Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 360 – 372
© TÜBİTAK
doi:10.3906/mat-1703-61

Various centroids and some characterizations of catenary rotation hypersurfaces

Dong-Soo KIM[1], **Young Ho KIM**[2], **Dae Won YOON**[3,*]
[1]Department of Mathematics, Chonnam National University, Gwangju, Republic of Korea
[2]Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea
[3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Republic of Korea

Received: 16.03.2017 • **Accepted/Published Online:** 19.05.2017 • **Final Version:** 22.01.2018

Abstract: We study positive C^1 functions $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$. For $x = (x_1, \cdots, x_n)$ with nonzero numbers $x_1, \cdots, x_n$, we consider the rectangular domain $I(x) = I(x_1) \times \cdots \times I(x_n) \subset \mathbb{R}^n$, where $I(x_i) = [0, x_i]$ if $x_i > 0$ and $I(x_i) = [x_i, 0]$ if $x_i < 0$. We denote by V, S, $(\bar{x}_V, \bar{z}_V)$, and $(\bar{x}_S, \bar{z}_S)$ the volume of the domain under the graph of $z = f(x)$, the surface area S of the graph of $z = f(x)$, the geometric centroid of the domain under the graph of $z = f(x)$, and the surface centroid of the graph itself over the rectangular domain $I(x)$, respectively. In this paper, first we show that among C^2 functions with isolated singularities, $S = kV$, $k \in \mathbb{R}$ characterizes the family of catenary rotation hypersurfaces $f(x) = k\cosh(r/k)$, $r = |x|$. Next we show that the equality of n coordinates of $(\bar{x}_S, \bar{z}_S)$ and $(\bar{x}_V, 2\bar{z}_V)$ for every rectangular domain $I(x)$ characterizes the family of catenary rotation hypersurfaces among C^2 functions with isolated singularities.

Key words: Centroid, surface centroid, volume, surface area, catenary rotation hypersurface

1. Introduction

Let us consider the catenary given by $f(x) = k\cosh((x-c)/k), x \in \mathbb{R}$ with a positive constant k. It is well known that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. In other words, for a positive C^1 function $f(x)$ defined on an interval I and an interval $[a, b] \subset I$, we consider the area $A(a, b)$ over the interval $[a, b]$ and the arc length $L(a, b)$ of the graph of $f(x)$. Then the function $f(x) = k\cosh((x-c)/k), k > 0$ satisfies for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$. This property characterizes the family of catenaries among nonconstant C^2 functions as follows ([19]).

Proposition 1.1 *For a nonconstant positive C^2 function $f(x)$ defined on an interval I, the following are equivalent.*

(1) There exists a positive constant k such that for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$.

(2) The function $f(x)$ satisfies $f(x) = k\sqrt{1 + f'(x)^2}$, where k is a positive constant.

*Correspondence: dwyoon@gnu.ac.kr
2010 *AMS Mathematics Subject Classification:* 53A07, 26A30
Supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (NRF-2015R1D1A3A01020387).

(3) For some $k > 0$ *and* $c \in \mathbb{R}$*,*

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

When $f(x)$ is a positive C^1 function defined on an interval I and $[a,b]$ is an interval contained in the interval I, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a,b), \bar{y}_A(a,b))$ and $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a,b), \bar{y}_L(a,b))$ the geometric centroid of the area under the graph of $f(x)$ and the centroid of the graph itself over the interval $[a,b]$, respectively. Then, for a catenary curve, we have the following ([19]).

Proposition 1.2 *A catenary curve* $f(x) = k\cosh((x-c)/k)$ *satisfies the following.*

(1) For every interval $[a,b] \subset \mathbb{R}$*,* $\bar{x}_L(a,b) = \bar{x}_A(a,b)$*.*

(2) For every interval $[a,b] \subset \mathbb{R}$*,* $\bar{y}_L(a,b) = 2\bar{y}_A(a,b)$*.*

Conversely, in a recent paper [15], it was shown that one of $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$ for every interval $[a,b]$ characterizes the family of catenaries among nonconstant positive C^2 functions. See also the recent paper [1].

For hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ given by the graph of a function $z = f(x), x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, it is quite natural to ask the following question:

Question "Which hypersurfaces $z = f(x), x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ satisfy the above properties?"

In this paper, we study positive C^1 functions $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$. We consider rectangular domains with a fixed end point, say the origin, as follows. For a nonzero real number x, we denote by $I(x)$ the interval defined by

$$I(x) = \begin{cases} [0,x], & \text{if } x > 0, \\ [x,0], & \text{if } x < 0. \end{cases} \tag{1.1}$$

For nonzero real numbers $x_1, \cdots, x_n$, we put $x = (x_1, \cdots, x_n)$ and consider the rectangular domain defined by

$$I(x) = I(x_1) \times \cdots \times I(x_n) \subset \mathbb{R}^n. \tag{1.2}$$

We denote by $V(x)$, $S(x)$, $(\bar{x}_V(x), \bar{z}_V(x)) = (\bar{x}_{1V}, \cdots, \bar{x}_{nV}, \bar{z}_V)$, and $(\bar{x}_S(x), \bar{z}_S(x)) = (\bar{x}_{1S}, \cdots, \bar{x}_{nS}, \bar{z}_S)$ the volume of the domain under the graph of $z = f(x)$, the surface area of the graph of $z = f(x)$, the geometric centroid of the domain under the graph of $z = f(x)$, and the surface centroid of the graph itself over the rectangular domain $I(x)$, respectively.

