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Modules whose p-submodules are direct summands

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Abstract: In this article we deal with modules with the property that all p-submodules are direct summands. In contrast to CLS-modules, it is shown that the former property is closed under finite direct sums, but it is not inherited by direct summands. Hence we focus on when the direct summands of aforementioned modules enjoy the property. Moreover, we characterize the forenamed class of modules in terms of lifting homomorphisms.

Key words: Extending module, CLS modules, π-extending module

1. Introduction

In this paper, R and M will denote a ring with unity and a right R-module, respectively. Note that a submodule K of M is called *complement* in M if K has no proper essential extension in M. Recall that a module M is CS or *extending* if every submodule is essential in a direct summand equivalently, every complement submodule is a direct summand [4, 10].

In recent studies, there are many generalizations of extending modules with respect to various sets of submodules. A submodule N of M is called *projection invariant* if $f(N) \subseteq N$ for all $f^2 = f \in \text{End}(M_R)$. Hence a module M is called *π-extending* [3], if every projection invariant submodule is essential in a direct summand. Even though the class of π-extending modules is closed under direct sums, the former property is not inherited by direct summands (see, [3, Example 5.5]). Recall from [10], a submodule N of M is called a *z-closed submodule* of M if M/N is nonsingular. These submodules are named closed in [9] and complement in [5]. A module M is a *CLS-module* [9], if every z-closed submodule of M is a direct summand of M.

In this paper, a submodule N of M is called a *p-submodule* if N is a projection invariant submodule in M and M/N is nonsingular. We investigate some certain properties of p-submodules. Observe that the class of p-submodules of M is a sublattice of the lattice of submodules of M. We explain the connections between complements, p-submodules, and z-closed submodules. Moreover, we deal with lifting properties on p-submodules. We call a module M a *PD-module*, if every p-submodule of M is a direct summand of M. We obtain that PD-modules are generalizations of both CLS-modules and π-extending modules. Furthermore, we provide examples that demonstrate the class of PD-modules is different from the classes of CLS-modules and π-extending modules. Contrary to CLS-modules, we obtain that finite direct sums of PD-modules are PD-modules. Additionally, we present an example that shows that any direct summand of PD-modules need not be a PD-module. To this end, we determine when the PD condition is inherited by direct summands. Finally, we characterize this class of modules using lifting homomorphisms.

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Let $K \subseteq M$. Then $K \leq M$, $K \leq_e M$, $K \leq_d M$, $K \trianglelefteq_p M$, $Z(M)$, $End(M_R)$ and $M_n(R)$ denote K is a submodule of M, K is an essential submodule of M, K is a direct summand of M, K is a projection invariant submodule of M, the singular submodule of M, the endomorphism ring of M_R, and the n-by-n full matrix ring over R, respectively. Recall that a ring R is called *Abelian* if every idempotent of R is central. For unknown terminology and notation, see [2, 4, 5, 10].

2. Lifting properties on p-submodules

In this section, we examine lifting properties on p-submodules. Let us begin with the basic results for p-submodules.

Lemma 2.1 (i) *Any intersection of p-submodules of M_R is a p-submodule of M_R.*

(ii) *Let Y_1 and Y_2 be submodules of M_R such that $Y_1 \leq Y_2$. If Y_1 is a p-submodule of Y_2 and Y_2 is a p-submodule of M_R, then Y_1 is a p-submodule of M_R.*

Proof (i) Let N_1 and N_2 be p-submodules of M. Then $N_1, N_2 \trianglelefteq_p M$ and $Z(M/N_1) = 0$, $Z(M/N_2) = 0$. It is clear that $N_1 \cap N_2$ is projection invariant in M. Observe that $(M/N_1) \oplus (M/N_2) \cong M/(N_1 \cap N_2)$. Thus $M/(N_1 \cap N_2)$ is nonsingular, and hence $N_1 \cap N_2$ is a p-submodule of M.

