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The equation $dd' + d'd = D^2$ for derivations on C*-algebras

Sayed Khalil EKRAMI[1,*], **Madjid MIRZAVAZIRI**[2]
[1]Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran
[2]Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran

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Abstract: Let $\mathcal{A}$ be an algebra. A linear mapping $d:\mathcal{A}\to\mathcal{A}$ is called a *derivation* if $d(ab)=d(a)b+ad(b)$ for each $a,b\in\mathcal{A}$. Given two derivations d and d' on a C*-algebra $\mathcal{A}$, we prove that there exists a derivation D on $\mathcal{A}$ such that $dd'+d'd=D^2$ if and only if d and d' are linearly dependent.

Key words: Derivation; C*-algebra

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $d:\mathcal{A}\to\mathcal{A}$ is called a *derivation* if it satisfies the Leibniz rule $d(ab)=d(a)b+ad(b)$ for each $a,b\in\mathcal{A}$. When $\mathcal{A}$ is a $*$-algebra, d is called a $*$*-derivation* if $d(a^*)=d(a)^*$ for each $a\in\mathcal{A}$.

As a typical example of a nonzero derivation in a noncommutative algebra, we can consider the *inner derivation* δ_a *implemented by* an element $a\in\mathcal{A}$, which is defined as $\delta_a(x)=xa-ax$ for all $x\in\mathcal{A}$. There are known algebras $\mathcal{A}$ such that each derivation on $\mathcal{A}$ is inner, which is implemented by an element of the algebra $\mathcal{A}$ or an algebra $\mathcal{B}$ containing $\mathcal{A}$. For example, each derivation on a von Neumann algebra $\mathcal{M}$ is inner and is implemented by an element of $\mathcal{M}$. Moreover, each derivation on a C*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is inner and implemented by an element of the weak closure $\mathcal{M}$ of $\mathcal{A}$ in $B(\mathcal{H})$ (see [4, 10]).

Even for an inner derivation δ_a on an algebra $\mathcal{A}$, it is very probable that δ_a^2 is *not* a derivation. In fact, if d is a $*$-derivation on a C*-algebra $\mathcal{A}$, then d^2 is a derivation if and only if $d=0$. To see this, note that d^2 is a derivation if and only if

$$d^2(x)y+2d(x)d(y)+xd^2(y)=d^2(xy)=d^2(x)y+xd^2(y).$$

The latter is equivalent to the fact that $d(x)d(y)=0$ for each $x,y\in\mathcal{A}$. Thus $d(x)d(x)^*=d(x)d(x^*)=0$ for each $x\in\mathcal{A}$. Hence $\|d(x)\|^2=\|d(x)d(x)^*\|=0$. This shows that $d(x)=0$ for each $x\in\mathcal{A}$.

These considerations show that the set of derivations on an algebra $\mathcal{A}$ is not in general closed under product. There are various studies seeking some conditions under which the product of two derivations will be again a derivation. Posner [9] was the first to study the product of two derivations on a prime ring. He showed that if the product of two derivations on a prime ring, with characteristic not equal to 2, is a derivation then

*Correspondence: ekrami@pnu.ac.ir
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one of them must be equal to zero. The same question has been investigated by several authors on various algebras; see for example [1–3, 5–8] and references therein. In the realm of C*-algebras, Mathieu [5] showed that if the product of two derivations d and d' on a C*-algebra is a derivation then $dd' = 0$. The same result was proved by Pedersen [8] for unbounded densely defined derivations on a C*-algebra.

Let $\mathcal{A}$ be a C*-algebra. In the present paper, we show that given two derivations d, d' on $\mathcal{A}$, there exists a derivation D on $\mathcal{A}$ satisfying $dd' + d'd = D^2$, if and only if d and d' are linearly dependent. We prove the main result in two steps; we first deal with derivations on the matrix algebra $M_n(\mathbb{C})$, and in the final section for derivations on an arbitrary C*-algebra, where the result is derived with similar techniques.

2. The equation for the case of matrix algebras

In this section we are mainly concerned with the structure of derivations on the matrix algebra $M_n(\mathbb{C})$. We commence with the next elementary technical lemma.

