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On H-antimagicness of Cartesian product of graphs

Martin BAČA[1,*], Andrea SEMANIČOVÁ-FEŇOVČÍKOVÁ[1], Muhammad Awais UMAR[2], Des WELYYANTI[3]

[1]Department of Applied Mathematics and Informatics, Technical University, Košice, Slovakia
[2]Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
[3]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia

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Abstract: A graph $G = (V(G), E(G))$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. A graph G admitting an H-covering is called (a, d)-H-*antimagic* if there is a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that, for all subgraphs H' of G isomorphic to H, the H-weights, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$, constitute an arithmetic progression with the initial term a and the common difference d. In this paper we provide some sufficient conditions for the Cartesian product of graphs to be H-antimagic. We use partitions subsets of integers for describing desired H-antimagic labelings.

Key words: H-covering, super (a, d)-H-antimagic graph, partition of set, Cartesian product

1. Introduction

Let $G = (V, E)$ be a finite simple graph without isolated vertices. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. Then it is said that G admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering*. If every H_i is isomorphic to a given graph H, then G admits an H-*covering*.

For a (p, q)-graph G with p vertices and q edges, a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is a total labeling of G. Suppose that G admits an H-covering. Then for the subgraph H under the total labeling f, we define the associated H-*weight* as

$$wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

The graph G is called (a, d)-H-*antimagic* if there exists a total labeling f such that, for all subgraphs H' of G isomorphic to H, the H-weights constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. If $f(V) = \{1, 2, \ldots, p\}$, G is said to be *super* (a, d)-H-*antimagic*. If G is a (super) (a, d)-H-antimagic graph then the corresponding total labeling f is called the *(super)* (a, d)-H-*antimagic labeling*. For $d = 0$, the (super) (a, d)-H-antimagic graph is called H-*(super)magic*.

*Correspondence: martin.baca@tuke.sk

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The H-(super)magic labelings were first studied by Gutiérrez and Lladó [12] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [18] and Enomoto et al. [10], respectively. In [12] are considered star-(super)magic and path-(super)magic labelings of some connected graphs and it is proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Maryati et al. [22] gave P_h-(super)magic labelings of some trees such as shrubs, subdivision of shrubs, and banana tree graphs. Lladó and Moragas [20] investigated C_n-(super)magic graphs and proved that wheels, windmills, books, and prisms are C_h-magic for some h. Ngurah et al. [25] proved that chains, wheels, triangles, ladders, and grids are cycle-supermagic. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [17]. Inayah et al. [14] gave a connection between graceful trees and antimagic H-decomposition of complete graphs. Maryati et al. [23] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h. Maryati et al. [24] and Salman et al. [26] proved that certain families of trees are path-supermagic.

Motivated by H-(super)magic labelings, Inayah et al. [15] introduced the (a,d)-H-antimagic labeling. In [16] they investigated the super (a,d)-H-antimagic labelings for some shackles of a connected graph H. In [5] it was proved that wheels are cycle-antimagic. In [3, 8] was investigated the existence of super (a,d)-H-antimagic labelings for disconnected graphs. There it is proved that if a graph G admits a (super) (a,d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b,d)-H-antimagic labeling as well.

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [19]. Further information on super d-antimagic labelings can be found in [2, 7].

If H is isomorphic to K_2, then (super) (a,d)-K_2-antimagic total labelings are also called (super) (a,d)-edge-antimagic total. These labelings are the generalization of the edge-magic and super edge-magic labelings that were introduced by Kotzig and Rosa [18] and Enomoto et al. [10], respectively. However, it is worthwhile mentioning that a type of graph called strongly indexable had already been defined in [1] by Acharya and Hedge and it turns out that strongly indexable graphs are equivalent to super edge-magic graphs. For further information on (super) edge-antimagic total labelings, one can see [4, 6, 9, 11, 13, 21].