(ii) Let Y_1 be a p-submodule of Y_2 and Y_2 a p-submodule of M. Thus $Y_1 \trianglelefteq_p Y_2$, $Y_2 \trianglelefteq_p M$ and $Z(Y_2/Y_1) = 0$, $Z(M/Y_2) = 0$. It is clear that $Y_1 \trianglelefteq_p M$. Since $(M/Y_1)/(Y_2/Y_1) \cong M/Y_2$, $Z(M/Y_2) = 0$ and $Z(Y_2/Y_1) = 0$, it follows that $Z(M/Y_1) = 0$. Hence Y_1 is a p-submodule of M. □

The following lemma explains the connections between p-submodules, complements, and z-closed submodules.

Lemma 2.2 (i) *Every p-submodule of M_R is a complement in M_R.*

(ii) *If M_R is an indecomposable module, then p-submodules and z-closed submodules coincide.*

Proof (i) Let B be a p-submodule of M. Then $B \trianglelefteq_p M$ and $Z(M/B) = 0$. Assume that $B \leq_e T \leq M$ for some $T \leq M$. Then T/B is singular, and hence $T/B \leq Z(M/B)$. Thus $T = B$ and so $B \leq_c M$.

(ii) Every submodule of an indecomposable module is projection invariant; hence we get the result. □

The next example shows that there is a complement submodule that is not a p-submodule.

Example 2.3 *([9, Example, 2]) Let F be a field and V_F be a vector space over the field F with $dim(V_F) \geq 2$. Consider the commutative and indecomposable ring $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F, v \in V \right\}$. Let $I_v = \begin{bmatrix} 0 & Fv \\ 0 & 0 \end{bmatrix}$ be the ideal of R for any $v \in V$. Now I_v is a complement submodule in R_R but it is not a z-closed submodule by [9, Example 2]. Thus I_v is not a p-submodule by Lemma 2.2(ii).*

Following the idea in [10], we call p-submodule N of M a *p-lifting submodule* for X in M, if for any $\varphi : N \to X$, there exists $\theta : M \to X$ such that $\varphi = \theta|_N$ for any modules X_R and M_R. Let $\underline{P}$ stand for the collection of p-submodules of M. We denote the set of p-lifting submodules with $\underline{P}Lift_X(M)$. Now we investigate some certain module theoretical properties of the class of p-lifting submodules.

Proposition 2.4 *Let $M_1, M_2 \leq M$ and $M = M_1 \oplus M_2$. Then $M_1 \in \underline{P}Lift_X(M)$.*

Proof Let X_1 be a p-submodule of M_1 and $f : X_1 \to X$ a homomorphism. Thus $X_1 \oplus M_2$ is a projection invariant submodule of M by [3, Lemma 4.13]. Note that $M/(X_1 \oplus M_2) \cong M_1/X_1$, which is nonsingular. Hence $X_1 \oplus M_2$ is a p-submodule of M. Now let $\pi : X_1 \oplus M_2 \to X_1$ be the projection map and $\varphi : X_1 \oplus M_2 \to X$ defined by $\varphi = f\pi$. Observe that $M \in \underline{P}lift_X(M)$. Therefore, there exists $\theta : M \to X$ such that $\theta|_{X_1 \oplus M_2} = \varphi$. Now define $\gamma : M_1 \to X$ such that $\gamma = \theta\iota$, where $\iota : M_1 \to M$ is an inclusion map. Let $x_1 \in X_1$. Then $\gamma(x_1) = \theta(x_1) = f\pi(x_1) = f(x_1)$, and so $\gamma|_{X_1} = f$. Thus $M_1 \in \underline{P}Lift_X(M)$. □

Proposition 2.5 *The class $\underline{P}Lift_X(M)$ is closed under finite direct sums.*