As a result, first of all, in Section 3 we prove the following characterization theorem.

Theorem 1.3 *Suppose that* $z = f(x), x = (x_1, \cdots, x_n)$ *denotes a positive* C^2 *function defined on* $\mathbb{R}^n$ *with isolated singularities. Then the following are equivalent.*

(1) There exists a positive constant k *such that for every rectangular domain* $I(x)$*,* $V(x) = kS(x)$*.*

(2) By a Euclidean motion of $\mathbb{R}^n$ *if necessary, we have*

$$f(x) = k\cosh\left(\frac{\sqrt{x_1^2+\cdots+x_n^2}}{k}\right).$$

Next, in Sections 4 and 5, we prove the following characterization theorem.

Theorem 1.4 *Suppose that* $z = f(x), x = (x_1, \cdots, x_n)$ *denotes a positive* C^2 *function defined on* $\mathbb{R}^n$ *with isolated singularities. Then the following are equivalent.*

(1) For every rectangular domain $I(x)$*, we have*

$$(\bar{x}_{1S}, \cdots, \bar{x}_{nS}) = (\bar{x}_{1V}, \cdots, \bar{x}_{nV}).$$

(2) There exists $i \in \{1, 2, \cdots, n\}$ *such that for every rectangular domain* $I(x)$*, we have*

$$(\bar{x}_{1S}, \cdots, \check{\bar{x}}_{iS}, \cdots, \bar{x}_{nS}, \bar{z}_S) = (\bar{x}_{1V}, \cdots, \check{\bar{x}}_{iV}, \cdots, \bar{x}_{nV}, 2\bar{z}_V),$$

where ˇ *denotes a missing term.*

(3) By a Euclidean motion of $\mathbb{R}^n$ *if necessary, we have*

$$f(x) = k\cosh\left(\frac{\sqrt{x_1^2+\cdots+x_n^2}}{k}\right).$$

Remark 1.5 *For a positive constant* $k \in \mathbb{R}$ *and* $\ell \in \{1, 2, \cdots, n\}$*, we put*

$$f_{n,\ell}(x_1, \cdots, x_n) = k\cosh\left(\frac{\sqrt{x_1^2+\cdots+x_\ell^2}}{k}\right). \tag{1.3}$$

Then the function $f = f_{n,\ell}$ *satisfies* $f = kw$*, where* w *is defined by* $w(x) = \sqrt{1+|\nabla f(x)|^2}$*. Hence, for every rectangular domain* $I(x)$*, we have*

$$V(x) = kS(x) \quad \textit{and} \quad (\bar{x}_S(x), \bar{z}_S(x)) = (\bar{x}_V(x), 2\bar{z}_V(x)).$$

Let us denote by C_n the n-dimensional catenary rotation hypersurface given by the graph of the function $f_{n,n}$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Then, for $1 \le \ell \le n-1$, the graph of the function $f_{n,\ell}$ is nothing but the product $C_\ell \times \mathbb{R}^{n-\ell}$ of the ℓ-dimensional catenary rotation hypersurface $C_\ell \subset \mathbb{R}^{\ell+1}$ and the Euclidean space $\mathbb{R}^{n-\ell}$. Note that the functions $f_{n,\ell}$ have isolated singularities only when $\ell = n$.

In order to prove the above-mentioned main theorems, first of all we need the following (the main theorem of [11]):

Proposition 1.6 *Suppose that a* C^2 *function* $f : \mathbb{R}^n \to \mathbb{R}$ *with isolated critical points satisfies* $|\nabla f(x_1, \cdots, x_n)| = \phi(f(x_1, \cdots, x_n))$*, where* ϕ *is a function. Then* f *is a function of either a distance function* $r = |p - o|$ *from a fixed point* o *or a linear function. That is, the level sets are either concentric hyperspheres or parallel hyperplanes.*

We proceed as follows. In Section 2, we prove some lemmas that are crucial in the proof of Theorem 1.4. In particular, Lemma 2.1 establishes a sufficient and necessary condition for two positive C^1 functions to be proportional. In Section 3, with the help of Proposition 1.6, we prove Theorem 1.3. Finally, in Sections 4 and 5, using Proposition 1.6 and the lemmas in Section 2, we prove Theorem 1.4.

Two higher dimensional generalizations of Proposition 1.1 were established in [2]. In [13], it was shown that among C^2 functions $f : \mathbb{R}^2 \to \mathbb{R}$ with isolated singularities, $S = kV$, $k \in \mathbb{R}$ characterizes the family of catenary rotation surfaces $f(x, y) = k \cosh(r/k)$, $r = |(x, y)|$, where V and S denote the volume and the surface area of the graph of $z = f(x, y)$ over a rectangular domain of the form $[a, b] \times [c, d]$.