As can be seen also from the previous survey most known results related to the study of H-magic and H-antimagic labelings deal with some special classes of graphs. In this paper we describe some sufficient conditions that guarantee the existence of the H-supermagic or super H-antimagic labelings for the Cartesian product of two graphs. We prove that if there exists appropriate edge-covering in $G_1 \square G_2$ then the existence of H-(anti)magic labeling of $G_1 \square G_2$ depends only on some parity conditions for orders and sizes of graphs G_1 and G_2. We will use a technique of partitioning sets of integers in order to construct the desired labelings.

2. Preliminaries

The constructions of labelings will be made by using partition subsets of integers. Let n, k, and i be positive integers. Consider the partition $\mathcal{P}_k^n$ of the set of integers $\{1, 2, \dots, kn\}$ into k-tuples such that the ith k-tuple in the partition is defined in the following way:

For k even, $k \geq 2$, we define

$$\mathcal{P}_k^n(i) = \{i, 2n+1-i, 2n+i, 4n+1-i, \dots, (k-2)n+i, kn+1-i\}. \tag{1}$$

For n, k odd, $k \geq 3$, we define

$$\mathcal{P}_k^n(i) = \begin{cases} \{\frac{n+1}{2} + \frac{i-1}{2}, n+1+\frac{i-1}{2}, 3n+1-i, 3n+i, 5n+1-i, \\ \qquad 5n+i, 7n+1-i, \dots, (k-2)n+i, kn+1-i\} \\ \qquad \text{for } i \text{ odd}, \\ \{\frac{i}{2}, n+\frac{n+1}{2}+\frac{i}{2}, 3n+1-i, 3n+i, 5n+1-i, \\ \qquad 5n+i, 7n+1-i, \dots, (k-2)n+i, kn+1-i\} \\ \qquad \text{for } i \text{ even}. \end{cases} \tag{2}$$

It is easy to see that for both cases the sum of all numbers in the ith k-tuple is equal to

$$\sigma\left(\mathcal{P}_k^n(i)\right) = \sum_{i=1}^{k} \mathcal{P}_k^n(i) = \frac{(1+kn)k}{2}, \tag{3}$$

for $i = 1, 2, \dots, n$. Let us recall that from the divisibility it follows that if k is odd then n has to be odd too. By the notation $\mathcal{P}_k^n(i) \oplus c$ we will mean that the constant c is added to every number in $\mathcal{P}_k^n(i)$.

A *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \Box G_2$, is the graph with vertex set $V(G_1) \Box V(G_2)$, where two vertices (u, u') and (v, v') are adjacent if and only if $u = v$ and $u'v' \in E(G_2)$ or $u' = v'$ and $uv \in E(G_1)$.

Let G_1 be a (p_1, q_1)-graph and G_2 be a (p_2, q_2)-graph. Let the symbol v_i^j denote the vertex in $G_1 \Box G_2$ corresponding to the vertex $v_i \in V(G_1)$, $i = 1, 2, \dots, p_1$, in the jth copy of G_1, $j = 1, 2, \dots, p_2$. Let the symbol e^j denote the edge in $G_1 \Box G_2$ corresponding to the edge $e \in E(G_1)$ in the jth copy of G_1, $j = 1, 2, \dots, p_2$ and let the symbol e_i denote the edge in $G_1 \Box G_2$ corresponding to the edge $e \in E(G_2)$ in the ith copy of G_2, $i = 1, 2, \dots, p_1$. Thus the vertex set and the edge set of $G_1 \Box G_2$ are as follows:

$$\begin{aligned} V(G_1 \Box G_2) =& \{v_i^j : i = 1, 2, \dots, p_1, \ j = 1, 2, \dots, p_2\} \\ E(G_1 \Box G_2) =& \{e^j : e \in E(G_1), \ j = 1, 2, \dots, p_2\} \\ & \cup \{e_i : e \in E(G_2), \ i = 1, 2, \dots, p_1\}. \end{aligned}$$

The graph $G_1 \Box G_2$ is of order $p_1 p_2$ and of size $p_1 q_2 + p_2 q_1$.