Lemma 2.1 *Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leqslant i, k, \ell, j \leqslant n$ if and only if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$.*

Proof Let $\{E_{ij}\}_{1\leqslant i,j\leqslant n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. If $a_{ik}b_{\ell j} = b_{ik}a_{\ell j}$ for all $1 \leqslant i, k, \ell, j \leqslant n$ then we can write

$$(E_{ii}AE_{k\ell})(E_{\ell\ell}BE_{jj}) = a_{ik}b_{\ell j}E_{ij} = b_{ik}a_{\ell j}E_{ij} = (E_{ii}BE_{k\ell})(E_{\ell\ell}AE_{jj}).$$

We thus have

$$(\sum_{i=1}^{n} E_{ii})AE_{k\ell}B(\sum_{j=1}^{n} E_{jj}) = (\sum_{i=1}^{n} E_{ii})BE_{k\ell}A(\sum_{j=1}^{n} E_{jj}).$$

This shows that $AE_{k\ell}B = BE_{k\ell}A$ for each $1 \leqslant k, \ell \leqslant n$. We can therefore deduce that $AXB = BXA$ for all $X \in M_n(\mathbb{C})$.

On the other hand, if $AXB = BXA$ for all $X \in M_n(\mathbb{C})$, then setting $X = E_{jk}E_{kk}$, we get

$$a_{ij}b_{k\ell}E_{i\ell} = (E_{ii}AE_{jk})(E_{kk}BE_{\ell\ell}) = E_{ii}(AE_{jk}E_{kk}B)E_{\ell\ell} = (E_{ii}BE_{jk})(E_{kk}AE_{\ell\ell}) = b_{ij}a_{k\ell}E_{i\ell}.$$

□

Let $A = [a_{ij}] \in M_n(\mathbb{C})$. We denote the diagonal matrix whose diagonal entries are a_{ii} by A^D.

Proposition 2.2 *Let $A = [a_{ij}], B = [b_{ij}] \in M_n(\mathbb{C})$. Then there exists a $C = [c_{ij}] \in M_n(\mathbb{C})$ such that $\delta_A\delta_B + \delta_B\delta_A = {\delta_C}^2$ if and only if $\alpha A = \beta B + rI$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$.*

Proof We can assume that $a_{11} = b_{11} = c_{11} = 0$. This is due to the fact that $\delta_{A-a_{11}I} = \delta_A$, $\delta_{B-b_{11}I} = \delta_B$, and $\delta_{C-c_{11}I} = \delta_C$.

Let $\{E_{ij}\}_{1\leqslant i,j\leqslant n}$ be the standard system of matrix units for $M_n(\mathbb{C})$. Then $\delta_A\delta_B + \delta_B\delta_A = {\delta_C}^2$ if and only if

$$\delta_A(\delta_B(E_{k\ell})) + \delta_B(\delta_A(E_{k\ell})) = \delta_C(\delta_C(E_{k\ell}))$$

for each $1 \leqslant k, \ell \leqslant n$ or equivalently

$$E_{k\ell}(AB + BA) - 2AE_{k\ell}B - 2BE_{k\ell}A + (AB + BA)E_{k\ell} = E_{k\ell}C^2 - 2CE_{k\ell}C + C^2E_{k\ell},$$

for each $1 \leqslant k,\ell \leqslant n$. This is equivalent to the fact that

$$E_{ii}(E_{k\ell}(AB+BA)-2AE_{k\ell}B-2BE_{k\ell}A+(AB+BA)E_{k\ell})E_{jj}=E_{ii}(E_{k\ell}C^2-2CE_{k\ell}C+C^2E_{k\ell})E_{jj},$$

for each $1 \leqslant i,j,k,\ell \leqslant n$. Now for $i \neq k$ and $j \neq \ell$ we have, as in lemma 2.1

$$(a_{ik}b_{\ell j}+b_{ik}a_{\ell j})E_{ij}=(c_{ik}c_{\ell j})E_{ij}. \tag{2.1}$$

Similarly, for $i \neq k$ and $j=\ell$ we have

$$\big(-2a_{ik}b_{\ell\ell}-2b_{ik}a_{\ell\ell}+\sum_{m=1}^{n}(a_{im}b_{mk}+b_{im}a_{mk})\big)E_{i\ell}=\big(-2c_{ik}c_{\ell\ell}+\sum_{m=1}^{n}c_{im}c_{mk}\big)E_{i\ell}. \tag{2.2}$$

Moreover, for $i=k$ and $j \neq \ell$ we have

$$\big(\sum_{m=1}^{n}(a_{\ell m}b_{mj}+b_{\ell m}a_{mj})-2a_{kk}b_{\ell j}-2b_{kk}a_{\ell j}\big)E_{kj}=\big(\sum_{m=1}^{n}c_{\ell m}c_{mj}-2c_{kk}c_{\ell j}\big)E_{kj}. \tag{2.3}$$