To find the centroid of polygons, see [4]. For the perimeter centroid of a polygon, we refer to [3]. In [17], mathematical definitions of the centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [18]. The relationships between various centroids of a quadrangle were given in [7, 14].

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections [20]. Some characterizations of parabolas using these properties were given in [6, 10, 12]. Furthermore, Archimedes also proved the volume properties of the region surrounded by a paraboloid of rotation and a plane [20]. For characterizations of spheres, ellipsoids, elliptic paraboloids, or elliptic hyperboloids with respect to these volume properties, we refer to [5, 8, 9, 16].

2. Some Lemmas

In this section, we prove some lemmas that are useful in the proof of Theorem 1.4 stated in Section 1.

The following lemma plays an important role in this paper.

Lemma 2.1 *We denote by $f(x)$ and $g(x)$ two positive C^1 functions defined on an interval I containing $0 \in \mathbb{R}$. Suppose that $f(x)$ and $g(x)$ satisfy the following:*

$$\frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt} = \frac{\int_0^x tg(t)dt}{\int_0^x g(t)dt}, \quad x \in I, x \neq 0. \tag{2.1}$$

Then the ratio of $f(x)$ and $g(x)$ is constant. That is, for some constant $k \in \mathbb{R}$, we have $f(x) = kg(x)$.

Proof Suppose that $f(x)$ and $g(x)$ satisfy (2.1). Then for all $x \in I_0 = \{x \in I | x \neq 0\}$ we get

$$\int_0^x tf(t)dt = h(x) \int_0^x f(t)dt \tag{2.2}$$

and

$$\int_0^x tg(t)dt = h(x) \int_0^x g(t)dt, \tag{2.3}$$

where $h(x)$ is a C^2 function defined on the open set I_0.

By differentiating (2.2) and (2.3) respectively with respect to the variable x, we obtain for all $x \in I_0$

$$f(x)(x - h(x)) = h'(x) \int_0^x f(t)dt \tag{2.4}$$

and

$$g(x)(x - h(x)) = h'(x) \int_0^x g(t)dt. \tag{2.5}$$

We put

$$j(x) = \frac{x - h(x)}{h'(x)}. \tag{2.6}$$

Then $j(x)$ is a C^1 function on the open set I_1 defined by $I_1 = \{x \in I_0 | h'(x) \neq 0\}$, which satisfies

$$f(x)j(x) = \int_0^x f(t)dt \tag{2.7}$$

and

$$g(x)j(x) = \int_0^x g(t)dt. \tag{2.8}$$

Differentiating (2.7) and (2.8) respectively with respect to the variable x yields for all $x \in I_1$

$$(f(x)j(x))' = f(x) \tag{2.9}$$

and

$$(g(x)j(x))' = g(x). \tag{2.10}$$

It follows from (2.9) and (2.10) that on the open set I_1 the ratio $k(x) = f(x)/g(x)$ satisfies

$$k'(x)j(x) = 0. \tag{2.11}$$

Now suppose that the open set I_2 defined by $I_2 = \{x \in I_1 | k'(x) \neq 0\}$ is nonempty. Then (2.11) shows that on the open set I_2 the function $j(x)$ vanishes. Together with (2.9), this shows that the function $f(x)$ also vanishes on the open set I_2. This contradiction implies that the open set I_2 must be empty. That is, on the open set I_1 we have $k'(x) = 0$.

Finally, we claim that the open set I_1 is dense in the interval I. Otherwise, choose a connected component K of the nonempty interior of the complement I_1^c of the open set I_1. Then, on the component K, we have $h'(x) = 0$. Together with (2.4), this shows that $h(x) = x$. This contradiction completes the proof of the claim.

Since the open set $I_1 = \{x \in I_0 | h'(x) \neq 0\}$ is dense in the interval I and $k'(x)$ vanishes on the open set I_1, by continuity $k'(x)$ vanishes on the whole interval I. This completes the proof of Lemma 2.1. □

Now we consider a positive C^1 function $f(x)$ and a positive continuous function $w(x)$, which are defined on an interval I containing $0 \in \mathbb{R}$. Then we prove the following lemma, which is crucial in the proof of Theorem 1.4.

Lemma 2.2 *We consider a positive C^1 function $f(x)$ and a positive continuous function $w(x)$, which are defined on an interval I containing $0 \in \mathbb{R}$. Suppose that $f(x)$ and $w(x)$ satisfy the following:*

$$\int_0^x f(t)dt = k(x) \int_0^x w(t)dt \tag{2.12}$$

and

$$\int_0^x f(t)^2 dt = k(x) \int_0^x f(t)w(t)dt, \tag{2.13}$$

where $k(x)$ is a function defined on the open set $I_0 = I \setminus \{0\}$. Then, on the open set $I_1 = \{x \in I_0 | k'(x) \neq 0\}$, we have $f'(x) = 0$.

