}$ of bouquets of G is said to be semistrongly disjoint in G if the following conditions are satisfied:*

1. *$V(B_k) \cap V(B_l) = \phi$ for all $k \neq l$.*
2. *Any two vertices belonging to $R(\mathbf{B})$ are not adjacent in G.*

In the sequel by $G_1 \sqcup G_2$ we mean G_1 intersects G_2 only at one of its endpoints and by $\theta_{n_1,\dots,n_k} \setminus L_{n_i}$ we mean the graph obtained from $\theta_{n_1,\dots,n_k}$ by removing all vertices and edges of L_{ni} except its endpoints. Throughout this paper, we assume that x and y are the common vertices.

Now we are ready to compute the projective dimension of the graph $\theta_{n_1,\dots,n_k}$. In any case, to obtain an upper bound for $pd(\theta_{n_1,\dots,n_k})$, we use Theorem 2.1 and Lemma 2.2.

Theorem 2.6 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1}$. Then*

$$pd(G) = bightI(G) = 2\sum_{i=1}^{k_1} r_i = \sum_{i=1}^{k_1} pd(L_{3r_i+1}).$$

Proof We have $G' = G\setminus\{x\} = L_{3r_1}\sqcup\ldots\sqcup L_{3r_{k_1}}$ and $G'' = G\setminus N[x] = L_{3(r_1-1)+2}\sqcup\ldots\sqcup L_{3(r_{k_1}-1)+2}$. Using [6, Theorem 2.6], we obtain that

$$\begin{aligned} pd(G') =& \frac{2|V_{G'}|+1-k_1}{3} = \frac{2(3r_1+\ldots+3r_{k_1}-(k_1-1))+1-k_1}{3} \\ =& 2\sum_{i=1}^{k_1} r_i - k_1 + 1. \end{aligned}$$

By [6, Theorem 2.5], we get

$$\begin{aligned} pd(G'') =& \frac{2(|V_{G''}|-1)+k_1}{3} = \frac{2(3(r_1-1)+\ldots+3(r_{k_1}-1)+k_1+1-1)+k_1}{3} \\ =& 2\sum_{i=1}^{k_1} r_i - k_1. \end{aligned}$$

Hence, we have $pd(G)\le max\{2\sum_{i=1}^{k_1} r_i - k_1 + 2, 2\sum_{i=1}^{k_1} r_i\}$. Since $k_1\ge 3$, then $2-k_1<0$ and we conclude that $pd(G)\le 2\sum_{i=1}^{k_1} r_i$.

On the other hand, by [1, Theorem 3.3] and Lemma 2.3, we have that $d'_G = bightI(G)\le pd(G)$. It suffices to construct a semistrongly disjoint set $\mathbf{B} = \{B_1, B_2, \ldots, B_j\}$ of bouquets of G with $|F(\mathbf{B})| = 2\sum_{i=1}^{k_1} r_i$. Suppose that $B_1 = N[x]$, $B_2 = N[y]$ and $\mathbf{B_3} = \{B_3, B_4, \ldots, B_j\}$ are the semistrongly disjoint set of bouquets of type $(2,1)$ on the disjoint lines $L_{3r_1+1-4},\ldots,L_{3r_{k_1}+1-4}$, which can be expressed as $3r_i+1-4 = 3(r_i-1)$ for any $1\le i\le k_1$. Hence there exist r_i-1 bouquets with two flowers and one root in L_{3r_i+1-4} for any $1\le i\le k_1$. Putting $\mathbf{B} = B_1\cup B_2\cup \mathbf{B_3}$, we obtain $|F(\mathbf{B})| = 2\sum_{i=1}^{k_1} r_i$; then $2\sum_{i=1}^{k_1} r_i \le bightI(G)$ and complete the proof. □

Theorem 2.7 *Let G be the graph $\theta_{n_1,\ldots,n_{k_2}}$ consisting of lines $L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$. Then*

$$pd(G) = bightI(G) = 2\sum_{i=1}^{k_2} t_i + 1 = \sum_{i=1}^{k_2} pd(L_{3t_i+2}) - k_2 + 1.$$

Proof We have $G' = G\setminus\{x\} = L_{3t_1+1}\sqcup\ldots\sqcup L_{3t_{k_2}+1}$ and $G'' = G\setminus N[x] = L_{3t_1}\sqcup\ldots\sqcup L_{3t_{k_2}}$. Using [6, Corollary 2.8], we derive

$$\begin{aligned} pd(G') =& \frac{2|V_{G'}|-2}{3} = \frac{2(3t_1+\ldots+3t_{k_2}+1)-2}{3} \\ =& 2\sum_{i=1}^{k_2} t_i. \end{aligned}$$

By [6, Theorem 2.6], we get

$$pd(G'') = \frac{2|V_{G''}| + 1 - k_2}{3} = \frac{2(3t_1 + \ldots + 3t_{k_2} - (k_2 - 1)) + 1 - k_2}{3}$$
$$= 2\sum_{i=1}^{k_2} t_i - k_2 + 1.$$

Hence, it follows that $pd(G) \leq max\{2\sum_{i=1}^{k_2} t_i + 1, 2\sum_{i=1}^{k_2} t_i + 1\} = 2\sum_{i=1}^{k_2} t_i + 1$. It suffices to construct a semistrongly disjoint set $\mathbf{B} = \{B_1, B_2, \ldots, B_j\}$ of bouquets of G with $|F(\mathbf{B})| = 2\sum_{i=1}^{k_2} t_i + 1$. Suppose that $B_1 = N[x]$ and $\mathbf{B_2} = \{B_2, B_3, \ldots, B_j\}$ are the semistrongly disjoint set of bouquets of the lines $L_{3t_1+2-2}, L_{3t_2+2-3}, \ldots, L_{3t_{k_2}+2-3}$, where $3t_1 + 2 - 2 = 3t_1$, $3t_i + 2 - 3 = 3(t_i - 1) + 2$ for any $2 \leq i \leq k_2$. Hence there exist t_1 bouquets with two flowers and one root in L_{3t_1+2-2}, $t_i - 1$ bouquets with two flowers and one root, and one bouquet with one flower and one root in L_{3t_i+2-3} for any $2 \leq i \leq k_2$. Putting $\mathbf{B} = B_1 \cup \mathbf{B_2}$, we obtain $|F(\mathbf{B})| = 2\sum_{i=1}^{k_2} t_i + 1$; then $2\sum_{i=1}^{k_2} t_i + 1 \leq bightI(G)$ and we conclude the desired equality. □

Theorem 2.8 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_3}}$ consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1}+1}, L_{3s_1}, \ldots, L_{3s_{k_3}}$ such that $k_1, k_3 > 0$. Then*

$$pd(G) = bightI(G) = 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_3 = \sum_{i=1}^{k_1} pd(L_{3r_i+1}) + \sum_{i=1}^{k_3} pd(L_{3s_i}) - k_3.$$

Proof We have $G' = G \setminus \{x\} = L_{3r_1} \sqcup \ldots \sqcup L_{3r_{k_1}} \sqcup L_{3(s_1-1)+2} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+2}$ and $G'' = G \setminus N[x] = L_{3(r_1-1)+2} \sqcup \ldots \sqcup L_{3(r_{k_1}-1)+2} \sqcup L_{3(s_1-1)+1} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+1}$. By [6, Theorem 2.7], we obtain that

$$pd(G') = \frac{2|V_{G'}| - 2 + k_3 - k_1}{3}$$
$$= \frac{2(3r_1 + \ldots, 3r_{k_1} - k_1 + 3(s_1 - 1) + \ldots + 3(s_{k_3} - 1) + (k_3 - 1) + 2) - 2 + k_3 - k_1}{3}$$
$$= 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_1 - k_3.$$

Using [6, Theorem 2.5] we get

$$pd(G'') = \frac{2|V_{G''}| - 2 + k_1}{3}$$
$$= \frac{2(3(r_1 - 1) + \ldots + 3(r_{k_1} - 1) + k_1 + 3(s_1 - 1) + \ldots + 3(s_{k_3} - 1) + 1) - 2 + k_1}{3}$$
$$= 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_1 - 2k_3.$$

Hence, we have $pd(G) \leq max\{2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_1 - k_3 + 1, 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_3\}$, since $k_1 > 0$, then $1 - k_1 \leq 0$ and hence $pd(G) \leq 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_3$. Using similar arguments of the proof of Theorem 2.6, we derive $2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_3} s_i - k_3 \leq bightI(G)$, which yields the asserted equality. □