Finally for $i=k$ and $j=\ell$ we have

$$\begin{aligned}&\big(\sum_{m=1}^{n}(a_{\ell m}b_{m\ell}+b_{\ell m}a_{m\ell})-2a_{kk}b_{\ell\ell}-2b_{kk}a_{\ell\ell}+\sum_{m=1}^{n}(a_{km}b_{mk}+b_{km}a_{mk})\big)E_{k\ell}\\&=\big(\sum_{m=1}^{n}c_{\ell m}c_{m\ell}-2c_{kk}c_{\ell\ell}+\sum_{m=1}^{n}c_{km}c_{mk}\big)E_{k\ell}.\end{aligned} \tag{2.4}$$

If $k \neq \ell$ then putting $i=\ell$ and $j=k$ in the equation (2.1) we have $c_{\ell k}^2=2a_{\ell k}b_{\ell k}$. Thus for $i \neq k$ and $j \neq \ell$ we have $(a_{ik}b_{\ell j}+b_{ik}a_{\ell j})^2=c_{ik}^2c_{\ell j}^2=4a_{ik}b_{ik}a_{\ell j}b_{\ell j}$. This implies that

$$a_{ik}b_{\ell j}=b_{ik}a_{\ell j},\ \text{for } i \neq k, j \neq \ell. \tag{2.5}$$

Now, if $b_{\ell j} \neq 0$ for some $1 \leq \ell, j \leq n$ with $\ell \neq j$, then the equation

$$a_{ik}=\frac{a_{\ell j}}{b_{\ell j}}b_{ik},\ \text{for } i \neq k,$$

implies the existence of some α and β with $|\alpha|+|\beta| \neq 0$ such that

$$\alpha(A-A^D)=\beta(B-B^D). \tag{2.6}$$

If $b_{\ell j}=0$ for all $1 \leq \ell, j \leq n$ with $\ell \neq j$, then $B=B^D$ and so the equation (2.6) holds for $\alpha=0$ and any nonzero $\beta \in \mathbb{C}$.

Putting $\ell=k$ in (2.4) we get

$$\sum_{m=1}^{n}(a_{km}b_{mk}+b_{km}a_{mk})-2a_{kk}b_{kk}=\sum_{m=1}^{n}c_{km}c_{mk}-c_{kk}c_{kk}.$$

Thus it follows from (2.4) that

$$2a_{\ell\ell}b_{\ell\ell} - 2a_{kk}b_{\ell\ell} - 2b_{kk}a_{\ell\ell} + 2a_{kk}b_{kk} = c_{\ell\ell}c_{\ell\ell} - 2c_{kk}c_{\ell\ell} + c_{kk}c_{kk},$$

or simply

$$2(a_{\ell\ell} - a_{kk})(b_{\ell\ell} - b_{kk}) = (c_{\ell\ell} - c_{kk})^2.$$

For $\ell = 1$ we have

$$c_{kk}^2 = 2a_{kk}b_{kk}, \text{ for each } 1 \leq k \leq n,$$

and then

$$a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell} = c_{kk}c_{\ell\ell}. \tag{2.7}$$

Thus for all $1 \leq k, \ell \leq n$ we have $(a_{kk}b_{\ell\ell} + b_{kk}a_{\ell\ell})^2 = c_{kk}^2 c_{\ell\ell}^2 = 4a_{kk}b_{kk}a_{\ell\ell}b_{\ell\ell}$. This implies that

$$a_{kk}b_{\ell\ell} = b_{kk}a_{\ell\ell}, \text{ for all } 1 \leq k, \ell \leq n. \tag{2.8}$$

A similar argument as about the equation (2.5) implies the existence of some α' and β' with $|\alpha'| + |\beta'| \neq 0$ such that

$$\alpha' A^D = \beta' B^D.$$

Returning to the fact that $a_{im}b_{mk} = b_{im}a_{mk} = \frac{1}{2}c_{im}c_{mk}$ for $m \neq i, k$, we have

$$\sum_{\substack{m=1 \\ m\neq i,k}}^{n} (a_{im}b_{mk} + b_{im}a_{mk}) = \sum_{\substack{m=1 \\ m\neq i,k}}^{n} c_{im}c_{mk}, \text{ for } i \neq k. \tag{2.9}$$