Proof Suppose that $f(x)$ and $w(x)$ satisfy (2.12). Then the function $k(x)$ is a C^1 function on the open set I_0. By differentiating (2.12) and (2.13) with respect to the variable x, respectively, we obtain

$$f(x) - k(x)w(x) = k'(x) \int_0^x w(t)dt \tag{2.14}$$

and

$$f(x)\{f(x) - k(x)w(x)\} = k'(x) \int_0^x f(t)w(t)dt. \tag{2.15}$$

Hence, on the open set I_1, for the function $j(x)$ defined by

$$j(x) = \frac{f(x) - k(x)w(x)}{k'(x)}, \tag{2.16}$$

we get

$$j(x) = \int_0^x w(t)dt \tag{2.17}$$

and

$$j(x)f(x) = \int_0^x f(t)w(t)dt. \tag{2.18}$$

It follows from (2.17) that the function $j(x)$ is a C^1 function on the open set I_1. Differentiating (2.17) and (2.18) with respect to the variable x gives

$$j'(x) = w(x) \tag{2.19}$$

and

$$(j(x)f(x))' = f(x)w(x), \tag{2.20}$$

respectively. Together with (2.19), (2.20) shows that

$$j(x)f'(x) = 0. \tag{2.21}$$

Suppose that the open set I_2 defined by $I_2 = \{x \in I_1 | f'(x) \neq 0\}$ is nonempty. Then (2.21) implies that on I_2 $j(x)$ vanishes, and hence $j'(x)$ also vanishes there. Together with (2.19), this contradicts the positivity of the function $w(x)$. This contradiction shows that the open set I_2 must be empty. That is, $f'(x)$ vanishes on the open set I_1. This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 stated in Section 1.

We consider a positive C^2 function $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$ with isolated singularities. Suppose that the function $z = f(x), x = (x_1, \cdots, x_n)$ satisfies for some $k > 0$ $V(x) = kS(x)$ on every rectangular domain $I(x)$. Then for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$, we have

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) dt_1 \cdots dt_n = k \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, \cdots, t_n) dt_1 \cdots dt_n, \tag{3.1}$$

where w is a function defined by

$$w(x_1, \cdots, x_n) = \sqrt{1 + |\nabla f(x_1, \cdots, x_n)|^2}. \tag{3.2}$$

By differentiating (3.1) with respect to $x_1, \cdots, x_n$ successively, the fundamental theorem of calculus gives $f(x_1, \cdots, x_n) = kw(x_1, \cdots, x_n)$ for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$. By continuity, $f(x_1, \cdots, x_n) = kw(x_1, \cdots, x_n)$ for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$. That is, we get a partial differential equation

$$f(x_1, \cdots, x_n) = k\sqrt{1 + |\nabla f(x_1, \cdots, x_n)|^2}. \tag{3.3}$$

This shows that the function $z = f(x_1, \cdots, x_n)$ satisfies

$$|\nabla f(x_1, \cdots, x_n)| = \phi(f(x_1, \cdots, x_n)), \quad \phi(t) = \frac{\sqrt{t^2 - k^2}}{k}. \tag{3.4}$$

Now it follows from Proposition 1.6 that by a Euclidean motion, if necessary, the function $f(x_1, \cdots, x_n)$ is either a radial function $f(x_1, \cdots, x_n) = h(r), r = |(x_1, \cdots, x_n)|$ or a function $f(x_1, \cdots, x_n) = h(x_1)$ of only x_1.

We consider two cases as follows.

Case 1. $f(x_1, \cdots, x_n) = h(r)$**.** In this case, we have from (3.4)

$$h'(r) = \pm\frac{\sqrt{h(r)^2 - k^2}}{k}, \tag{3.5}$$

which shows that for some real number $c \in \mathbb{R}$

$$f(x_1, \cdots, x_n) = h(r) = k\cosh\left(\frac{r - c}{k}\right). \tag{3.6}$$

Since the function $f(x_1, \cdots, x_n)$ has isolated singularities and $|\nabla f(x_1, \cdots, x_n)| = |h'(r)| = |\sinh((r-c)/k)|$ vanishes where $r(x_1, \cdots, x_n) = c$, the constant c must be nonpositive, but if c is negative, the function $f(x_1, \cdots, x_n)$ cannot be differentiable at the origin. This implies that $c = 0$, and hence

$$f(x_1, \cdots, x_n) = k\cosh\left(\frac{r}{k}\right), \quad r = |(x_1, \cdots, x_n)|. \tag{3.7}$$

Case 2. $f(x_1, \cdots, x_n) = h(x_1)$**.** In this case, we have from (3.4)

$$h'(x_1) = \pm\frac{\sqrt{h(x_1)^2 - k^2}}{k}, \tag{3.8}$$

which shows that for some real number $c \in \mathbb{R}$

$$f(x_1, \cdots, x_n) = h(x_1) = k \cosh\left(\frac{x_1 - c}{k}\right). \tag{3.9}$$

Since $|\nabla f(x_1, \cdots, x_n)|$ vanishes on the hyperplane $x_1 = c$, this case is impossible.

Summarizing the above two cases, we see that (1) $\Rightarrow$ (2).

Conversely, Remark 1.5 shows that (2) $\Rightarrow$ (1). This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.4: (1) $\Rightarrow$ (3)

In this section, with the help of Lemmas in Section 2, we prove (1) $\Rightarrow$ (3) of Theorem 1.4 stated in Section 1.