Proof Suppose that $M_1, M_2 \in \underline{P}lift_X(M)$ and $T = M_1 \oplus M_2$. Let K be a p-submodule of T and $g : K \to X$ a homomorphism. Since $K \trianglelefteq_p T$, $K = (K \cap M_1) \oplus (K \cap M_2)$ by [1, Proposition 3.1 (5)]. Note that $K_i \trianglelefteq_p M_i$ and $Z(M_i/K_i) = 0$, where $K_i = N \cap M_i$ for $i = 1, 2$. Thus K_i is a p-submodule of M_i for $i = 1, 2$. Consider the following maps. Let $\alpha_1 : K_1 \to X$ be defined by $\alpha_1 = g\iota_1$ and $\alpha_2 : K_2 \to X$ be defined by $\alpha_2 = g\iota_2$, where $\iota_1 : K_1 \to K$ and $\iota_2 : K_2 \to K$ are inclusion maps. Hence there exist $\theta_1 : M_1 \to X$ and $\theta_2 : M_2 \to X$ such that $\theta_1|_{K_1} = \alpha_1$ and $\theta_2|_{K_2} = \alpha_2$ by hypothesis. Now define $\gamma : T \to X$ by $\gamma = \theta_1\pi_1 + \theta_2\pi_2$, where $\pi_i : T \to M_i$ is the i-th projection map for $i = 1, 2$. Let $k \in K$. Thus $k = k_1 + k_2$ such that $k_1 \in K_1$ and $k_2 \in K_2$. Hence $\gamma(k) = \theta_1\pi_1(k) + \theta_2\pi_2(k) = \alpha_1(k_1) + \alpha_2(k_2) = g(k_1) + g(k_2) = g(k)$. Therefore, g extends to γ, which yields that $T \in \underline{P}Lift_X(M)$. The proof follows from the induction argument. □

3. PD modules

Now we focus on the class of modules whose p-submodules are direct summands. We obtain examples that demonstrate the class of PD-modules differs from the classes of CLS-modules and π-extending modules. It is proved that the class of PD-modules is closed under finite direct sums, but the aforementioned property is not inherited by direct summands. Furthermore, there is a characterization for the class PD-modules using lifting homomorphisms.

Observe that every singular module satisfies the PD condition, but the converse of this fact is not true: let $M_{\mathbb{Z}} = \mathbb{Z}$. Hence $M_{\mathbb{Z}}$ is a PD-module that is nonsingular. Our first result gives the relations between the classes of PD-modules, CLS-modules, and π-extending modules.

Proposition 3.1 *Consider the following for a module M_R,*

(1) *M is a CS-module.*

(2) *M is a CLS-module.*

(3) *M is a π-extending module.*

(4) *M is a PD-module.*

Then $(1) \Rightarrow (2) \Rightarrow (4)$ *and* $(1) \Rightarrow (3) \Rightarrow (4)$*, but these implications are not reversible in general.*

Proof $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. These implications are clear from [9, Corollary 5] and [3, Proposition 3.7], respectively.

$(2) \Rightarrow (4)$. It is obvious from definitions.

$(3) \Rightarrow (4)$. Let V be a p-submodule of M. Thus $V \leq_e T \leq_d M$ for some $T \leq_d M$. Hence T/V is singular and so $T/V \leq Z(M/V)$. Then $V = T$. Consequently, M is a PD-module.

(2) $\not\Rightarrow$ (1) and (3) $\not\Rightarrow$ (1). Let $M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ for any prime p. Then $M_{\mathbb{Z}}$ is a CLS-module that is not extending by [9, Example 6]. On the other hand, it is a π-extending module by [7, page 1814] and [3, Proposition 3.7].

(4) $\not\Rightarrow$ (2). Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ be the upper triangular matrix ring over $\mathbb{Z}$. It is well known that R_R is π-extending and so it is a PD-module that is not CS. Since $Z(R_R) = 0$, R_R is not a CLS-module by [10, Corollary 5.60].