Thus letting $\ell = i$ in (2.2) we have

$$a_{ik}(b_{ii} - b_{kk}) + b_{ik}(a_{ii} - a_{kk}) = c_{ik}(c_{ii} - c_{kk}), \text{ for } i \neq k, \tag{2.10}$$

and then

$$\begin{aligned} (a_{ik}(b_{ii} - b_{kk}) + b_{ik}(a_{ii} - a_{kk}))^2 &= c_{ik}^2(c_{ii} - c_{kk})^2 \\ &= 4a_{ik}b_{ik}(b_{ii} - b_{kk})(a_{ii} - a_{kk}), \text{ for } i \neq k. \end{aligned}$$

This implies that

$$a_{ik}(b_{ii} - b_{kk}) = b_{ik}(a_{ii} - a_{kk}), \text{ for } i \neq k. \tag{2.11}$$

Using (2.11) and (2.8) we have

$$b_{jj}a_{ik}(b_{ii} - b_{kk}) = b_{ik}b_{jj}(a_{ii} - a_{kk}) = b_{ik}a_{jj}(b_{ii} - b_{kk}).$$

Now let $B^D \notin \mathbb{C}I$. Then $b_{ii} \neq b_{kk}$ for some i and k. This shows that $b_{jj}a_{ik} = a_{jj}b_{ik}$. Hence we have $\alpha = \alpha'$ and $\beta = \beta'$. By a similar argument we can say that if $A^D \notin \mathbb{C}I$ then $\alpha = \alpha'$ and $\beta = \beta'$. We therefore have

$$\text{if } A^D \notin \mathbb{C}I \text{ or } B^D \notin \mathbb{C}I \text{ then } \alpha A = \beta B \text{ for some } \alpha \text{ and } \beta \text{ with } |\alpha| + |\beta| \neq 0.$$

On the other hand, if $A^D = sI$ and $B^D = tI$ for some $s, t \in \mathbb{C}$ then

$$\alpha' A^D + \alpha(A - A^D) = s(\alpha' - \alpha)I + \alpha A,$$

and

$$\beta' B^D + \beta(B - B^D) = t(\beta' - \beta)I + \beta B.$$

Therefore, $s(\alpha' - \alpha)I + \alpha A = t(\beta' - \beta)I + \beta B$.

Summarizing these we can say that $\delta_A\delta_B + \delta_B\delta_A = {\delta_C}^2$ implies $\alpha A = \beta B + rI$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$.

Conversely, suppose that $\alpha A = \beta B + rI$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$. If $\alpha \neq 0$ put $C = \sqrt{2\beta\alpha^{-1}}B$, and if $\beta \neq 0$ put $C = \sqrt{2\alpha\beta^{-1}}A$. Then it is easy to see that $\delta_A\delta_B + \delta_B\delta_A = {\delta_C}^2$. □

Remark 2.3 *The condition $\alpha A = \beta B + rI$ for some $\alpha, \beta, r \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$ in $M_n(\mathbb{C})$ is equivalent to the fact that δ_A and δ_B are linearly dependent.*

A natural question is the following: Is it true in general that $dd' + d'd = D^2$ on an algebra $\mathcal{A}$ is equivalent to d and d' being linearly dependent? In this case we of course have $D = \sqrt{2\lambda}d = \sqrt{2\lambda'}d'$ for some $\lambda, \lambda' \in \mathbb{C}$. The following example shows that the answer is not affirmative in general.

Example 2.4 *Let $\mathcal{A}$ be the subalgebra of $M_4(\mathbb{C})$ generated by E_{11}, E_{12}, E_{34}, and E_{44}. If $d = \delta_{E_{12}}$ and $d' = \delta_{E_{34}}$, then for each $X = xE_{11} + yE_{12} + zE_{34} + wE_{44} \in \mathcal{A}$ we have*

$$(dd' + d'd)(X) = \delta_{E_{12}}(-wE_{34}) + \delta_{E_{34}}(xE_{12}) = 0.$$

Thus $dd' + d'd = \delta_0^2$. However, d and d' are linearly independent.