We consider a positive C^2 function $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$ with isolated singularities.

First, suppose that the function $z = f(x), x = (x_1, \cdots, x_n)$ satisfies $\bar{x}_{1S} = \bar{x}_{1V}$ on every rectangular domain $I(x)$. Then for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$, we have

$$\frac{\int_0^{x_1} \cdots \int_0^{x_n} t_1 w(t_1, \cdots, t_n) dt_1 \cdots dt_n}{\int_0^{x_1} \cdots \int_0^{x_n} w(t_1, \cdots, t_n) dt_1 \cdots dt_n} = \frac{\int_0^{x_1} \cdots \int_0^{x_n} t_1 f(t_1, \cdots, t_n) dt_1 \cdots dt_n}{\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) dt_1 \cdots dt_n}, \tag{4.1}$$

where we put

$$w(x_1, \cdots, x_n) = \sqrt{1 + |\nabla f(x_1, \cdots, x_n)|^2}. \tag{4.2}$$

For nonzero numbers $x_2, \cdots, x_n \in \mathbb{R}$, we let

$$\tilde{f}_{x_2 \cdots x_n}(t_1) = \int_0^{x_2} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_2 \cdots dt_n \tag{4.3}$$

and

$$\tilde{w}_{x_2 \cdots x_n}(t_1) = \int_0^{x_2} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_2 \cdots dt_n. \tag{4.4}$$

Then it follows from (4.1) that for every nonzero real number x_1

$$\frac{\int_0^{x_1} t_1 \tilde{f}_{x_2 \cdots x_n}(t_1) dt_1}{\int_0^{x_1} \tilde{f}_{x_2 \cdots x_n}(t_1) dt_1} = \frac{\int_0^{x_1} t_1 \tilde{w}_{x_2 \cdots x_n}(t_1) dt_1}{\int_0^{x_1} \tilde{w}_{x_2 \cdots x_n}(t_1) dt_1}. \tag{4.5}$$

Hence, Lemma 2.1 implies that for some constant $k_1 = k_1(x_2, \cdots, x_n)$ depending on the variables $x_2, \cdots, x_n$ we have

$$\tilde{f}_{x_2 \cdots x_n}(t_1) = k_1(x_2, \cdots, x_n) \tilde{w}_{x_2 \cdots x_n}(t_1). \tag{4.6}$$

Together with (4.3) and (4.4), this yields that

$$\int_0^{x_2} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_2 \cdots dt_n = k_1(x_2, \cdots, x_n) \int_0^{x_2} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_2 \cdots dt_n. \tag{4.7}$$

By integrating from 0 to x_1, we get from (4.7)

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n = k_1(x_2, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n. \tag{4.8}$$

Now suppose that for some $i \in \{2, 3, \cdots, n\}$ the function $z = f(x), x = (x_1, \cdots, x_n)$ satisfies $\bar{x}_{iS} = \bar{x}_{iV}$ on every rectangular domain $I(x)$. Then, just as in the above discussions, we may prove the following:

$$\begin{aligned} &\int_0^{x_1} \cdots \check{\int_0^{x_i}} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_1 \cdots \check{dt_i} \cdots dt_n \\ &= k_i(x_1, \cdots, \check{x}_i, \cdots, x_n) \int_0^{x_1} \cdots \check{\int_0^{x_i}} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_1 \cdots \check{dt_i} \cdots dt_n, \end{aligned} \tag{4.9}$$

where $k_i = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n)$ is a function depending on $x_1, \cdots, \check{x}_i, \cdots, x_n$. Hereafter $\check{}$ denotes a missing term. By integrating from 0 to x_i, we obtain from (4.9)

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n. \tag{4.10}$$

Finally, suppose that for all $i \in \{1, 2, \cdots, n\}$ the function $z = f(x), x = (x_1, \cdots, x_n)$ satisfies $\bar{x}_{iS} = \bar{x}_{iV}$ on every rectangular domain $I(x)$. Then it follows from (4.8) and (4.10) that for each $i \in \{2, \cdots, n-1\}$

$$k_1(x_2, \cdots, x_n) = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n) = k_n(x_1, \cdots, x_{n-1}). \tag{4.11}$$

This shows that the function $k_1 = k_1(x_2, \cdots, x_n)$ is nothing but a constant k. Thus, (4.8) implies that

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n = k \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n. \tag{4.12}$$

By differentiating (4.12) with respect to $x_1, \cdots, x_n$ successively, we get for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$

$$f(x_1, \cdots, x_n) = kw(x_1, \cdots, x_n). \tag{4.13}$$

By continuity, (4.13) holds for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Therefore, the same argument as in the proof of Theorem 1.3 completes the proof of $(1) \Rightarrow (3)$.

5. Proof of Theorem 1.4: $(2) \Rightarrow (3)$

In this section, with the help of the lemmas in Section 2, we prove $(2) \Rightarrow (3)$ of Theorem 1.4 stated in Section 1.