Theorem 2.9 *Let G be the graph $\theta_{n_1,\ldots,n_{k_2+k_3}}$ consisting of lines $L_{3s_1},\ldots,L_{3s_{k_3}},L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$ such that $k_2,k_3>0$. Then*

$$pd(G)=bightI(G)=2\sum_{i=1}^{k_2}t_i+2\sum_{i=1}^{k_3}s_i-k_3+1=\sum_{i=1}^{k_2}pd(L_{3t_i+2})+\sum_{i=1}^{k_3}pd(L_{3s_i})-k_2-k_3+1.$$

Proof We have $G'=G\setminus\{x\}=L_{3(s_1-1)+2}\sqcup\ldots\sqcup L_{3(s_{k_3}-1)+2}\sqcup L_{3t_1+1}\sqcup\ldots\sqcup L_{3t_{k_2}+1}$ and $G''=G\setminus N[x]=L_{3(s_1-1)+1}\sqcup\ldots\sqcup L_{3(s_{k_3}-1)+1}\sqcup L_{3t_1}\sqcup\ldots\sqcup L_{3t_{k_2}}$. By [6, Theorem 2.5] we get that

$$\begin{aligned}pd(G')=&\frac{2|V_{G'}|-2+k_3}{3}\\=&\frac{2(3(s_1-1)+\ldots+3(s_{k_3}-1)+k_3+3t_1+\ldots+3t_{k_2}+1)-2+k_3}{3}\\=&2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-k_3.\end{aligned}$$

Using [6, Theorem 2.6], we obtain that

$$\begin{aligned}pd(G'')=&\frac{2|V_{G'}|+1-k_2}{3}\\=&\frac{2(3(s_1-1)+\ldots+3(s_{k_3}-1)+1+3t_1+\ldots+3t_{k_2}-k_2)+1-k_2}{3}\\=&2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-2k_3-k_2+1.\end{aligned}$$

Therefore,

$$pd(G)\le max\{2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-k_3+1,2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-k_3+1\}=2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-k_3+1.$$

A similar argument as Theorem 2.7 shows that $2\sum_{i=1}^{k_3}s_i+2\sum_{i=1}^{k_2}t_i-k_3+1\le bightI(G)$, as required. □

Theorem 2.10 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_2+k_3}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$, $L_{3s_1},\ldots,L_{3s_{k_3}}$ such that $k_1,k_2,k_3>0$. Then*

$$\begin{aligned}pd(G)=bightI(G)=2\sum_{i=1}^{k_1}r_i+2\sum_{i=1}^{k_2}t_i+2\sum_{i=1}^{k_3}s_i-k_3=&\sum_{i=1}^{k_1}pd(L_{3r_i+1})+\sum_{i=1}^{k_2}pd(L_{3t_i+2})\\&+\sum_{i=1}^{k_3}pd(L_{3s_i})-k_2-k_3.\end{aligned}$$

Proof We have

$$G'=G\setminus\{x\}=L_{3r_1}\sqcup\ldots\sqcup L_{3r_{k_1}}\sqcup L_{3t_1+1}\sqcup\ldots\sqcup L_{3t_{k_2}+1}\sqcup L_{3(s_1-1)+2}\sqcup\ldots\sqcup L_{3(s_{k_3}-1)+2}$$

and

$$G'' = G \setminus N[x] = L_{3(r_1-1)+2} \sqcup \ldots \sqcup L_{3(r_{k_1}-1)+2} \sqcup L_{3t_1} \sqcup \ldots \sqcup L_{3t_{k_2}} \sqcup L_{3(s_1-1)+1} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+1}.$$

By [6, Theorem 2.7], we obtain that

$$\begin{aligned} pd(G') =& \frac{2|V_{G'}| - 2 + k_3 - k_1}{3} \\ =& \frac{2(3r_1 + \ldots + 3r_{k_1} - k_1 + 3t_1 + \ldots + 3t_{k_2} + 1 + 3s_1 + \ldots + 3s_{k_3} - 2k_3) - 2}{3} \\ &+ \frac{k_3 - k_1}{3} = 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_1 - k_3. \end{aligned}$$

Moreover, by [6, Theorem 2.7] we get

$$\begin{aligned} pd(G'') =& \frac{2|V_{G''}| - 2 + k_1 - k_2}{3} \\ =& \frac{2(3r_1 + \ldots + 3r_{k_1} - 2k_1 + 3t_1 + \ldots + 3t_{k_2} - k_2 + 3s_1 + \ldots + 3s_{k_3} - 3k_3 + 1) - 2}{3} \\ &+ \frac{k_1 - k_2}{3} = 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_1 - k_2 - 2k_3. \end{aligned}$$

Therefore, we have

$$pd(G) \leq max\{2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_1 - k_3 + 1, 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_3\}.$$

Since $k_1 > 0$, then $1 - k_1 \leq 0$ and we conclude $pd(G) \leq 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_3$. A similar argument of the proof of Theorem 2.7 shows that $2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + 2\sum_{i=1}^{k_3} s_i - k_3 \leq bightI(G)$, which yields the asserted equality. □

Theorem 2.11 *Let G be the graph $\theta_{n_1,\ldots,n_{k_3}}$ consisting of lines $L_{3s_1}, \ldots, L_{3s_{k_3}}$. Then*

$$pd(G) = 2\sum_{i=1}^{k_3} s_i - k_3 + 1 = \sum_{i=1}^{k_3} pd(L_{3s_i}) - k_3 + 1 = bightI(G) + 1.$$

Proof We have that $G' = G \setminus \{x\} = L_{3(s_1-1)+2} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+2}$ and $G'' = G \setminus N[x] = L_{3(s_1-1)+1} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+1}$. By [6, Theorem 2.5], we obtain that

$$\begin{aligned} pd(G') =& \frac{2|V_{G'}| - 2 + k_3}{3} \\ =& \frac{2(3(s_1 - 1) + \ldots + 3(s_{k_3} - 1) + k_3 + 1) - 2 + k_3}{3} \\ =& 2\sum_{i=1}^{k_3} s_i - k_3. \end{aligned}$$

Using [6, Corollary 2.8], we derive

$$pd(G'') = \frac{2|V_{G''}| - 2}{3} = \frac{2(3(s_1 - 1) + \ldots + 3(s_{k_3} - 1) + 1) - 2}{3}$$
$$= 2\sum_{i=1}^{k_3} s_i - 2k_3.$$

Hence

$$pd(G) \leq max\{2\sum_{i=1}^{k_3} s_i - k_3 + 1, 2\sum_{i=1}^{k_3} s_i - k_3\} = 2\sum_{i=1}^{k_3} s_i - k_3 + 1.$$

On the other hand, we can see that $\beta_{i,|V(G)|}(G) \neq 0$ if and only if $i = 2\sum_{i=1}^{k_3} s_i - k_3 + 1$; thus $2\sum_{i=1}^{k_3} s_i - k_3 + 1 \leq pd(G)$. It follows that $pd(G) = 2\sum_{i=1}^{k_3} s_i - k_3 + 1$, as desired.

Now we show that $bightI(G) = 2\sum_{i=1}^{k_3} s_i - k_3$. Suppose that $\mathbf{B} = \{B_1, B_2, \ldots, B_l\}$ is a semistrongly disjoint set of bouquets of G. Consider the following cases:

Case (1) : $x, y \notin R(\mathbf{B}) \cup F(\mathbf{B})$. We may find the maximum cardinality of $F(\mathbf{B})$ in the disjoint lines $L_{3s_1-2}, \ldots, L_{3s_{k_3}-2}$. Since $3s_i - 2 = 3(s_i - 1) + 1$, then one can choose $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for any $1 \leq i \leq k_3$. Hence we obtain that $|F(\mathbf{B})| = 2\sum_{i=1}^{k_3} s_i - 2k_3$.