There are several known classes of *cycle*-supermagic graphs obtained by the Cartesian product of two graphs. Lladó and Moragas [20] showed that the graph $G \Box P_2$ is C_4-supermagic if G is a C_4-free supermagic graph of odd size. Ngurah et al. in [25] proved that ladder $P_n \Box P_2$ and book $K_{1,n} \Box P_2$ are C_4-supermagic for any integer n. Moreover, they proved that the grid $P_n \Box P_m$ is C_4-supermagic for any integer $m \geq 3$ and $n = 3, 4, 5$.

3. Constructions of $(H \Box G)$-supermagic labelings

In this section we examine the existence of $(H \Box G_2)$-supermagic labelings of the Cartesian product $G_1 \Box G_2$, where G_1 and G_2 satisfy certain conditions.

Theorem 1 *Let G_1 be a graph admitting an H-covering given by t subgraphs isomorphic to H. If G_2 is a graph of even order and even size and the graph $G_1 \Box G_2$ contains exactly t subgraphs isomorphic to $H \Box G_2$ then the graph $G_1 \Box G_2$ is $(H \Box G_2)$-supermagic.*

Proof Let G_1 be a (p_1, q_1)-graph. Let G_2 be a (p_2, q_2)-graph where $p_2 \equiv 0 \pmod 2$ and $q_2 \equiv 0 \pmod 2$. Assume that G_1 admits an H-covering containing t subgraphs $H_1, H_2, \dots, H_t$. Let f be any total labeling of G_1, $f : V(G_1) \cup E(G_1) \to \{1, 2, \dots, p_1 + q_1\}$, such that the vertices of G_1 are labeled with the values $1, 2, \dots, p_1$.

Let the graph $G_1 \square G_2$ contain exactly t subgraphs, say $H_1 \square G_2, H_2 \square G_2, \dots, H_t \square G_2$. Note that they are isomorphic to the subgraph $H \square G_2$.

Define the labeling g of $G_1 \square G_2$ in the following way:

$$
\begin{aligned}
&g(v_i^j) = (j-1)p_1 + f(v_i) && \text{if } i = 1, 2, \dots, p_1, \\
& && \quad j = 1, 2, \dots, \frac{p_2}{2}, \\
&g(v_i^j) = jp_1 + 1 - f(v_i) && \text{if } i = 1, 2, \dots, p_1, \\
& && \quad j = \frac{p_2}{2} + 1, \frac{p_2}{2} + 2, \dots, p_2, \\
&g(e^j) = (p_2 - 1)p_1 + (j-1)q_1 + f(e) && \text{if } e \in E(G_1), j = 1, 2, \dots, \frac{p_2}{2}, \\
&g(e^j) = (p_2 + 1)p_1 + jq_1 + 1 - f(e) && \text{if } e \in E(G_1), \\
& && \quad j = \frac{p_2}{2} + 1, \frac{p_2}{2} + 2, \dots, p_2, \\
&\{g(e_i) : e \in E(G_2)\} = \mathcal{P}_{q_2}^{p_1}(i) \oplus (p_1 p_2 + p_2 q_1) && \text{if } i = 1, 2, \dots, p_1.
\end{aligned}
$$

Since $f(V(G_1)) = \{1, 2, \dots, p_1\}$ and $f(E(G_1)) = \{p_1 + 1, p_1 + 2, \dots, p_1 + q_1\}$, the labeling g assigns the values $1, 2, \dots, p_1$ to the vertices $v_1^1, v_2^1, \dots, v_{p_1}^1$, the values $p_1 + 1, p_1 + 2, \dots, 2p_1$ to the vertices $v_1^2, v_2^2, \dots, v_{p_1}^2$, $\dots$, and the values $(p_2 - 1)p_1 + 1, (p_2 - 1)p_1 + 2, \dots, p_1 p_2$ to the vertices $v_1^{p_2}, v_2^{p_2}, \dots, v_{p_1}^{p_2}$.