We consider a positive C^2 function $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$ with isolated singularities. As before, we denote by $w(x)$ the function defined by

$$w(x_1, \cdots, x_n) = \sqrt{1 + |\nabla f(x_1, \cdots, x_n)|^2}, \tag{5.1}$$

where ∇f is the gradient vector of the function f.

First of all, we prove the following lemma.

Lemma 5.1 *We consider a positive C^2 function $z = f(x), x = (x_1, \cdots, x_n)$ defined on the n-dimensional Euclidean space $\mathbb{R}^n$. Suppose that the function $z = f(x)$ satisfies $(\bar{x}_{1S}, \cdots, \bar{x}_{(n-1)S}, \bar{z}_S) = (\bar{x}_{1V}, \cdots, \bar{x}_{(n-1)V}, 2\bar{z}_V)$ on every rectangular domain $I(x)$. Then we have for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$*

$$\int_0^{x_n} f(x_1, \cdots, x_{n-1}, t_n) dt_n = k(x_n) \int_0^{x_n} w(x_1, \cdots, x_{n-1}, t_n) dt_n, \tag{5.2}$$

where $k(x)$ is a function defined on the open set $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Furthermore, on the open set $\mathbb{R}_1 = \{t \in \mathbb{R}_0 | k'(t) \neq 0\}$, we have

$$\frac{\partial f}{\partial x_n}(x_1, \cdots, x_{n-1}, t_n) = 0, \quad x_1, \cdots, x_{n-1} \in \mathbb{R}_0, \quad t_n \in \mathbb{R}_1. \tag{5.3}$$

Proof First, suppose that the function $z = f(x), x = (x_1, \cdots, x_n)$ satisfies $\bar{z}_S = 2\bar{z}_V$ on every rectangular domain $I(x)$. Then for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$, we have

$$\frac{\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) w(t_1, \cdots, t_n) dt_1 \cdots dt_n}{\int_0^{x_1} \cdots \int_0^{x_n} w(t_1, \cdots, t_n) dt_1 \cdots dt_n} = \frac{\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n)^2 dt_1 \cdots dt_n}{\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) dt_1 \cdots dt_n}, \tag{5.4}$$

where the function $w(x_1, \cdots, x_n)$ is defined in (5.1).

Now suppose that $\bar{x}_{1S} = \bar{x}_{1V}$ on every rectangular domain $I(x)$. Then we have from (4.8)

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n = k_1(x_2, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n, \tag{5.5}$$

where $k_1(x_2, \cdots, x_n)$ is a function of $x_2, \cdots, x_n$. Together with (5.4), this shows that

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n)^2 dt_1 \cdots dt_n = k_1(x_2, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) w(t_1, \cdots, t_n) dt_1 \cdots dt_n. \tag{5.6}$$

Finally, suppose that $\bar{x}_{iS} = \bar{x}_{iV}$ for all $i \in \{2, \cdots, n-1\}$ on every rectangular domain $I(x)$. Then, just as in the case of $\bar{x}_{1S} = \bar{x}_{1V}$, we may show that

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n, \tag{5.7}$$

and

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n)^2 dt_1 \cdots dt_n = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n) \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) w(t_1, \cdots, t_n) dt_1 \cdots dt_n, \tag{5.8}$$

where $k_i = k_i(x_1, \cdots, \check{x}_i, \cdots, x_n)$ is a function depending on $x_1, \cdots, \check{x}_i, \cdots, x_n$. Hence, it follows from (5.5) and (5.7) that the functions $k_i, i = 1, 2, \cdots, n-1$ are all the same function and hence it is a function of x_n only. That is, we have $k_i(x) = k(x_n), i = 1, 2, \cdots, n-1$ for some function k of x_n. Together with (5.7) and (5.8), this implies

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n = k(x_n) \int_0^{x_1} \cdots \int_0^{x_n} w(t_1, t_2 \cdots, t_n) dt_1 \cdots dt_n, \tag{5.9}$$

and

$$\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n)^2 dt_1 \cdots dt_n = k(x_n) \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n) w(t_1, \cdots, t_n) dt_1 \cdots dt_n, \tag{5.10}$$

By differentiating (5.9) with respect to $x_1, \cdots, x_{n-1}$ successively, we get for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$

$$\int_0^{x_n} f(x_1, \cdots, x_{n-1}, t_n) dt_n = k(x_n) \int_0^{x_n} w(x_1, \cdots, x_{n-1}, t_n) dt_n. \tag{5.11}$$

Successively differentiating (5.10) with respect to $x_1, \cdots, x_{n-1}$ also gives for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$

$$\int_0^{x_n} f(x_1, \cdots, x_{n-1}, t_n)^2 dt_n = k(x_n) \int_0^{x_n} f(x_1, \cdots, x_{n-1}, t_n) w(x_1, \cdots, x_{n-1}, t_n) dt_n. \tag{5.12}$$

Now we use Lemma 2.2. Together with (5.11) and (5.12), Lemma 2.2 shows that for the open set $\mathbb{R}_1$ defined by $\mathbb{R}_1 = \{t \in \mathbb{R} | k'(t) \neq 0, t \neq 0\}$ and for all nonzero $x_1, \cdots, x_{n-1} \in \mathbb{R}$ and $t \in \mathbb{R}_1$, $\partial f / \partial x_n(x_1, \cdots, x_{n-1}, t)$ vanishes. This completes the proof of Lemma 5.1. □