Lemma 2.5 *Let $\mathcal{A}$ be the subalgebra of $M_4(\mathbb{C})$ generated by E_{11}, E_{12}, E_{34}, and E_{44}. Then each derivation on $\mathcal{A}$ is of the form $d = \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}}$ for some $\alpha, \beta, \gamma, \theta \in \mathbb{C}$.*

Proof Let $d : \mathcal{A} \to \mathcal{A}$ be a derivation. For each projection P,

$$d(P) = d(P^2) = d(P)P + Pd(P). \tag{2.12}$$

If $d(E_{11}) = \alpha E_{11} + \beta E_{12} + \gamma E_{34} + \theta E_{44}$ for some $\alpha, \beta, \gamma, \theta \in \mathbb{C}$, we have from (2.12) that $\alpha = \gamma = \theta = 0$, so that $d(E_{11}) = \beta E_{12}$ for some $\beta \in \mathbb{C}$. Similarly, we get $d(E_{44}) = \gamma E_{34}$ for some $\gamma \in \mathbb{C}$. Therefore,

$$d(E_{12}) = d(E_{11}E_{12}) = d(E_{11})E_{12} + E_{11}d(E_{12}) = E_{11}d(E_{12}),$$

so that $d(E_{12}) = \lambda E_{11} + \alpha E_{12}$ for some $\lambda, \alpha \in \mathbb{C}$. Then

$$0 = d(E_{12}E_{11}) = d(E_{12})E_{11} + E_{12}d(E_{11}) = d(E_{12})E_{11} = \lambda E_{11}.$$

Thus $\lambda = 0$ and so $d(E_{12}) = \alpha E_{12}$ for some $\alpha \in \mathbb{C}$. Similarly, we get $d(E_{34}) = \theta E_{34}$ for some $\theta \in \mathbb{C}$.

For each $X \in \mathcal{A}$ we have

$$\begin{aligned} d(X) &= d(xE_{11} + yE_{12} + zE_{34} + wE_{44}) = xd(E_{11}) + yd(E_{12}) + zd(E_{34}) + wd(E_{44}) \\ &= (\beta x + \alpha y)E_{12} + (\theta z + \gamma w)E_{34} = \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}}(X). \end{aligned}$$

Therefore, $d = \delta_{\beta E_{12} - \alpha E_{11} - \gamma E_{34} + \theta E_{44}}$ for some $\alpha, \beta, \gamma, \theta \in \mathbb{C}$. □

Remark 2.6 *Let $\mathcal{A}$ be the subalgebra of $M_4(\mathbb{C})$ generated by E_{11}, E_{12}, E_{34}, and E_{44} and let d, d', and D be derivations on $\mathcal{A}$. By Lemma 2.5, we can assume that $d = \delta_{\beta E_{12}-\alpha E_{11}-\gamma E_{34}+\theta E_{44}}$, $d' = \delta_{\beta' E_{12}-\alpha' E_{11}-\gamma' E_{34}+\theta' E_{44}}$, and $D = \delta_{sE_{12}-rE_{11}-tE_{34}+uE_{44}}$ for some $\alpha, \beta, \gamma, \theta, \alpha', \beta', \gamma', \theta', r, s, t, u \in \mathbb{C}$. Then $dd' + d'd = D^2$ if and only if $rs = \alpha\beta' + \alpha'\beta$, $tu = \gamma\theta' + \gamma'\theta$, $r^2 = 2\alpha\alpha'$, and $u^2 = 2\theta\theta'$.*

3. Derivations on C*-algebras

In this section, let $\mathcal{H}$ be a Hilbert space with the orthonormal basis $\{\xi_i\}_{i\in\mathbb{I}}$. For a bounded operator $T \in B(\mathcal{H})$, let $t_{ij} = \langle T\xi_j, \xi_i\rangle$ for $i, j \in \mathbb{I}$. We thus have $T\xi_j = \sum_{i\in\mathbb{I}} t_{ij}\xi_i$ and we can write $T = [t_{ij}]_{i,j\in\mathbb{I}}$. The latter is called the matrix representation of the operator $T \in B(\mathcal{H})$.

For $i, j \in \mathbb{I}$, let $E_{ij} \in B(\mathcal{H})$ be the operator defined by $E_{ij}\xi_j = \xi_i$ and $E_{ij}\xi_k = 0$ for $k \neq j$. Then $E_{ij}E_{jk} = E_{ik}$ and $E_{ij}E_{\ell k} = 0$ for $j \neq \ell$. Using the fact that $E_{ij}(x) = \langle x, \xi_j\rangle\, \xi_i$ for all $x \in \mathcal{H}$, we get

$$T = \sum_{p\in\mathbb{I}}\sum_{q\in\mathbb{I}} t_{qp}E_{qp} \tag{3.1}$$

for every $T \in B(\mathcal{H})$. Moreover, putting $r_{ij} = \langle R\xi_j, \xi_i\rangle, s_{ij} = \langle S\xi_j, \xi_i\rangle$ we have

$$RS = \sum_{p\in\mathbb{I}}\sum_{q\in\mathbb{I}}\sum_{m\in\mathbb{I}} r_{qm}s_{mp}E_{qp} \tag{3.2}$$

for every $R, S \in B(\mathcal{H})$. It follows from (3.1) and (3.2) that

$$E_{ij}TE_{k\ell} = t_{jk}E_{i\ell}, \quad E_{ij}RSE_{k\ell} = \sum_{p\in\mathbb{I}} r_{jp}s_{pk}E_{i\ell} \quad (T, R, S \in B(\mathcal{H})).$$

Using these facts, we are ready to prove the next theorem.