Case (2): $x, y \in F(\mathbf{B})$. Suppose that x and y lie in the bouquets with two flowers and one root.

i) If the bouquets containing x and y are in the same line, as L_{3s_1}, then we have $3s_1 - 6 = 3(s_1 - 2)$ and $3s_i - 2 = 3(s_i - 1) + 1$ for any $2 \leq i \leq k_3$. Hence, there exist $s_1 - 2$ bouquets with two flowers and one root in L_{3s_1-6} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $2 \leq i \leq k_3$. It follows that $|F(\mathbf{B})| = 2(s_1 - 2) + 2\sum_{i=2}^{k_3}(s_i - 1) + 4 = 2\sum_{i=1}^{k_3} s_i - 2k_3 + 2$.

ii) If the bouquets containing x and y are in different lines, as L_{3s_1} and L_{3s_2}, then we have $3s_1 - 4 = 3(s_1 - 2) + 2$, $3s_2 - 4 = 3(s_2 - 2) + 2$ and $3s_i - 2 = 3(s_i - 1) + 1$ for any $3 \leq i \leq k_3$. Hence, there exist $s_i - 2$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_i-4}, for $i = 1, 2$ and also there exist $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for any $3 \leq i \leq k_3$. Therefore, we obtain that $|F(\mathbf{B})| = 2(s_1 - 2) + 2(s_1 - 2) + 2 + 2\sum_{i=3}^{k_3}(s_i - 1) + 4 = 2\sum_{i=1}^{k_3} s_i - 2k_3 + 2$.

Case (3): $x \in R(\mathbf{B})$ and $y \in F(\mathbf{B})$. Assume that x lies in a bouquet with k_3 flowers and one root and y lies in a bouquet with two flowers and one root of L_{3s_1}; then we have $3s_1 - 5 = 3(s_1 - 2) + 1$ and $3s_i - 3 = 3(s_i - 1)$ for $2 \leq i \leq k_3$. Hence, there exist $s_1 - 2$ bouquets with two flowers and one root in L_{3s_1-5} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-3} for $2 \leq i \leq k_3$. Thus we get $|F(\mathbf{B})| = 2(s_1 - 2) + 2 + 2\sum_{i=2}^{k_3}(s_i - 1) + k_3 = 2\sum_{i=1}^{k_3} s_i - k_3$.

Case (4): $y \notin F(\mathbf{B}) \cup R(\boldsymbol{B})$, but $x \in F(\mathbf{B}) \cup R(\mathbf{B})$. Then there exist k_3 lines of length $3s_i - 1$ that have a common vertex x. By [6, Theorem 2.5], we get

$$bightI(G) = \frac{2|V_G| - 2 + k_3}{3} = 2\sum_{i=1}^{k_3} s_i - k_3.$$

Case (5): $x, y \in R(\mathbf{B})$. Assume that x and y lie in the bouquets with k_3 flowers and a root. Since $3s_i - 4 = 3(s_i - 2) + 2$, then there exist $s_i - 2$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_i-4} for $1 \leq i \leq k_3$. Hence, $|F(\mathbf{B})| = 2k_3 + k_3 + 2\sum_{i=1}^{k_3}(s_i - 2) = 2\sum_{i=1}^{k_3} s_i - k_3$.

One can easily check that in any of the above cases by our choice of other bouquets we have at most the given amount of flowers. Since we want to find the maximum number of flowers of a semistrongly disjoint set of bouquets of G, then by choosing any vertex z we try to consider the bouquets with the maximum number of flowers containing z. Note that the described cases above are satisfied if we interchange x and y. It follows that the maximum value for $F(\mathbf{B})$ is equal to $2\sum_{i=1}^{k_3} s_i - k_3$, as desired. □

Theorem 2.12 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_2}}$ consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1}+1}, L_{3t_1+2}, \ldots, L_{3t_{k_2}+2}$. If $k_1 = 1$, then we have*

$$pd(G) = 2r_1 + 2\sum_{i=1}^{k_2} t_i + 1 = bightI(G) + 1 = pd(L_{3r_1+1}) + \sum_{i=1}^{k_2} pd(L_{3t_i+2}) - k_2 + 1,$$

and for $k_1 \geq 2$, the following relation is satisfied:

$$pd(G) = bightI(G) = 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i = \sum_{i=1}^{k_1} pd(L_{3r_i+1}) + \sum_{i=1}^{k_2} pd(L_{3t_i+2}) - k_2.$$

Proof We have

$$G' = G \setminus \{x\} = L_{3r_1} \sqcup \ldots \sqcup L_{3r_{k_1}} \sqcup L_{3t_1+1} \sqcup \ldots \sqcup L_{3t_{k_2}+1}$$

and

$$G'' = G \setminus N[x] = L_{3(r_1-1)+2} \sqcup \ldots \sqcup L_{3(r_{k_1}-1)+2} \sqcup L_{3t_1} \sqcup \ldots \sqcup L_{3t_{k_2}}.$$

By [6, Theorem 2.6], we get that

$$\begin{aligned} pd(G') =& \frac{2|V_{G'}| + 1 - k_1}{3} \\ =& \frac{2(3r_1 + \ldots + 3r_{k_1} - k_1 + 3t_1 + \ldots + 3t_{k_2} + 1) + 1 - k_1}{3} \\ =& 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i - k_1 + 1. \end{aligned}$$

Moreover, by [6, Theorem 2.7], we have

$$\begin{aligned} pd(G'') =& \frac{2|V_{G''}| - 2 + k_1 - k_2}{3} \\ =& \frac{2(3r_1 + \ldots + 3r_{k_1} - 2k_1 + 3t_1 + \ldots + 3t_{k_2} - k_2 + 1) - 2 + k_1 - k_2}{3} \\ =& 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i - k_1 - k_2. \end{aligned}$$

Therefore, $pd(G) \leq max\{2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i - k_1 + 2, 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i\}$. If $k_1 = 1$, then $pd(G) \leq 2r_1 + 2\sum_{i=1}^{k_2} t_i + 1$, and for $k_1 \geq 2$, since $2 - k_1 \leq 0$, then it yields $pd(G) \leq 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i$. In the case $k_1 = 1$, we can see that $\beta_{i,|V(G)|}(G) \neq 0$ if and only if $i = 2r_1 + 2\sum_{i=1}^{k_2} t_i + 1$. It follows that $2r_1 + 2\sum_{i=1}^{k_2} t_i + 1 \leq pd(G)$; then $pd(G) = 2r_1 + 2\sum_{i=1}^{k_2} t_i + 1$.

Now suppose that $k_1 \geq 2$. In order to prove $bightI(G) = pd(G)$, we use similar arguments of the proof of Theorem 2.6; then we derive $2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i \leq bightI(G)$, which yields the asserted equality.

To complete the proof, it remains to prove $bightI(G) = pd(G) - 1 = 2r_1 + 2\sum_{i=1}^{k_2} t_i$ for $k_1 = 1$. Assume that $\boldsymbol{B} = \{B_1, \ldots, B_l\}$ is the semistrongly disjoint set of bouquets of G. The same argument as in the proof of Theorem 2.11 shows that the maximum value for $|F(\mathbf{B})|$ is equal to $2r_1 + 2\sum_{i=1}^{k_2} t_i$, as desired. □

Corollary 2.13 *Let G be the graph $\theta_{n_1,\ldots,n_k}$. Then we have $bightI(G) = pd(G)$ unless $n_i \equiv 0 \pmod 3$ for any $1 \leq i \leq k$ or there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$ and for any $1 \leq i \neq j \leq k$ we have $n_i \equiv 2 \pmod 3$; then it yields $pd(G) = bightI(G) + 1$.*

Theorem 2.14 *Let G be the graph $\theta_{n_1,\ldots,n_k}$. Unless in two cases $n_i \equiv 0 \pmod 3$ for any $1 \leq i \leq k$ or there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$ and for any $1 \leq i \neq j \leq k$, $n_i \equiv 2 \pmod 3$, we have*

$$depth(R/I(G)) = min\{|F| : F \subseteq V(G) \text{ is a maximal independent set in } G\}.$$

Moreover, $R/I(G)$ is Cohen–Macaulay if and only if G is unmixed.

Proof Using Corollary 2.13 and the Auslander–Buchsbaum formula, we have $depth(R/I(G)) = |V(G)| - bightI(G)$. By definition of big height of $I(G)$, there exists a minimal vertex cover S of G such that we have $|S| = bightI(G)$. Since the complement of S, $V(G) \setminus S$, is a maximal independent set of G having minimum cardinality, then we get

$$depth(R/I(G)) = min\{|F| : F \subseteq V(G) \text{ is a maximal independent set in } G\},$$

as desired. By [10, Corollary 5.3.11], we have

$$dim(R/I(G)) = max\{|F| : F \subseteq V(G) \text{ is an independent set in } G\};$$

hence $R/I(G)$ is Cohen–Macaulay if and only if all maximal independent sets of G have the same cardinality or equivalently all minimal vertex covers of G have the same cardinality. This completes the proof. □

3. Regularity

The aim of this section is to study the regularity of the graph $\theta_{n_1,\ldots,n_k}$ and investigate the equality in Lemma 2.3 (1) for this class of graphs. To obtain an appropriate upper bound for $reg(\theta_{n_1,\ldots,n_k})$, we use Theorem 2.1 and Lemma 2.2.