Under the labeling g, the edges in the first copy of G_1 successively attain values $p_1 p_2 + 1, p_1 p_2 + 2, \dots, p_1 p_2 + q_1$, the edges in the second copy of G_1 successively assume values $p_1 p_2 + q_1 + 1, p_1 p_2 + q_1 + 2, \dots, p_1 p_2 + 2q_1$, $\dots$, and the edges in the p_2th copy of G_1 successively assume values $p_1 p_2 + (p_2 - 1)q_1 + 1, p_1 p_2 + (p_2 - 1)q_1 + 2, \dots, p_1 p_2 + p_2 q_1$. Values $p_1 p_2 + p_2 q_1 + 1, p_1 p_2 + p_2 q_1 + 2, \dots, p_1 p_2 + p_2 q_1 + p_1 q_2$ are assigned to the edges in the copies of G_2. Thus g is a total labeling of $G_1 \square G_2$, where the smallest possible labels are assigned to the vertices.

For the $(H \square G_2)$-weight of the subgraph $H_l \square G_2$, $l = 1, 2, \dots, t$, we have

$$
\begin{aligned}
wt_g(H_l \square G_2) =& \sum_{j=1}^{p_2} \sum_{i:\, v_i \in V(H_l)} g(v_i^j) + \sum_{j=1}^{p_2} \sum_{e \in E(H_l)} g(e^j) + \sum_{i:\, v_i \in V(H_l)} \sum_{e \in E(G_2)} g(e_i) \\
=& \left(\sum_{j=1}^{\frac{p_2}{2}} \sum_{i:\, v_i \in V(H_l)} ((j-1)p_1 + f(v_i)) \right) + \left(\sum_{j=\frac{p_2}{2}+1}^{p_2} \sum_{i:\, v_i \in V(H_l)} (jp_1 + 1 - f(v_i)) \right) \\
&+ \left(\sum_{j=1}^{\frac{p_2}{2}} \sum_{e \in E(H_l)} ((p_2 - 1)p_1 + (j-1)q_1 + f(e)) \right)
\end{aligned}
$$

$$+\left(\sum_{j=\frac{p_2}{2}+1}^{p_2}\sum_{e\in E(H_l)}((p_2+1)p_1+jq_1+1-f(e))\right)+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)$$

$$=\frac{(1+p_1p_2)p_2}{2}|V(H)|+\left(p_2^2p_1+\frac{p_2}{2}+\frac{q_1p_2^2}{2}\right)|E(H)|+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i).$$

We express the next term as follows:

$$\begin{aligned}\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)&=\sum_{i:\,v_i\in V(H_l)}(\sigma(\mathcal{P}_{q_2}^{p_1}(i)\oplus(p_1p_2+p_2q_1)))\\&=\sum_{i:\,v_i\in V(H_l)}(\sigma(\mathcal{P}_{q_2}^{p_1}(i))+(p_1p_2+p_2q_1)q_2)\\&=\sum_{i:\,v_i\in V(H_l)}(\sigma(\mathcal{P}_{q_2}^{p_1}(i)))+(p_1p_2+p_2q_1)q_2|V(H)|.\end{aligned}$$

According to (3) we get

$$\sum_{i:\,v_i\in V(H_l)}(\sigma(\mathcal{P}_{q_2}^{p_1}(i)))=\frac{(1+p_1q_2)q_2}{2}|V(H)|.$$

Thus

$$\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)=\left(p_1p_2+p_2q_1+\frac{1+p_1q_2}{2}\right)q_2|V(H)|.$$

In the previous part we used the argument that all H_l are isomorphic to H and thus $|V(H_l)|=|V(H)|$, $|E(H_l)|=|E(H)|$, for $l=1,2,\dots,t$.