Finally, we prove (2) ⇒ (3) of Theorem 1.4 as follows. Without loss of generality, we may assume that the function $z = f(x)$ satisfies $(\bar{x}_{1S}, \cdots, \bar{x}_{(n-1)S}, \bar{z}_S) = (\bar{x}_{1V}, \cdots, \bar{x}_{(n-1)V}, 2\bar{z}_V)$ on every rectangular domain $I(x)$ because the other cases can be treated similarly. Then it follows from Lemma 5.1 that we have for all nonzero $x_1, \cdots, x_n \in \mathbb{R}$

$$\int_0^{x_n} f(x_1, \cdots, x_{n-1}, t_n) dt_n = k(x_n) \int_0^{x_n} w(x_1, \cdots, x_{n-1}, t_n) dt_n, \tag{5.13}$$

where $k(x)$ is a function defined on the open set $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. Furthermore, for the open set $\mathbb{R}_1 = \{t \in \mathbb{R}_0 | k'(t) \neq 0\}$, we have

$$\frac{\partial f}{\partial x_n}(x_1, \cdots, x_{n-1}, t_n) = 0, \quad x_1, \cdots, x_{n-1} \in \mathbb{R}_0, \quad t_n \in \mathbb{R}_1. \tag{5.14}$$

We consider two cases as follows.

Case 1. $\mathbb{R}_1 = \{t \in \mathbb{R}_0 | k'(t) \neq 0\} = \phi$. In this case, the function $k(t)$ is a positive constant k. By differentiating (5.13) with respect to the variable x_n, we get

$$f(x_1, \cdots, x_{n-1}, x_n) = kw(x_1, \cdots, x_{n-1}, x_n). \tag{5.15}$$

Hence, just as in the proof of Theorem 1.3, it follows from Proposition 1.6 that by a Euclidean motion if necessary, the function $z = f(x_1, \cdots, x_n)$ is given by

$$f(x_1, \cdots, x_n) = k \cosh\left(\frac{r}{k}\right), \quad r = |(x_1, \cdots, x_n)|. \tag{5.16}$$

Case 2. $\mathbb{R}_1 = \{t \in \mathbb{R}_0 | k'(t) \neq 0\} \neq \phi$. In this case, choose a connected component (a, b) of the nonempty open set $\mathbb{R}_1$. We may assume that $a > 0$ because the case of $b < 0$ can be treated similarly. When $t \in (a, b)$, from (5.3) we have

$$f(x_1, \cdots, x_{n-1}, t) = f(x_1, \cdots, x_{n-1}) \tag{5.17}$$

and hence we get

$$w(x_1, \cdots, x_{n-1}, t) = \sqrt{1 + |\nabla f(x_1, \cdots, x_{n-1})|^2} = w(x_1, \cdots, x_{n-1}). \tag{5.18}$$

Thus, it follows from (5.2) that for $t \in (a, b)$

$$\alpha(x_1, \cdots, x_{n-1}) + \int_a^t f(x_1, \cdots, x_{n-1}) dx_n = k(t)\{\beta(x_1, \cdots, x_{n-1}) + \int_a^t w(x_1, \cdots, x_{n-1}) dx_n\}, \tag{5.19}$$

where we put

$$\alpha = \alpha(x_1, \cdots, x_{n-1}) = \int_0^a f(x_1, \cdots, x_{n-1}, x_n)dx_n \tag{5.20}$$

and

$$\beta = \beta(x_1, \cdots, x_{n-1}) = \int_0^a w(x_1, \cdots, x_{n-1}, x_n)dx_n. \tag{5.21}$$

By integrating, we get from (5.19)

$$\alpha(x_1, \cdots, x_{n-1}) + (t-a)f(x_1, \cdots, x_{n-1}) = k(t)\{\beta(x_1, \cdots, x_{n-1}) + (t-a)w(x_1, \cdots, x_{n-1})\}, \tag{5.22}$$

It follows from (5.22) that

$$k(t) = \frac{tf(x_1, \cdots, x_{n-1}) + \gamma(x_1, \cdots, x_{n-1})}{tw(x_1, \cdots, x_{n-1}) + \delta(x_1, \cdots, x_{n-1})}, \tag{5.23}$$

where we put

$$\gamma = \alpha(x_1, \cdots, x_{n-1}) - af(x_1, \cdots, x_{n-1}) \tag{5.24}$$

and

$$\delta = \beta(x_1, \cdots, x_{n-1}) - aw(x_1, \cdots, x_{n-1}). \tag{5.25}$$

Note that (5.23) may be rewritten as

$$k(t) = (\frac{f}{w})\left(1 + \frac{\bar{\gamma} - \bar{\delta}}{t + \bar{\delta}}\right), \tag{5.26}$$

where we let $\bar{\gamma} = \gamma/f$ and $\bar{\delta} = \delta/w$.