Theorem 3.1 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1}}$ consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1}+1}$. Then*

$$reg(G) = c_G = \sum_{i=1}^{k_1} r_i.$$

Proof Assume that the edges of L_{3r_i+1} are labeled by $e_1^{(i)}, e_2^{(i)}, \ldots, e_{3r_i}^{(i)}$, where $e_j^{(i)} = \{x_j^{(i)}, x_{j+1}^{(i)}\}$, $x_1^{(i)} = x$, and $x_{3r_i+1}^{(i)} = y$. We consider the edges $e_2^{(i)}, e_5^{(i)}, \ldots, e_{3r_i-1}^{(i)}$ of L_{3r_i+1}, for any $1 \leq i \leq k_1$. It is seen that $\{e_2^{(1)}, e_5^{(1)}, \ldots, e_{3r_1-1}^{(1)}, \ldots, e_2^{(k_1)}, e_5^{(k_1)}, \ldots, e_{3r_{k_1}-1}^{(k_1)}\}$ is pairwise 3-disjoint in G. Hence, it follows that $\sum_{i=1}^{k_1} r_i \leq c_G$.

To complete the proof, it suffices to detect an appropriate upper bound for $reg(G')$ and $reg(G'')$. We have $G' = G \setminus \{x\} = L_{3r_1} \sqcup \ldots \sqcup L_{3r_{k_1}}$ and $G'' = G \setminus N_G[x] = L_{3(r_1-1)+2} \sqcup \ldots \sqcup L_{3(r_{k_1}-1)+2}$. By Lemma 2.2, $reg(G') \leq max\{reg(G' \setminus \{y\}), reg(G' \setminus N_{G'}[y]) + 1\}$, where $G' \setminus \{y\}$ is the disjoint union of $L_{3(r_1-1)+2}, \ldots, L_{3(r_{k_1}-1)+2}$ and $G' \setminus N_{G'}[y]$ is the disjoint union of $L_{3(r_1-1)+1}, \ldots, L_{3(r_{k_1}-1)+1}$. Then we obtain $reg(G' \setminus \{y\}) = \sum_{i=1}^{k_1} r_i$ and $reg(G' \setminus N_{G'}[y]) = \sum_{i=1}^{k_1} r_i - k_1$. Hence, it follows that $reg(G') \leq \sum_{i=1}^{k_1} r_i$.

Again, using Lemma 2.2, $reg(G'') \leq max\{reg(G'' \setminus \{y\}), reg(G'' \setminus N_{G''}[y]) + 1\}$, where $G'' \setminus \{y\}$ is the disjoint union of $L_{3(r_1-1)+1}, \ldots, L_{3(r_{k_1}-1)+1}$ and $G'' \setminus N_{G''}[y]$ is the disjoint union of $L_{3(r_1-1)}, \ldots, L_{3(r_{k_1}-1)}$. Thus, we get $reg(G'' \setminus \{y\}) = \sum_{i=1}^{k_1} r_i - k_1$ and $reg(G'' \setminus N_{G''}[y]) = \sum_{i=1}^{k_1} r_i - k_1$. Therefore, $reg(G'') \leq \sum_{i=1}^{k_1} r_i - k_1 + 1$. Since $k_1 \geq 3$, then it immediately yields that $reg(G) \leq \sum_{i=1}^{k_1} r_i$, as required. □

Theorem 3.2 *Let G be the graph $\theta_{n_1,\ldots,n_{k_2+k_3}}$ consisting of lines $L_{3s_1}, \ldots, L_{3s_{k_3}}, L_{3t_1+2}, \ldots, L_{3t_{k_2}+2}$ such that $k_2, k_3 > 0$. Then*

$$reg(G) = c_G = \sum_{i=1}^{k_2} t_i + \sum_{i=1}^{k_3} s_i - k_3 + 1.$$

Proof Suppose that the edges of L_{3s_i} are labeled by $e_1^{(i)}, \ldots, e_{3s_i-1}^{(i)}$, where $e_j^{(i)} = \{x_j^{(i)}, x_{j+1}^{(i)}\}$, $x_1^{(i)} = x$ and $x_{3s_i}^{(i)} = y$ and the edges of L_{3t_i+2} are labeled by $e_1^{(k_3+i)}, \ldots, e_{3t_i+1}^{(k_3+i)}$, where $e_j^{(k_3+i)} = \{x_j^{(k_3+i)}, x_{j+1}^{(k_3+i)}\}$, $x_1^{(k_3+i)} = x$ and $x_{3t_i+2}^{(k_3+i)} = y$. Observe that

$$\{e_1^{(1)}, e_4^{(1)}, \ldots, e_{3s_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \ldots, e_{3s_2-3}^{(2)}, \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3s_{k_3}-3}^{(k_3)}, e_3^{(k_3+1)},$$
$$e_6^{(k_3+1)}, \ldots, e_{3t_1}^{(k_3+1)}, \ldots, e_3^{(k_3+k_2)}, e_6^{(k_3+k_2)}, e_{3t_{k_2}}^{(k_3+k_2)}\}$$

is a pairwise 3-disjoint in G. Then we get $\sum_{i=1}^{k_2} t_i + \sum_{i=1}^{k_3} s_i - k_3 + 1 \leq c_G$. To complete the proof, we need to achieve an upper bound for $reg(G')$ and $reg(G'')$. Using Lemma 2.2, one has

$$reg(G') \leq max\{reg(G' \setminus \{y\}), reg(G' \setminus N_{G'}[y]) + 1\},$$

where $G' = G \setminus \{x\} = L_{3(s_1-1)+2} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+2} \sqcup L_{3t_1+1} \sqcup \ldots \sqcup L_{3t_{k_2}+1}$, $G' \setminus \{y\}$ is the disjoint union of $L_{3(s_1-1)+1}, \ldots, L_{3(s_{k_3}-1)+1}, L_{3t_1}, \ldots, L_{3t_{k_2}}$ and $G' \setminus N_{G'}[y]$ is the disjoint union of $L_{3(s_1-1)}, \ldots, L_{3(s_{k_3}-1)}, L_{3(t_1-1)+2}, \ldots, L_{3(t_{k_2}-1)+2}$. Hence, we obtain that

$$reg(G' \setminus \{y\}) = \sum_{i=1}^{k_3}(s_i - 1) + \sum_{i=1}^{k_2} t_i = \sum_{i=1}^{k_3} s_i + \sum_{i=1}^{k_2} t_i - k_3$$

and

$$reg(G' \setminus N_{G'}[y]) = \sum_{i=1}^{k_3}(s_i - 1) + \sum_{i=1}^{k_2} t_i = \sum_{i=1}^{k_3} s_i + \sum_{i=1}^{k_2} t_i - k_3.$$

It follows that

$$reg(G') \leq \sum_{i=1}^{k_3}(s_i-1)+\sum_{i=1}^{k_2}t_i+1=\sum_{i=1}^{k_3}s_i+\sum_{i=1}^{k_2}t_i-k_3+1.$$

Again, using Lemma 2.2, we have $reg(G'') \leq max\{reg(G''\setminus\{y\}), reg(G''\setminus N_{G''}[y])+1\}$, where $G''=G\setminus N_G[x]=L_{3(s_1-1)+1}\sqcup\ldots\sqcup L_{3(s_{k_3}-1)+1}\sqcup L_{3t_1}\sqcup\ldots\sqcup L_{3t_{k_2}}$, $G''\setminus\{y\}$ is the disjoint union of $L_{3(s_1-1)},\ldots,L_{3(s_{k_3-1})},L_{3(t_1-1)+2}$, $\ldots,L_{3(t_{k_2}-1)+2}$ and $G''\setminus N_{G''}[y]$ is the disjoint union of $L_{3(s_1-2)+2},\ldots,L_{3(s_{k_3-2})+2},L_{3(t_1-1)+1},\ldots,L_{3(t_{k_2}-1)+1}$. Then we obtain $reg(G''\setminus\{y\})=\sum_{i=1}^{k_3}s_i+\sum_{i=1}^{k_2}t_i-k_3$ and $reg(G''\setminus N_{G''}[y])=\sum_{i=1}^{k_3}s_i+\sum_{i=1}^{k_2}t_i-k_3-k_2$. Hence $reg(G'')\leq\sum_{i=1}^{k_3}s_i+\sum_{i=1}^{k_2}t_i-k_3$. One derives the equality $reg(G)=c_G=\sum_{i=1}^{k_3}s_i+\sum_{i=1}^{k_2}t_i-k_3+1$. □