Summarizing all the corresponding expressions we get

$$\begin{aligned}wt_g(H_l\square G_2)=&\frac{(1+p_1p_2)p_2+(2p_1p_2+2p_2q_1+p_1q_2+1)q_2}{2}|V(H)|\\&+\left(p_1p_2^2+\frac{p_2}{2}+\frac{q_1p_2^2}{2}\right)|E(H)|,\end{aligned}$$

for $l=1,2,\dots,t$. It means that all $(H_l\square G_2)$-weights are the same. This concludes the proof. □

When G_2 is a graph of odd size, and of even order, then by using a similar method as in the previous theorem we are able to prove the existence of the $(H\square G_2)$-supermagic labeling of the graph $G_1\square G_2$. In this case the graph G_1 has to be of odd order.

Theorem 2 *Let G_1 be a graph of odd order admitting an H-covering given by t subgraphs isomorphic to H. If G_2 is a graph of even order and odd size and the graph $G_1\square G_2$ contains exactly t subgraphs isomorphic to $H\square G_2$ then the graph $G_1\square G_2$ is $(H\square G_2)$-supermagic.*

Proof In order to obtain the result it is sufficient to use the same total labeling defined in the proof of Theorem 1 and partition (2). □

4. Constructions of super $(H\square G)$-antimagic labelings

In this section we study the existence of the super $(H\square G_2)$-antimagic labelings of $G_1\square G_2$, where we suppose that graph G_1 admits a super H-antimagic labeling and $G_1\square G_2$ admits an $(H\square G_2)$-covering. For the construction of the desired labelings we use the partitions (1) and (2). Let us recall that according to (3) the odd size of G_2 necessitates the odd order of G_1.

Theorem 3 *Let G_1 be a super (a,d)-H-antimagic graph containing t subgraphs isomorphic to H. If G_2 is a graph of even size and the graph $G_1\square G_2$ contains exactly t subgraphs isomorphic to $H\square G_2$ then the graph $G_1\square G_2$ is super $(b,|V(G_2)|d)$-$(H\square G_2)$-antimagic, where the parameter b depends on the parameter a and on orders and sizes of graphs G_1, G_2, and H.*

Proof Let $f: V(G_1)\cup E(G_1)\to\{1,2,\dots,p_1+q_1\}$ be a super (a,d)-H-antimagic labeling of a (p_1,q_1)-graph G_1 and let $H_1,H_2,\dots,H_t$ be the family of all subgraphs of G_1 isomorphic to H. Clearly, the set of all H-weights is as follows:

$$\{wt_f(H_l): l=1,2,\dots,t\}=\{a,a+d,\dots,a+(t-1)d\} \tag{4}$$

and the smallest possible labels $1,2,\dots,p_1$ appear on the vertices of G_1.

Suppose that G_2 is a (p_2,q_2)-graph with $q_2\equiv 0 \pmod 2$ and the graph $G_1\square G_2$ contains exactly t subgraphs, say $H_1\square G_2, H_2\square G_2,\dots,H_t\square G_2$, all isomorphic to a subgraph $H\square G_2$.

Define the labeling g of $G_1\square G_2$ in the following way:

$$\begin{aligned}
g(v_i^j)&=(j-1)p_1+f(v_i) && \text{if } i=1,2,\dots,p_1,\\
& && \quad j=1,2,\dots,p_2,\\
g(e^j)&=(p_2-1)p_1+(j-1)q_1+f(e) && \text{if } e\in E(G_1), j=1,2,\dots,p_2,\\
\{g(e_i): e\in E(G_2)\}&=\mathcal{P}_{q_2}^{p_1}(i)\oplus(p_1p_2+p_2q_1) && \text{if } i=1,2,\dots,p_1.
\end{aligned}$$