For each $i \in \{1, \cdots, n-1\}$, let us differentiate (5.26) with respect to x_i. Then we obtain for all $t \in (a, b)$ and for each $i = 1, 2, \cdots, n-1$

$$\frac{\partial}{\partial x_i}(\frac{f}{w})t^2 + \eta t + \nu = 0, \tag{5.27}$$

where η and ν are some functions of $x_1, \cdots, x_{n-1}$. This implies that for each $i = 1, 2, \cdots, n-1$ the function $\partial(f/w)/\partial x_i(x_1, \cdots, x_{n-1}, t) = 0$ for all nonzero $x_1, \cdots, x_{n-1}$ and $t \in (a, b)$. Together with (5.17) and (5.18), this shows that the gradient of the ratio f/w vanishes on the open set $\mathbb{R}^{n-1} \times \mathbb{R}_1 \subset \mathbb{R}^n$.

On a fixed connected component K of the interior $int(\mathbb{R}_1^c)$ of the complement $\mathbb{R}_1^c$ of the open set $\mathbb{R}_1$, the function $k(t)$ is a constant k_0. Hence, it follows from (5.13) that for all $t \in K$

$$\int_0^t f(x_1, \cdots, x_{n-1}, x_n)dx_n = k_0 \int_0^t w(x_1, \cdots, x_{n-1}, x_n)dx_n. \tag{5.29}$$

By differentiating (5.29) with respect to the variable t, we get

$$f(x_1, \cdots, x_{n-1}, t) = k_0 w(x_1, \cdots, x_{n-1}, t), \quad t \in K. \tag{5.30}$$

This implies that the gradient of the ratio f/w vanishes on the open set $\mathbb{R}^{n-1} \times int(\mathbb{R}_1^c)$.

Summarizing the above discussions, by continuity we see that the gradient of the ratio f/w vanishes on the whole space $\mathbb{R}^n$. That is, $f = kw$ for some positive constant k.

Just as in the proof of Case 1, it follows from Proposition 1.6 that by a Euclidean motion, if necessary, the function $z = f(x_1, \cdots, x_n)$ is given by

$$f(x_1, \cdots, x_n) = k \cosh\left(\frac{r}{k}\right), \quad r = |(x_1, \cdots, x_n)|. \tag{5.31}$$

This completes the proof of (2) $\Rightarrow$ (3).

Conversely, Remark 1.5 shows that (3) $\Rightarrow$ (1) and (2). This completes the proof of Theorem 1.4.

References

[1] Coll V, Dodd J. A characteristic averaging property of the catenary. Am Math Mon 2016; 123: 683-688.

[2] Coll V, Harrison M. Two generalizations of a property of the catenary. Am Math Mon 2014; 121: 109-119.

[3] Kaiser MJ. The perimeter centroid of a convex polygon. Appl Math Lett 1993; 6: 17-19.

[4] Khorshidi B. A new method for finding the center of gravity of polygons. J Geom 2009; 96: 81-91.

[5] Kim DS. Ellipsoids and elliptic hyperboloids in the Euclidean space E^{n+1}. Linear Algebra Appl 2015; 471: 28-45.

[6] Kim DS, Kim DS. Centroid of triangles associated with a curve. B Korean Math Soc 2015; 52: 571-579.

[7] Kim DS, Kim W, Lee K, Yoon DW. Various centroids of polygons and some characterizations of rhombi. Commun Korean Math Soc 2017; 32: 135-145.

[8] Kim DS, Kim YH. Some characterizations of spheres and elliptic paraboloids. Linear Algebra Appl 2012; 437: 113-120.

[9] Kim DS, Kim YH. Some characterizations of spheres and elliptic paraboloids II. Linear Algebra Appl 2013; 438: 1356-1364.

[10] Kim DS, Kim YH. On the Archimedean characterization of parabolas. B Korean Math Soc 2013; 50: 2103-2114.

[11] Kim DS, Kim YH. A characterization of concentric hyperspheres in R^n. B Korean Math Soc 2014; 51: 531-538.

[12] Kim DS, Kim YH, Park S. Center of gravity and a characterization of parabolas. Kyungpook Math J 2015; 55: 473-484.

[13] Kim DS, Kim YH, Yoon DW. Some characterizations of catenary rotation surfaces. Kyungpook Math J (in press).

[14] Kim DS, Lee KS, Lee KB, Lee YI, Son S, Yang JK, Yoon DW. Centroids and some characterizations of parallelograms. Commun Korean Math Soc 2016; 31: 637-645.

[15] Kim DS, Moon HT, Yoon DW. Centroids and some characterizations of catenary curves. Commun Korean Math Soc 2017; 32: 709-714.

[16] Kim DS, Song B. A characterization of elliptic hyperboloids. Honam Math J 2013; 35: 37-49.

[17] Krantz SG. A matter of gravity. Am Math Mon 2013; 110: 465-481.

[18] Krantz SG, McCarthy JE, Parks HR. Geometric characterizations of centroids of simplices. J Math Anal Appl 2006; 316: 87-109.

[19] Parker E. A property characterizing the catenary. Math Mag 2010; 83: 63-64.

[20] Stein S. Archimedes. What Did He Do Besides Cry Eureka? Washington, DC, USA: Mathematical Association of America, 1999.