Theorem 3.3 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_2}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$ such that $k_1,k_2>0$. Then*

$$reg(G)=c_G=\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i.$$

Proof Suppose that the edges of L_{3r_i+1} are labeled by $e_1^{(i)},\ldots,e_{3r_i}^{(i)}$, where $e_j^{(i)}=\{x_j^{(i)},x_{j+1}^{(i)}\}$, $x_1^{(i)}=x$ and $x_{3r_i+1}^{(i)}=y$ and the edges of L_{3t_i+2} are labeled by $e_1^{(k_1+i)},\ldots,e_{3t_i+1}^{(k_1+i)}$, where $e_j^{(k_1+i)}=\{x_j^{(k_1+i)},x_{j+1}^{(k_1+i)}\}$, $x_1^{(k_1+i)}=x$ and $x_{3t_i+2}^{(k_1+i)}=y$. Since the set

$$\{e_2^{(1)},e_5^{(1)},\ldots,e_{3r_1-1}^{(1)},e_2^{(2)},e_5^{(2)},\ldots,e_{3r_2-4}^{(2)},e_{3r_2-1}^{(2)},\ldots,e_2^{(k_1)},e_5^{(k_1)},\ldots,e_{3r_{k_1}-4}^{(k_1)},$$
$$e_{3r_{k_1}-1}^{(k_1)},e_2^{(k_1+1)},e_5^{(k_1+1)},\ldots,e_{3t_1-1}^{(k_1+1)},\ldots,e_2^{(k_1+k_2)},e_5^{(k_1+k_2)},\ldots,e_{3t_{k_2}-1}^{(k_1+k_2)}\}$$

is pairwise 3-disjoint, then it follows that $\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i\leq c_G$. We have $G'=G\setminus\{x\}=L_{3r_1}\sqcup\ldots\sqcup L_{3r_{k_1}}\sqcup L_{3t_1+1}\sqcup\ldots\sqcup L_{3t_{k_2}+1}$ and $G''=G\setminus N_G[x]=L_{3(r_1-1)+2}\sqcup\ldots\sqcup L_{3(r_{k_1}-1)+2}\sqcup L_{3t_1}\sqcup\ldots\sqcup L_{3t_{k_2}}$. In order to use Lemma 2.2 for $reg(G')$ and $reg(G'')$, we have to compute $reg(G'\setminus\{y\})$, $reg(G'\setminus N_{G'}[y])+1$, $reg(G''\setminus\{y\})$ and $reg(G''\setminus N_{G''}[y])+1$, where $G'\setminus\{y\}$ is the disjoint union of $L_{3(r_1-1)+2},\ldots,L_{3(r_{k_1}-1)+2},L_{3t_1},\ldots,L_{3t_{k_2}}$, $G'\setminus N_{G'}[y]$ is the disjoint union of $L_{3(r_1-1)+1},\ldots,L_{3(r_{k_1-1})+1},L_{3(t_1-1)+2},\ldots,L_{3(t_{k_2}-1)+2}$, $G''\setminus\{y\}$ is the disjoint union of $L_{3(r_1-1)+1},\ldots,L_{3(r_{k_1-1})+1},L_{3(t_1-1)+2},\ldots,L_{3(t_{k_2}-1)+2}$, and $G''\setminus N_{G''}[y]$ is the disjoint union of $L_{3(r_1-1)},\ldots,L_{3(r_{k_1-1})},L_{3(t_1-1)+1},\ldots,L_{3(t_{k_2}-1)+1}$. Hence, we obtain that $reg(G'\setminus\{y\})=\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i$ and $reg(G'\setminus N_{G'}[y])=\sum_{i=1}^{k_1}t_i+\sum_{i=1}^{k_2}r_i-k_1$. Thus $reg(G')\leq\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i$. Moreover, $reg(G''\setminus\{y\})=\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i-k_1$ and $reg(G''\setminus N_{G''}[y])=\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i-k_1-k_2$. Therefore, $reg(G'')\leq\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i-k_1$. Since $1-k_1\leq 0$, then $reg(G)=c_G=\sum_{i=1}^{k_1}r_i+\sum_{i=1}^{k_2}t_i$, which proves the required equality. □

Theorem 3.4 *Let G be the graph $\theta_{n_1,\ldots,n_{k_3}}$ consisting of lines $L_{3s_1},\ldots,L_{3s_{k_3}}$. Then*

$$reg(G)=c_G=\sum_{i=1}^{k_3}s_i-k_3+1.$$

Proof Suppose that the edges of L_{3s_i} are labeled by $e_1^{(i)}, \ldots, e_{3s_i-1}^{(i)}$, where $e_j^{(i)} = \{x_j^{(i)}, x_{j+1}^{(i)}\}$, $x_1^{(i)} = x$ and $x_{3s_i}^{(i)} = y$. Observe that

$$\{e_1^{(1)}, e_4^{(1)}, \ldots, e_{3s_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \ldots, e_{3s_2-3}^{(2)}, \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3s_{k_3}-3}^{(k_3)}\}$$

is pairwise 3-disjoint in G; hence $\sum_{i=1}^{k_3} s_i - k_3 + 1 \leq c_G$. Since $G' = G \setminus \{x\} = L_{3(s_1-1)+2} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+2}$ and $G'' = G \setminus N_G[x] = L_{3(s_1-1)+1} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+1}$, then again using Lemma 2.2,

$$reg(G') \leq max\{reg(G' \setminus \{y\}), reg(G' \setminus N_{G'}[y]) + 1\},$$

where $G' \setminus \{y\}$ is the disjoint union of $L_{3(s_1-1)+1}, \ldots, L_{3(s_{k_3}-1)+1}$ and $G' \setminus N_{G'}[y]$ is the disjoint union of $L_{3(s_1-1)}, \ldots, L_{3(s_{k_3}-1)}$. It follows that $reg(G' \setminus \{y\}) = \sum_{i=1}^{k_3} s_i - k_3$ and $reg(G' \setminus N_{G'}[y]) = \sum_{i=1}^{k_3} s_i - k_3$. Hence $reg(G') \leq \sum_{i=1}^{k_3} s_i - k_3 + 1$. Since $G'' \setminus \{y\}$ is the disjoint union of $L_{3(s_1-1)}, \ldots, L_{3(s_{k_3}-1)}$ and $G'' \setminus N_{G''}[y]$ is the disjoint union of $L_{3(s_1-2)+2}, \ldots, L_{3(s_{k_3}-2)+2}$, then we obtain $reg(G'' \setminus \{y\}) = \sum_{i=1}^{k_3} s_i - k_3$ and $reg(G'' \setminus N_{G''}[y]) = \sum_{i=1}^{k_3} s_i - k_3$. Applying Lemma 2.2, we have $reg(G'') \leq \sum_{i=1}^{k_3} s_i - k_3 + 1$. On the other hand, the set

$$\{e_3^{(1)}, e_6^{(1)}, \ldots, e_{3s_1-3}^{(1)}, \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3s_{k_3}-3}^{(k_3)}\}$$

is pairwise 3-disjoint in G'' and hence $\sum_{i=1}^{k_3} s_i - k_3 \leq c_{G''}$. We claim that $reg(G'') = c_{G''} = \sum_{i=1}^{k_3} s_i - k_3$. To prove the fact, consider the strongly disjoint set $\mathbf{B} = \{B_1, \ldots, B_l\}$ of bouquets in G''. Any of the following cases may happen:

Case (1): $y \notin R(\mathbf{B}) \cup F(\mathbf{B})$. In this situation, the set

$$\{e_3^{(1)}, e_6^{(1)}, \ldots, e_{3(s_1-1)}^{(1)}, \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3(s_{k_3}-1)}^{(k_3)}\}$$

is pairwise 3-disjoint.