We can see that the labeling g assigns the values $1,2,\dots,\ p_1$ to the vertices $v_1^1,v_2^1,\dots,v_{p_1}^1$, the values $p_1+1,p_1+2,\dots,2p_1$ to the vertices $v_1^2,v_2^2,\dots,v_{p_1}^2$, $\dots$, and the values $(p_2-1)p_1+1,(p_2-1)p_1+2,\dots,p_1p_2$ to the vertices $v_1^{p_2},v_2^{p_2},\dots,v_{p_1}^{p_2}$. The edges in the first copy of G_1 successively assume values $p_1p_2+1,p_1p_2+2,\dots,p_1p_2+q_1$, the edges in the second copy of G_1 successively attain values $p_1p_2+q_1+1,p_1p_2+q_1+2,\dots,p_1p_2+2q_1$, $\dots$, and the edges in the p_2th copy of G_1 successively assume values $p_1p_2+(p_2-1)q_1+1,p_1p_2+(p_2-1)q_1+2\dots,p_1p_2+p_2q_1$. Values $p_1p_2+p_2q_1+1,p_1p_2+p_2q_1+2,\dots,p_1p_2+p_2q_1+p_1q_2$ are assigned to the edges in the copies of G_2. Thus the labeling g is a bijection from the vertex set and the edge set of $G_1\square G_2$ onto the set $\{1,2,\dots,p_1p_2+p_2q_1+p_1q_2\}$ and the vertices of $G_1\square G_2$ are labeled with the smallest possible numbers.

For the $(H\square G_2)$-weight of the subgraph $H_l\square G_2$, $l=1,2,\dots,t$, we have

$$\begin{aligned}
wt_g(H_l\square G_2)=&\sum_{j=1}^{p_2}\sum_{i:\,v_i\in V(H_l)} g(v_i^j)+\sum_{j=1}^{p_2}\sum_{e\in E(H_l)} g(e^j)+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)} g(e_i)\\
=&\sum_{j=1}^{p_2}\sum_{i:\,v_i\in V(H_l)}((j-1)p_1+f(v_i))\\
&+\sum_{j=1}^{p_2}\sum_{e\in E(H_l)}((p_2-1)p_1+(j-1)q_1+f(e))+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)} g(e_i)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{(p_2-1)p_1p_2}{2}|V(H)|+p_2\sum_{v_i\in V(H_l)}f(v_i)\\
&+\left((p_2-1)p_1p_2+\frac{(p_2-1)q_1p_2}{2}\right)|E(H)|\\
&+p_2\sum_{e\in E(H_l)}f(e)+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)\\
=&\frac{(p_2-1)p_1p_2}{2}|V(H)|+\left((p_2-1)p_1p_2+\frac{(p_2-1)q_1p_2}{2}\right)|E(H)|\\
&+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)+p_2\left(\sum_{v_i\in V(H_l)}f(v_i)+\sum_{e\in E(H_l)}f(e)\right)\\
=&\frac{(p_2-1)p_1p_2}{2}|V(H)|+\left((p_2-1)p_1p_2+\frac{(p_2-1)q_1p_2}{2}\right)|E(H)|\\
&+\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)+p_2wt_f(H_l).
\end{aligned}
$$

Analogously as in the proof of Theorem 1 we get

$$\sum_{i:\,v_i\in V(H_l)}\sum_{e\in E(G_2)}g(e_i)=\left(p_1p_2+p_2q_1+\frac{1+p_1q_2}{2}\right)q_2|V(H)|.$$

As every subgraph H_l is isomorphic to H, we have $|V(H_l)|=|V(H)|$ and $|E(H_l)|=|E(H)|$, for $l=1,2,\dots,t$.

Thus

$$
\begin{aligned}
wt_g(H_l\Box G_2)=&\frac{(p_2-1)p_1p_2}{2}|V(H)|+\left((p_2-1)p_1p_2+\frac{(p_2-1)q_1p_2}{2}\right)|E(H)|\\
&+\left(p_1p_2+p_2q_1+\frac{1+p_1q_2}{2}\right)q_2|V(H)|+p_2wt_f(H_l),
\end{aligned}
$$

for $l=1,2,\dots,t$.