(4) $\not\Rightarrow$ (3). Let R be the ring in Example 2.3 with the dimension of V_F being 2. Hence R_R is not uniform. Thus R_R is not π-extending by [3, Proposition 3.8]. It can be easily seen that R_R is the only p-submodule of R_R. Therefore it is a PD-module. □

Lemma 3.2 *CLS and PD conditions are equivalent for an indecomposable module.*

Proof Clear from Lemma 2.2 (ii). □

We obtain that any submodule of a PD-module need not be a PD-module: Let R be a domain that is not right Ore. Thus every nonzero ideal of R is essential in R. Note that R_R is an indecomposable module that is not uniform. Hence R_R is not extending and so it is not CLS by [10, Corollary 5.60]. Therefore, R_R is not a PD-module by Lemma 3.2. However, $E(R_R)$, the injective hull of R_R, is a PD-module by Proposition 3.1.

The next result explains when the aforementioned property is inherited by submodules.

Proposition 3.3 *If M_R is a PD-module, then every p-submodule A of M_R is a PD-module.*

Proof Let Y be a p-submodule of A and A be a p-submodule of M. Hence Y is a p-submodule of M by Lemma 2.1 (ii). Therefore, Y is a direct summand of M. Thus $M = Y \oplus Y'$ for some submodule Y' of M. It follows that $A = A \cap (Y \oplus Y') = Y \oplus (A \cap Y')$. Then Y is a direct summand of A. Thereupon A is a PD-module. □

It is shown in [10, page 269] that CLS-modules are not closed under direct sums. Contrary to CLS-modules, PD-modules enjoy the direct sums property.

Theorem 3.4 *Let $M = M_1 \oplus ... \oplus M_k$ for some submodules $M_1, ..., M_k$ of M. If M_i is a PD-module for all $1 \leq i \leq k$, then M is a PD-module.*

Proof It is sufficient to prove the result for the case $k = 2$. Let $M = M_1 \oplus M_2$ and Y be a p-submodule of M. Thus Y is a projection invariant submodule in M and M/Y is nonsingular. Hence $Y = (Y \cap M_1) \oplus (Y \cap M_2)$, where $Y \cap M_1 \trianglelefteq_p M_1$ and $Y \cap M_2 \trianglelefteq_p M_2$ by [1, Proposition 3.1 (5)]. Observe that $M_1/(Y \cap M_1) \cong (M_1 + Y)/Y$, which is nonsingular. Thus $Y \cap M_1$ is a p-submodule of M_1. Thereby $Y \cap M_1$ is a direct summand of M_1. Hence $M_1 = (Y \cap M_1) \oplus T_1$ for some $T_1 \leq M_1$. Following the previous steps, $Y \cap M_2$ is also a direct summand of M_2. Thus $M_2 = (Y \cap M_2) \oplus T_2$ for some $T_2 \leq M_2$. Therefore, we obtain $M = M_1 \oplus M_2 = (Y \cap M_1) \oplus (Y \cap M_2) \oplus T_1 \oplus T_2 = Y \oplus T$, where $T = T_1 \oplus T_2$. Hence M is a PD-module. Now we get the proof to apply the induction argument on k. □

It is proven in [10, Lemma 5.61] that any direct summand of CLS-modules is a CLS-module. PD-modules do not behave the same as CLS-modules with respect to direct summand property.

Example 3.5 *([3, Example 5.5] or [8, Example 4]) Let S be the polynomial ring $\mathbb{R}[x_1, x_2, ..., x_n]$, where $\mathbb{R}$ is the real field and $n \geq 3$ is an odd integer. Consider the ring $R = S/Ss$ for $s = (\sum_{i=1}^{n} x_i^2) - 1$. Hence $M_R = \oplus_{i=1}^{n} R$ is a free PD-module such that M_R has a direct summand that does not have PD condition.*

Proof R is a commutative Noetherian domain and so R_R is a PD-module. Thus $M_R = \oplus_{i=1}^{n} R$ is a PD-module by Theorem 3.4. However, M has an indecomposable direct summand K with the dimension $n-1 \geq 2$. Hence K_R is not an extending module by [3, Proposition 3.8]. Suppose that K_R is a PD-module. Thus K_R satisfies the CLS condition by Lemma 3.2. Furthermore, K_R is nonsingular. Therefore, K_R is an extending module by [10, Corollary 5.60], a contradiction. Hence K_R is not a PD-module. □

Now we concentrate on when the direct summand of PD-modules is a PD-module.