Case (2): $y \in F(\mathbf{B})$. Observe that the set

$$\{e_4^{(1)}, e_7^{(1)}, \ldots, e_{3s_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \ldots, e_{3(s_2-1)}^{(2)} \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3(s_{k_3}-1)}^{(k_3)}\}$$

is pairwise 3-disjoint in G''.

Case (3): $y \in R(\mathbf{B})$. In this case, the set

$$\{e_3^{(1)}, e_6^{(1)}, \ldots, e_{3(s_1-1)}^{(1)}, \ldots, e_3^{(k_3)}, e_6^{(k_3)}, \ldots, e_{3(s_{k_3}-1)}^{(k_3)}\}$$

is pairwise 3-disjoint in G''.

It is easily checked that the considered sets have the maximum cardinality of a pairwise 3-disjoint set in G''.

Altogether and by [11, Theorem 2.18], $c_{G''} = reg(G'') = \sum_{i=1}^{k_3} s_i - k_3$, as claimed. Hence one derives $reg(G) \leq \sum_{i=1}^{k_3} s_i - k_3 + 1$ and so the result holds. □

Theorem 3.5 *Let G be the graph $\theta_{n_1,\dots,n_{k_1+k_2+k_3}}$ consisting of lines $L_{3r_1+1},\dots,L_{3r_{k_1}+1},L_{3t_1+2},\dots,L_{3t_{k_2}+2}$, $L_{3s_1},\dots,L_{3s_{k_3}}$ such that $k_1,k_2,k_3>0$. Then*

$$reg(G)=c_G=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_3.$$

Proof Suppose that the edges of G are labeled by $e_j^{(i)}=\{x_j^{(i)},x_{j+1}^{(i)}\}$ such that $x_1^{(i)}=x$ for any i and $x_{3r_l+1}^{(l)}=x_{3t_m+2}^{(k_1+m)}=x_{3s_n}^{(k_1+k_2+n)}=y$ for $1\le l\le k_1$, $1\le m\le k_2$ and $1\le n\le k_3$. It is easily seen that the set

$$\{e_2^{(1)},e_5^{(1)},\dots,e_{3r_1-1}^{(1)},\dots,e_2^{(k_1)},e_5^{(k_1)},\dots,e_{3r_{k_1}-1}^{(k_1)},e_2^{(k_1+1)},e_5^{(k_1+1)},\dots,e_{3t_1-1}^{(k_1+1)},\dots,$$
$$e_2^{(k_1+k_2)},e_5^{(k_1+k_2)},\dots,e_{3t_{k_2}-1}^{(k_1+k_2)},e_2^{(k_1+k_2+1)},e_5^{(k_1+k_2+1)},\dots,e_{3s_1-4}^{(k_1+k_2+1)},\dots,e_2^{(k_1+k_2+k_3)},$$
$$e_5^{(k_1+k_2+k_3)},\dots,e_{3s_{k_3}-4}^{(k_1+k_2+k_3)}\}$$

is pairwise 3-disjoint in G. It follows that $\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_3\le c_G$. We have $G'=G\setminus\{x\}=L_{3r_1}\sqcup\dots\sqcup L_{3r_{k_1}}\sqcup L_{3t_1+1}\sqcup\dots\sqcup L_{3t_{k_2}+1}\sqcup L_{3s_1-1}\sqcup\dots\sqcup L_{3s_{k_3}-1}$ and $G''=G\setminus N_G[x]=L_{3r_1-1}\sqcup\dots\sqcup L_{3r_{k_1}-1}\sqcup L_{3t_1}\sqcup\dots\sqcup L_{3t_{k_2}}\sqcup L_{3s_1-2}\sqcup\dots\sqcup L_{3s_{k_3}-2}$. Using Lemma 2.2, $reg(G')\le max\{reg(G'\setminus\{y\}),reg(G'\setminus N_{G'}[y])+1\}$, where $G'\setminus\{y\}$ is the disjoint union of $L_{3(r_1-1)+2},\dots,L_{3(r_{k_1}-1)+2},L_{3t_1},\dots,L_{3t_{k_2}},L_{3(s_1-1)+1},\dots,L_{3(s_{k_3}-1)+1}$ and $G'\setminus N_{G'}[y]$ is the disjoint union of $L_{3(r_1-1)+1},\dots,L_{3(r_{k_1}-1)+1},L_{3(t_1-1)+2},\dots,L_{3(t_{k_2}-1)+2},L_{3(s_1-1)},\dots,L_{3(s_{k_3}-1)}$. This yields that $reg(G'\setminus\{y\})=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_3$ and $reg(G'\setminus N_{G'}[y])=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_1-k_3$. Since $1-k_1\le 0$, then one obtains $reg(G')\le\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_3$. On the other hand, $G''\setminus\{y\}$ is the disjoint union of $L_{3(r_1-1)+1},\dots,L_{3(r_{k_1}-1)+1},L_{3(t_1-1)+2},\dots,L_{3(t_{k_2}-1)+2},L_{3(s_1-1)},\dots,L_{3(s_{k_3}-1)}$ and $G''\setminus N_{G''}[y]$ is the disjoint union of $L_{3(r_1-1)},\dots,L_{3(r_{k_1}-1)},L_{3(t_1-1)+1},\dots,L_{3(t_{k_2}-1)+1},L_{3(s_1-2)+2},\dots,L_{3(s_{k_3}-1)+2}$. One derives that $reg(G''\setminus\{y\})=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_1-k_3$ and $reg(G''\setminus N_{G''}[y])=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_1-k_2-k_3$. Since $1-k_2\le 0$ we conclude that $reg(G'')\le\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_1-k_3$. Assumption $k_1\ge 1$ forces $reg(G)=\sum_{i=1}^{k_1} r_i+\sum_{i=1}^{k_2} t_i+\sum_{i=1}^{k_3} s_i-k_3$. □

Theorem 3.6 *Let G be the graph $\theta_{n_1,\dots,n_{k_2}}$ consisting of lines $L_{3t_1+2},\dots,L_{3t_{k_2}+2}$. Then*

$$reg(G)=\sum_{i=1}^{k_2} t_i+1=c_G+1.$$

Proof Suppose that the edges of G are labeled by $e_j^{(i)}=\{x_j^{(i)},x_{j+1}^{(i)}\}$ such that $x_1^{(i)}=x$ and $x_{3t_i+2}^{(i)}=y$. It suffices to find an appropriate upper bound for $reg(G')$ and $reg(G'')$, where $G'=L_{3t_1+1}\sqcup\dots\sqcup L_{3t_{k_2}+1}$ and $G''=L_{3t_1}\sqcup\dots\sqcup L_{3t_{k_2}}$. Since $G'\setminus\{y\}$ is the disjoint union of $L_{3t_1},\dots,L_{3t_{k_2}}$ and $G'\setminus N_{G'}[y]$ is the disjoint union of $L_{3(t_1-1)+2},\dots,L_{3(t_{k_2}-1)+2}$, then one concludes that $reg(G'\setminus\{y\})=\sum_{i=1}^{k_2} t_i$ and $reg(G'\setminus N_{G'}[y])=\sum_{i=1}^{k_2} t_i$. Again, using Lemma 2.2, we get $reg(G')\le\sum_{i=1}^{k_2} t_i+1$.

Furthermore, $G''\setminus\{y\}$ is the disjoint union of $L_{3(t_1-1)+2},\dots,L_{3(t_{k_2}-1)+2}$ and $G''\setminus N_{G''}[y]$ is the disjoint union of $L_{3(t_1-1)+1},\dots,L_{3(t_{k_2}-1)+1}$. Thus, $reg(G''\setminus\{y\})=\sum_{i=1}^{k_2} t_i$ and $reg(G''\setminus N_{G''}[y])=\sum_{i=1}^{k_2} t_i-k_2$.