If we denote

$$
\begin{aligned}
A=&\left[\left(p_1p_2+p_2q_1+\frac{1+p_1q_2}{2}\right)q_2+\frac{(p_2-1)p_1p_2}{2}\right]|V(H)|\\
&+\left((p_2-1)p_1p_2+\frac{(p_2-1)q_1p_2}{2}\right)|E(H)|,
\end{aligned}
$$

then

$$wt_g(H_l\Box G_2)=A+p_2wt_f(H_l).$$

According to (4) we get that the set of all the $(H\Box G_2)$-weights under the labeling g is

$$
\begin{aligned}
&\{wt_g(H_l\Box G_2):l=1,2,\dots,t\}\\
&\qquad=\{A+p_2a,A+p_2a+p_2d,\dots,A+p_2a+(t-1)p_2d\}.
\end{aligned}
$$

This means that the graph $G_1\Box G_2$ is super $(A+p_2a,p_2d)$-$(H\Box G_2)$-antimagic. □

If we use the total labeling g considered in the proof of Theorem 3 and the partition (2) then we are able to prove the following theorem.

Theorem 4 *Let G_1 be a super (a,d)-H-antimagic graph of odd order containing t subgraphs isomorphic to H. If G_2 is a graph of odd size and the graph $G_1 \square G_2$ contains exactly t subgraphs isomorphic to $H \square G_2$ then the graph $G_1 \square G_2$ is super $(b, |V(G_2)|d)$-$(H \square G_2)$-antimagic, where the parameter b depends on the parameter a and on orders and sizes of graphs G_1, G_2, and H.*

The following theorem shows the existence of a super $(H \square G_2)$-antimagic labeling with difference d for $G_1 \square G_2$ if graph G_2 has odd order and $G_1 \square G_2$ admits an $(H \square G_2)$-covering.

Theorem 5 *Let G_1 be a super (a,d)-H-antimagic graph containing t subgraphs isomorphic to H. If G_2 is a graph of odd order and even size and the graph $G_1 \square G_2$ contains exactly t subgraphs isomorphic to $H \square G_2$ then the graph $G_1 \square G_2$ is super (b,d)-$(H \square G_2)$-antimagic, where the parameter b depends on the parameter a and on orders and sizes of graphs G_1, G_2, and H.*

Proof Let f be a super (a,d)-H-antimagic labeling of the (p_1,q_1)-graph G_1 and let $H_1, H_2, \ldots, H_t$ be the family of all subgraphs of G_1 isomorphic to H. Assume that G_2 is a (p_2,q_2)-graph with $p_2 \equiv 1 \pmod 2$, $q_2 \equiv 0 \pmod 2$, and also that the graph $G_1 \square G_2$ contains exactly t subgraphs isomorphic to the subgraph $H \square G_2$.

Define the labeling g of $G_1 \square G_2$ as follows:

$$g(v_i^j) = (j-1)p_1 + f(v_i) \qquad \text{if } i = 1,2,\ldots,p_1, \; j = 1,2,\ldots,\frac{p_2-1}{2}, p_2$$

$$g(v_i^j) = jp_1 + 1 - f(v_i) \qquad \text{if } i = 1,2,\ldots,p_1, \; j = \frac{p_2-1}{2}+1, \frac{p_2-1}{2}+2, \ldots, p_2-1,$$

$$g(e^j) = (p_2-1)p_1 + (j-1)q_1 + f(e) \qquad \text{if } e \in E(G_1), \; j = 1,2,\ldots,\frac{p_2-1}{2}, p_2$$

$$g(e^j) = (p_2+1)p_1 + jq_1 + 1 - f(e) \qquad \text{if } e \in E(G_1), \; j = \frac{p_2-1}{2}+1, \frac{p_2-1}{2}+2, \ldots, p_2-1,$$

$$\{g(e_i) : e \in E(G_2)\} = \mathcal{P}_{q_2}^{p_1}(i) \oplus (p_1p_2 + p_2q_1) \qquad \text{if } i = 1,2,\ldots,p_1.$$