Proposition 3.6 *Let $M_1, M_2 \leq M$ and $M = M_1 \oplus M_2$ be a PD-module. If M_1 is a projection invariant submodule of M, then M_1 and M_2 are PD-modules.*

Proof Let $M = M_1 \oplus M_2$ be a PD-module and M_1 a projection invariant submodule of M. Let X_1 be a p-submodule of M_1. Then $X_1 \trianglelefteq_p M_1$ and M_1/X_1 is nonsingular. Hence $X_1 \trianglelefteq_p M$ and M/X_1 is nonsingular by [5, Proposition 1.22]. Thus X_1 is a p-submodule of M and so X_1 is a direct summand of M. Then X_1 is a direct summand of M_1. Hence M_1 is a PD-module. Now let X_2 be a p-submodule of M_2. Further $M_1 \oplus X_2$ is a projection invariant in M by [3, Lemma 4.11]. Observe that $M/(M_1 \oplus X_2) \cong M_2/X_2$ is nonsingular. Hence $M_1 \oplus X_2$ is a p-submodule of M. Thus $M_1 \oplus X_2$ is a direct summand of M and so X_2 is a direct summand of M_2. Hence M_2 is a PD-module. □

Corollary 3.7 *Let M be a PD-module with an Abelian endomorphism ring. Then every direct summand of M is a PD-module.*

Proof Since M has an Abelian endomorphism ring, every direct summand of M is projection invariant. Hence apply Proposition 3.6 to get the proof. □

Consider the free module M_R in Example 3.5. Although M is a PD-module, it has a direct summand that is not a PD-module. Observe that $End(M_R) \cong M_n(R)$. Hence the endomorphism ring of M_R is not Abelian. It shows that we cannot remove the condition of Abelian in Corollary 3.7.

Theorem 3.8 *Suppose M_R has an Abelian endomorphism ring and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ for modules M_k where $1 \leq k \leq n$. Then M is a PD-module if and only if M_k is a PD-module for each $1 \leq k \leq n$.*

Proof Apply Theorem 3.4 and Corollary 3.7 to get the proof. □

One might ask whether the essential extension of a PD-module is a PD-module or not. However, the following example explains this question in a negative way.

Example 3.9 *Let R be a principal ideal ring, but not a complete discrete valuation ring. Hence there exists an indecomposable torsion-free R-module L of rank 2 by [6, Theorem 19]. Hence $N_1 \oplus N_2 \leq_e L$ for some uniform submodules N_1 and N_2 of L. It follows that $N_1 \oplus N_2$ is a PD-module by Theorem 3.4. However, L_R is not a PD-module by [10, Corollary 5.60] and Lemma 3.2.*

Finally we characterize PD-modules using lifting homomorphisms from p-submodules to the module.

Proposition 3.10 *Let $\underline{P}$ be a nonempty set of all p-submodules of M_R. Then the following statements are equivalent.*

(1) *M_R is a PD-module.*

(2) *$\underline{P} \subseteq \underline{P}Lift_Y(M)$ for all Y_R.*

(3) *$\underline{P} \subseteq \underline{P}Lift_Y(M)$ for all $Y \in \underline{P}$.*

Proof (1) $\Rightarrow$ (2). Let M be a PD-module and $K \in \underline{P}$. Then K is a direct summand of M. Let $f : K \to Y$ be a homomorphism. Define $g : M \to Y$ by $g = f\pi$, where $\pi : M \to K$ is projection map. Hence $g|_K = f$ and so $K \in \underline{P}Lift_Y(M)$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). Let K be a p-submodule of M. Then $K \in \underline{P}Lift_Y(M)$ for all $Y \in \underline{P}$. Hence $\iota : K \to K$ identity map can be extended to $g : M \to K$. Therefore, $M = K \oplus \ker g$. Consequently, M_R is a PD-module. □

Open Problem. Whether Theorem 3.4 is true for any number of modules or not?

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