Since $1-k_2 \leq 0$, then $reg(G'') \leq \sum_{i=1}^{k_2} t_i$. Both facts and Lemma 2.2 imply $reg(G) \leq \sum_{i=1}^{k_2} t_i+1$. On the other hand, we can see that $\beta_{2\sum_{i=1}^{k_2} t_i+1,|V_G|}(G) \neq 0$; therefore $\sum_{i=1}^{k_2} t_i+1 \leq reg(G)$ and hence $reg(G) = \sum_{i=1}^{k_2} t_i+1$, as desired. It remains to prove $c_G = \sum_{i=1}^{k_2} t_i$. In order to show this fact, consider the strongly disjoint set $\mathbf{B} = \{B_1, \dots, B_l\}$ of bouquets in G. Any of the following situations may happen:

Case (1) : $x, y \notin F(\mathbf{B}) \cup R(\mathbf{B})$. Then there exist t_i bouquets with two flowers and one root in any line. The set $\{e_2^{(1)}, e_5^{(1)}, \dots, e_{3t_1-1}^{(1)}, e_2^{(k_2)}, e_5^{(k_2)}, \dots, e_{3t_{k_2}-1}^{(k_2)}\}$ is pairwise 3-disjoint in G.

Case (2) : $x, y \in F(\mathbf{B})$.

i. If the bouquets containing x and y are in the same line, as L_{3t_1+2}, then there exist $t_1 - 2$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3t_1+2} and for $2 \leq i \leq k_2$ there exist t_i bouquets with two flowers and one root in L_{3t_i+2}. Observe that the set

$$\{e_1^{(1)}, e_4^{(1)}, \dots, e_{3t_1-5}^{(1)}, e_{3t_1}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3t_2}^{(2)}, \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_{k_2}}^{(k_2)}\}$$

is pairwise 3-disjoint in G.

ii. If the bouquets containing x and y are in different lines, as L_{3t_1+2} and L_{3t_2+2}, then one has t_i bouquets with two flowers and one root in L_{3t_i+2} for $1 \leq i \leq k_2$. It is seen that the set $\{e_1^{(1)}, e_4^{(1)}, \dots, e_{3t_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3t_2}^{(2)}, \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_{k_2}}^{(k_2)}\}$ is pairwise 3-disjoint in G.

Case (3) : $x \in R(\mathbf{B})$ and $y \notin F(\mathbf{B}) \cup R(\mathbf{B})$. Suppose that x is the root of a bouquet with k_2 flowers. Moreover, we have $t_i - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3t_i+2-3} for $1 \leq i \leq k_2$. The set $\{e_3^{(1)}, e_6^{(1)}, \dots, e_{3t_1}^{(1)}, \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_{k_2}}^{(k_2)}\}$ is pairwise 3-disjoint in G.

Case (4) : $x \in F(\mathbf{B})$ and $y \notin F(\mathbf{B}) \cup R(\mathbf{B})$. Suppose that the bouquets containing x are in L_{3t_1+2}; then there exist t_i bouquets with two flowers and one root in L_{3t_i+2} for $1 \leq i \leq k_2$. Observe that

$$\{e_1^{(1)}, e_4^{(1)}, \dots, e_{3t_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3t_2}^{(2)} \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_{k_2}}^{(k_2)}\}$$

is pairwise 3-disjoint in G.

Case (5) : $x \in R(\mathbf{B})$ and $y \in F(\mathbf{B})$. Suppose that x is the root of a bouquet with k_2 flowers and the bouquet containing y with two flowers and one root lies in L_{3t_1+2}. Moreover, we have $t_1 - 1$ other bouquets with two flowers and one root in L_{3t_1+2-5} and $t_i - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3t_i+2-3} for $2 \leq i \leq k_2$. It is seen that $\{e_3^{(1)}, e_6^{(1)}, \dots, e_{3t_1}^{(1)}, \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_{k_2}}^{(k_2)}\}$ is pairwise 3-disjoint in G.

Case (6) : $x \in R(\mathbf{B})$ and $y \in R(\mathbf{B})$. Suppose that x and y lie in bouquets with k_2 flowers. Moreover, there exist $t_i - 1$ bouquets with two flowers and one root in L_{3t_i+2-4} for $1 \leq i \leq k_2$. Observe that

$$\{e_1^{(1)}, e_4^{(1)}, \dots, e_{3t_1-2}^{(1)}, e_{3t_1+1}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3t_2-3}^{(2)}, \dots, e_3^{(k_2)}, e_6^{(k_2)}, \dots, e_{3t_2-3}^{(k_2)}\}$$

is pairwise 3-disjoint in G.

It is easily seen that the considered sets have the maximum cardinality of a pairwise 3-disjoint set in G. Note that the above described cases are satisfied if we interchange x and y. Furthermore, one can check that the number of flowers of bouquets containing x or y as discussed above has no effect on the value of c_G. Altogether, one has $c_G = \sum_{i=1}^{k_2} t_i$ and so the result holds. □

Theorem 3.7 *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_3}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3s_1},\ldots,L_{3s_{k_3}}$ such that $k_1,k_3>0$. If $k_1=1$ then*

$$reg(G) = c_G + 1 = r_1 + \sum_{i=1}^{k_3} s_i - k_3 + 1,$$

and for $k_1 \geq 2$, the following relation is satisfied:

$$reg(G) = c_G = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3.$$

Proof Assume that the edges of L_{3r_i+1} are labeled by $e_j^{(i)} = \{x_j^{(i)}, x_{j+1}^{(i)}\}$ such that $x_1^{(i)} = x$ and $x_{3r_i+1}^{(i)} = y$ for $1 \leq i \leq k_1$ and the edges of L_{3s_i} are labeled by $e_j^{(k_1+i)} = \{x_j^{(k_1+i)}, x_{j+1}^{(k_1+i)}\}$ such that $x_1^{(k_1+i)} = x$ and $x_{3s_i}^{(k_1+i)} = y$ for $1 \leq i \leq k_3$. Let $G' = G \setminus \{x\}$ and $G'' = G \setminus N_G[x]$. Using Lemma 2.2,

$$c_G \leq reg(G) \leq max\{reg(G'), reg(G'')+1\},$$

where $G' = L_{3r_1} \sqcup \ldots \sqcup L_{3r_{k_1}} \sqcup L_{3(s_1-1)+2} \sqcup \ldots \sqcup L_{3(s_{k_3}-1)+2}$ and $G'' = L_{3(r_1-1)+2} \sqcup \ldots \sqcup L_{3(r_{k_1}-1)+2} \sqcup L_{3(s_1-1)+1} \sqcup \ldots \sqcup L_{3(s_1-1)+1}$. Since $G' \setminus \{y\}$ is the disjoint union of $L_{3(r_1-1)+2},\ldots,L_{3(r_{k_1}-1)+2},L_{3(s_1-1)+1},\ldots,L_{3(s_{k_3}-1)+1}$ and $G' \setminus N_{G'}[y]$ is the disjoint union of $L_{3(r_1-1)+1},\ldots,L_{3(r_{k_1}-1)+1},L_{3(s_1-1)},\ldots,L_{3(s_{k_3}-1)}$, then we get $reg(G' \setminus \{y\}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3$ and $reg(G' \setminus N_{G'}[y]) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_1 - k_3$. According to $1-k_1 \leq 0$ and using Lemma 2.2, one concludes that

$$reg(G') \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3.$$

Applying the same argument, $G'' \setminus \{y\}$ is the disjoint union of $L_{3(r_1-1)+1},\ldots,L_{3(r_{k_1}-1)+1},L_{3(s_1-1)},\ldots,L_{3(s_{k_3}-1)}$ and $G'' \setminus N_{G''}[y]$ is the disjoint union of $L_{3(r_1-1)},\ldots,L_{3(r_{k_1}-1)},L_{3(s_1-2)+2},\ldots,L_{3(s_{k_3}-2)+2}$. Hence, we obtain that $reg(G'' \setminus \{y\}) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_1 - k_3$ and $reg(G'' \setminus N_{G''}[y]) = \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_1 - k_3$. Then $reg(G'') \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_1 - k_3 + 1$. Altogether, we conclude that

$$reg(G) \leq max\{\sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3, \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_1 - k_3 + 2\}.$$

If $k_1 = 1$ then $reg(G) \leq r_1 + \sum_{i=1}^{k_3} s_i - k_3 + 1$, while for $k_1 \geq 2$ it follows that $reg(G) \leq \sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3$. We can see that $\beta_{2r_1+2\sum_{i=1}^{k_3} s_i - k_3, |V_G|}(G) \neq 0$. In the case $k_1 = 1$, one derives that $r_1 + \sum_{i=1}^{k_3} s_i - k_3 + 1 \leq reg(G)$ and hence $reg(G) = r_1 + \sum_{i=1}^{k_3} s_i - k_3 + 1$, as required.