The labeling g assigns the values $1,2,\ldots,p_1$ to the vertices $v_1^1, v_2^1, \ldots, v_{p_1}^1$, the values $p_1+1, p_1+2, \ldots, 2p_1$ to the vertices $v_1^2, v_2^2, \ldots, v_{p_1}^2$, $\ldots$, and the values $\frac{p_2-3}{2}p_1+1, \frac{p_2-3}{2}p_1+2, \ldots, \frac{p_2-1}{2}p_1$ to the vertices $v_1^{\frac{p_2-1}{2}}, \ldots, v_{p_1}^{\frac{p_2-1}{2}}$. Then the values $\frac{p_2-1}{2}p_1+1, \frac{p_2-1}{2}p_1+2, \ldots, \frac{p_2+1}{2}p_1$ are assigned to the vertices $v_1^{\frac{p_2+1}{2}}, \ldots, v_{p_1}^{\frac{p_2+1}{2}}$, $\ldots$, and the values $p_1(p_2-2)+1, p_1(p_2-2)+2, \ldots, p_1(p_2-1)$ are assigned to the vertices $v_1^{p_2-1}, \ldots, v_{p_1}^{p_2-1}$. Finally, the values $p_1(p_2-1)+1, p_1(p_2-1)+2, \ldots, p_1p_2$ are assigned to the vertices

$v_1^{p_2}, v_2^{p_2}, \dots, v_{p_1}^{p_2}$. The edges in the first copy of G_1 successively attain values $p_1p_2+1, p_1p_2+2, \dots, p_1p_2+q_1$, the edges in the second copy of G_1 successively assume values $p_1p_2+q_1+1, p_1p_2+q_1+2, \dots, p_1p_2+2q_1$, $\dots$, and the edges in the $\frac{p_2-1}{2}$ th copy of G_1 successively attain values $p_1p_2+\frac{p_2-3}{2}q_1+1, p_1p_2+\frac{p_2-3}{2}q_1+2\dots, p_1p_2+\frac{p_2-1}{2}q_1$. The edges in the $\frac{p_2+1}{2}$ th copy of G_1 successively assume values $p_1p_2+\frac{p_2-1}{2}q_1+1, \dots, p_1p_2+\frac{p_2+1}{2}q_1$, $\dots$, the edges in the (p_2-1) th copy of G_1 successively attain values $p_1p_2+q_1(p_2-2)+1, p_1p_2+q_1(p_2-2)+2, \dots, p_1p_2+q_1(p_2-1)$, and the edges in the p_2 th copy of G_1 successively assume values $p_1p_2+q_1(p_2-1)+1, p_1p_2+q_1(p_2-1)+2, \dots, p_1p_2+p_2q_1$. Values $p_1p_2+p_2q_1+1, p_1p_2+p_2q_1+2, \dots, p_1p_2+p_2q_1+p_1q_2$ are assigned to the edges in the copies of G_2.

Clearly, the values of g are $1, 2, \dots, p_1p_2+p_2q_1+p_1q_2$ and the vertex labels are the smallest possible labels. By a similar procedure as in the proof of Theorem 3 it is not difficult to check that $(H\square G_2)$-weights form an arithmetic progression with difference d. $\square$

Using the total labeling defined in the proof of Theorem 5 and the partition (2) we are able to prove the next theorem.

Theorem 6 *Let G_1 be a super (a,d)-H-antimagic graph of odd order containing t subgraphs isomorphic to H. If G_2 is a graph of odd order and odd size and the graph $G_1\square G_2$ contains exactly t subgraphs isomorphic to $H\square G_2$ then the graph $G_1\square G_2$ is super (b,d)-$(H\square G_2)$-antimagic, where the parameter b depends on the parameter a and on orders and sizes of graphs G_1, G_2, and H.*

5. Conclusion

In this paper we provided several sufficient conditions for Cartesian product $G_1\square G_2$ to be H-supermagic or to be super (a,d)-H-antimagic for several values of d. These conditions are based on parities of orders and sizes of graphs G_1 and G_2. We used partitions subsets of integers to obtain required labelings. However, it is not possible to use the described method for all combinations of parities and all feasible values of the parameter d. This is a topic for further investigation.

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