Theorem 3.1 *Let $\mathcal{A}$ be a C^*-algebra and let d, d' be two derivations on $\mathcal{A}$. Then there exists a derivation D on $\mathcal{A}$ such that $dd' + d'd = D^2$ if and only if d and d' are linearly dependent.*

Proof Let $\mathcal{A}$ act faithfully on the Hilbert space $\mathcal{H}$ with the orthonormal basis $\{\xi_i\}_{i\in\mathbb{I}}$. By the Kadison–Sakai theorem [4, 10], $d = \delta_R, d' = \delta_S$, and $D = \delta_T$ for some R, S, and T in $B(\mathcal{H})$.

Then $dd' + d'd = D^2$ if and only if $(dd' + d'd)(E_{k\ell}) = D^2(E_{k\ell})$ for each $k, \ell \in \mathbb{I}$, or equivalently

$$E_{k\ell}(RS + SR) - 2RE_{k\ell}S - 2SE_{k\ell}R + (RS + SR)E_{k\ell} = E_{k\ell}T^2 - 2TE_{k\ell}T + T^2E_{k\ell},$$

for each $k, \ell \in \mathbb{I}$. This is equivalent to the fact that

$$E_{ii}(E_{k\ell}(RS + SR) - 2RE_{k\ell}S - 2SE_{k\ell}R + (RS + SR)E_{k\ell})E_{jj} = E_{ii}(E_{k\ell}T^2 - 2TE_{k\ell}T + T^2E_{k\ell})E_{jj},$$

for each $i, j, k, \ell \in \mathbb{I}$. Now for $i \neq k$ and $j \neq \ell$ we have

$$r_{ik}s_{\ell j} + s_{ik}r_{\ell j} = t_{ik}t_{\ell j}.$$

Similarly, for $i \neq k$ and $j = \ell$ we have

$$-2r_{ik}s_{\ell\ell} - 2s_{ik}r_{\ell\ell} + \sum_{p\in\mathbb{I}}(r_{ip}s_{pk} + s_{ip}r_{pk}) = -2t_{ik}t_{\ell\ell} + \sum_{p\in\mathbb{I}} t_{ip}t_{pk}.$$

Furthermore, for $i = k$ and $j \neq \ell$ we have

$$\sum_{p\in\mathbb{I}}(r_{\ell p}s_{pj} + s_{\ell p}r_{pj}) - 2r_{kk}s_{\ell j} - 2s_{kk}r_{\ell j} = \sum_{p\in\mathbb{I}} t_{\ell p}t_{pj} - 2t_{kk}t_{\ell j}.$$

Finally for $i = k$ and $j = \ell$ we have

$$\sum_{p\in\mathbb{I}}(r_{\ell p}s_{p\ell} + s_{\ell p}r_{p\ell}) - 2r_{kk}s_{\ell\ell} - 2s_{kk}r_{\ell\ell} + \sum_{p\in\mathbb{I}}(r_{kp}s_{pk} + s_{kp}r_{pk})$$
$$= \sum_{p\in\mathbb{I}} t_{\ell p}t_{p\ell} - 2t_{kk}t_{\ell\ell} + \sum_{p\in\mathbb{I}} t_{kp}t_{pk}.$$

Now a similar verification as in Proposition 2.2 implies the result. □

Remark 3.2 *Note that the subalgebra $\mathcal{A}$ of $M_4(\mathbb{C})$ generated by E_{11}, E_{12}, E_{34}, and E_{44} is finite-dimensional (being a subalgebra of $M_4(\mathbb{C})$), and therefore it is complete with respect to the norm inherited from $M_4(\mathbb{C})$. Hence $\mathcal{A}$ is a Banach algebra. However, $\mathcal{A}$ is not a $*$-subalgebra of $M_4(\mathbb{C})$, since $X = E_{12} \in \mathcal{A}$, but $X^* = E_{21} \notin \mathcal{A}$. Therefore, $\mathcal{A}$ is not a C^*-algebra and Example 2.4 does not contradict the statement of Theorem 3.1.*

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