Now we want to clarify $c_G = r_1 + \sum_{i=1}^{k_3} s_i - k_3 = reg(G) - 1$. Consider the strongly disjoint set $\mathbf{B} = \{B_1, \ldots, B_l\}$ of bouquets in G. Any of the following situations may happen:

Case (1) : $x, y \notin F(\mathbf{B}) \cup R(\mathbf{B})$. Then there exist $r_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3r_1+1} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i} for $1 \le i \le k_3$. The set

$$\{e_2^{(1)}, e_5^{(1)}, \ldots, e_{3r_1-1}^{(1)}, e_3^{(2)}, e_6^{(2)}, \ldots, e_{3s_1-3}^{(2)}, \ldots, e_3^{(k_3+1)}, e_6^{(k_3+1)} \ldots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Case (2) : $x, y \in F(\mathbf{B})$.

i) If bouquets containing x and y lie in the same line, as L_{3r_1+1}, we can use the same argument as in the previous case.

ii) If bouquets containing x and y lie in the same line, as L_{3s_1}, then there exist $r_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3r_1+1-2} and s_1 bouquets with two flowers and one root in L_{3s_1} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $2 \le i \le k_3$. Hence, the set

$$\{e_2^{(1)}, e_5^{(1)}, \ldots, e_{3r_1-4}^{(1)}, e_2^{(2)}, e_5^{(2)}, \ldots, e_{3s_1-1}^{(2)}, e_2^{(3)}, e_5^{(3)}, \ldots, e_{3s_2-4}^{(3)}, \ldots, e_2^{(k_3+1)}, e_5^{(k_3+1)}, \ldots, e_{3s_{k_3}-4}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

iii) If bouquets containing x and y lie in the different lines, as L_{3r_1+1} and L_{3s_1}, then there exist r_1 bouquets with two flowers and one root in L_{3r_1+1-1} and $s_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_1} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $2 \le i \le k_3$. Then the set

$$\{e_1^{(1)}, e_4^{(1)}, \ldots, e_{3r_1-2}^{(1)}, e_4^{(2)}, e_7^{(2)}, \ldots, e_{3s_1-2}^{(2)}, e_3^{(3)}, e_6^{(3)}, \ldots, e_{3s_2-3}^{(3)}, \ldots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \ldots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

iv) If bouquets containing x and y lie in different lines, as L_{3s_1} and L_{3s_2}, then there exist $r_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3r_1+1-2} and $s_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_1-1}, $s_2 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_2-1} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $3 \le i \le k_3$. Then the set

$$\{e_2^{(1)}, e_5^{(1)}, \ldots, e_{3r_1-4}^{(1)}, e_{3r_1-1}^{(1)}, e_2^{(2)}, \quad e_5^{(2)}, \ldots, e_{3s_1-4}^{(2)}, e_4^{(3)}, e_7^{(3)}, \ldots, e_{3s_2-2}^{(3)}, e_2^{(4)}, e_5^{(4)}, \ldots, e_{3s_3-4}^{(4)}, \ldots,$$
$$e_2^{(k_3+1)}, e_5^{(k_3+1)}, \ldots, e_{3s_{k_3}-4}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Case (3) : $x \in R(\mathbf{B})$ and $y \notin R(\mathbf{B}) \cup R(\mathbf{B})$. Suppose that x is the root of a bouquet with $k_3 + 1$ flowers. Then there exist $r_1 - 1$ bouquets with two flowers and one root in L_{3r_1+1-3} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-3} for $1 \le i \le k_3$. Then the set

$$\{e_4^{(1)}, e_7^{(1)}, \dots, e_{3r_1-2}^{(1)}, e_1^{(2)}, e_4^{(2)}, \dots, e_{3s_1-2}^{(2)}, e_3^{(3)}, e_6^{(3)}, \dots, e_{3s_2-3}^{(3)}, \dots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \dots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Case (4) : $x \in F(\mathbf{B})$ and $y \notin F(\mathbf{B}) \cup R(\mathbf{B})$.

i) Assume that x lies in bouquets with two flowers and one root in L_{3r_1+1}. Hence, there exist r_1 bouquets with two flowers and one root in L_{3r_1} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $1 \le i \le k_3$. Then the set

$$\{e_1^{(1)}, e_4^{(1)}, \dots, e_{3r_1-2}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3s_1-3}^{(2)}, \dots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \dots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

ii) Assume that x lies in bouquets with two flowers and one root in L_{3s_1}. Hence, there exist $r_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3r_1+1-2}, $s_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_1-1}, and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-2} for $2 \le i \le k_3$. Then the set

$$\{e_2^{(1)}, e_5^{(1)}, \dots, e_{3r_1-4}^{(1)}, e_{3r_1-1}^{(1)}, e_2^{(2)}, e_5^{(2)}, \dots, e_{3s_1-4}^{(2)}, \dots, e_2^{(k_3+1)}, e_5^{(k_3+1)}, \dots, e_{3s_{k_3}-4}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Case (5) : $x \in R(\mathbf{B})$ and $y \in F(\mathbf{B})$. Assume that x is the root of a bouquet with $k_3 + 1$ flowers.

i) If the bouquet containing y lies in L_{3r_1+1}, then there exist $r_1 - 1$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3r_1+1-2} and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-3} for $1 \le i \le k_3$. Then the set

$$\{e_3^{(1)}, e_6^{(1)}, \dots, e_{3r_1-3}^{(1)}, e_{3r_1}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3s_1-3}^{(2)}, \dots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \dots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

ii) If the bouquet containing y lies in L_{3s_1}, then there exist $r_1 - 1$ bouquets with two flowers and one root in L_{3r_1+1-3}, $s_1 - 1$ bouquets with two flowers and one root in L_{3s_1-2}, and $s_i - 1$ bouquets with two flowers and one root in L_{3s_i-3} for $2 \le i \le k_3$. Then the set

$$\{e_3^{(1)}, e_6^{(1)}, \dots, e_{3r_1-3}^{(1)}, e_1^{(2)}, e_4^{(2)}, \dots, e_{3s_1-2}^{(2)}, e_3^{(3)}, e_6^{(3)}, \dots, e_{3s_2-3}^{(3)} \dots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \dots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Case (6) : $x, y \in R(\mathbf{B})$. Assume that x and y are the roots of the bouquets with $k_3 + 1$ flowers. Hence, there exist $r_1 - 1$ bouquets with two flowers and one root in L_{3r_1+1-4} and $s_i - 2$ bouquets with two flowers and one root and one bouquet with one flower and one root in L_{3s_i-4} for $1 \le i \le k_3$. Then the set

$$\{e_3^{(1)}, e_6^{(1)}, \dots, e_{3r_1-3}^{(1)}, e_{3r_1}^{(1)}, e_3^{(2)}, e_6^{(2)}, \dots, e_{3s_2-3}^{(2)}, \dots, e_3^{(k_3+1)}, e_6^{(k_3+1)}, \dots, e_{3s_{k_3}-3}^{(k_3+1)}\}$$

is pairwise 3-disjoint in G.

Thus, we can use the same argument as in the demonstration of the previous theorem and derive $c_G = r_1 + \sum_{i=1}^{k_3} s_i - k_3$, as required.

Suppose that $k_1 \geq 2$. We intend to show $\sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3 \leq c_G$. The set

$$\{e_2^{(1)}, e_5^{(1)}, \ldots, e_{3r_1-1}^{(1)}, \ldots, e_2^{(k_1)}, e_5^{(k_1)}, \ldots, e_{3r_{k_1}-1}^{(k_1)}, e_2^{(k_1+1)}, e_5^{(k_1+1)}, \ldots, e_{3s_1-4}^{(k_1+1)}, \ldots, e_2^{(k_1+k_3)}, e_5^{(k_1+k_3)}, \ldots, e_{3s_{k_3}-4}^{(k_1+k_3)}\}$$

is pairwise 3-disjoint in G and hence $\sum_{i=1}^{k_1} r_i + \sum_{i=1}^{k_3} s_i - k_3 \leq c_G$, as desired. □

Corollary 3.8 *Let G be the graph $\theta_{n_1,\ldots,n_k}$. Then we have $reg(G) = c_G$ unless $n_i \equiv 2 \pmod 3$ for any $1 \leq i \leq k$ or there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$ and for any $1 \leq i \neq j \leq k$ we have $n_i \equiv 0 \pmod 3$; then it yields $reg(G) = c_G + 1$.*

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