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# Semisymmetric contact metric manifolds of dimension $\geq 5$

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**Abstract:** We classify semisymmetric contact metric manifolds $M^{2n+1}(\varphi, \xi, \eta, g)$, $n \geq 2$ with $\xi$-parallel tensor $h$, where $2h$ denotes the Lie derivative of the structure tensor $\varphi$ in the direction of the characteristic vector field $\xi$.

**Key words:** Contact manifolds, semisymmetric spaces, conformally flat manifolds

## 1. Introduction

Cartan initiated the study of Riemannian symmetric spaces and he introduced the notion of locally symmetric space, that is, a Riemannian manifold for which the Riemannian curvature tensor $R$ is parallel [10]. Levy [12] showed that in these spaces the sectional curvature of every plane remains invariant under parallel transport of the plane along any curve. Semisymmetric spaces, as a direct generalization of the locally symmetric spaces, are the Riemannian manifolds that satisfy the condition $R(X,Y).R = 0$, where $X, Y \in \mathfrak{X}(M)$ and $R(X,Y)$ acts as a derivation on $R$. Haesen and Verstraelen proved that in these spaces the sectional curvature of every plane is invariant under parallel transport around any infinitesimal coordinate parallelogram [11]. The classification of semisymmetric manifolds was described by Szabó [15, 16].

Obviously locally symmetric spaces are semisymmetric, but in any dimension greater than two there are examples of semisymmetric spaces that are not locally symmetric [7]. Takahashi [17] studied semisymmetric Sasakian manifolds and he proved such manifolds have constant sectional curvature 1. In dimensions greater than three, semisymmetric contact metric manifolds with $\xi \in (\kappa, \mu)$-nullity distribution were studied by Papantoniou [13]. In 1992, Perrone classified 3-dimensional semisymmetric contact metric manifolds with $R(\xi, .)\xi = -k\varphi^2$ [14]. Perrone also proved that every 3-dimensional semisymmetric contact metric manifold having $\xi$-parallel tensor $h$ is either flat or of constant curvature [14]. On the other hand, Blair and Sharma [5] proved that every locally symmetric contact metric three-manifold has constant curvature 0 or 1. In 2006, Boeckx and Cho showed that every locally symmetric contact metric manifold is locally isometric to $S^{2n+1}(1)$ or $E^{n+1} \times S^n(4)$ [6]. The results that had been proven in 3 dimensions in [5, 13, 14] were extended by Calvaruso and Perrone [9]. They proved every semisymmetric contact metric three-manifold having constant Ricci curvature along the characteristic flow is locally symmetric.

In this paper we study semisymmetric contact metric manifolds of dim $\geq 5$ and we prove the following theorems:

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**Theorem 1** *Let $(M^{2n+1}, g), n \geq 2$ be an irreducible semisymmetric contact metric manifold. If the tensor $h$ is $\xi$-parallel, then $M$ is locally isometric to $S^{2n+1}(1)$.*

**Theorem 2** *Every 5-dimensional semisymmetric contact metric manifold having $\xi$-parallel tensor $h$ is locally isometric to either $E^3 \times S^2(4)$ or $S^5(1)$.*

## 2. Preliminaries

A contact manifold is an odd-dimensional $C^\infty$ manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field $\xi$ on $M^{2n+1}$ satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any $X \in \mathfrak{X}(M)$ is called the Reeb vector field or characteristic vector field of $\eta$. A Riemannian metric $g$ is said to be an associated metric if there exists a (1, 1)-tensor field $\varphi$ such that

$$d\eta(X,Y) = g(X, \varphi Y), \qquad \eta(X) = g(X, \xi), \qquad \varphi^2 = -I + \eta \otimes \xi. \tag{1}$$

The structure $(\varphi, \xi, \eta, g)$ is called a contact metric structure and a manifold $M^{2n+1}$ with a contact metric structure is said to be a contact metric manifold. We define a (1, 1)-tensor field $h$ by $h = (1/2)\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes Lie differentiation. It is shown that $h$ is a symmetric operator and anticommutes with $\varphi$ [3]. Hence, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$ then $-\lambda$ is also an eigenvalue of $h$ with eigenvector $\varphi X$.

The following formulas hold on contact metric manifolds [2, 3]:

$$\nabla_X \xi = -\varphi X - \varphi h X, \qquad h\varphi = -\varphi h, \tag{2}$$

$$\frac{1}{2}(R_{\xi X}\xi - \phi R_{\xi \varphi X}\xi) = h^2 X + \varphi^2 X, \tag{3}$$

$$(\nabla_\xi h)X = \varphi X - h^2 \varphi X - \varphi R_{X\xi}\xi, \tag{4}$$

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(X,Y)\xi - \eta(Y)(X + hX + \eta(X)\xi), \tag{5}$$

$$Ric(\xi, \xi) = 2n - trh^2. \tag{6}$$

**Theorem 3** *[4] Let $M^{2n+1}$ be a contact metric manifold and suppose that $R_{X,Y}\xi = 0$ for all vector fields $X$ and $Y$. Then $M^{2n+1}$ is locally the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive constant curvature 4.*

**Theorem 4** *[6] A locally symmetric contact metric manifold is locally isometric to $S^{2n+1}(1)$ or $E^{n+1} \times S^n(4)$.*

Szabó proved the local structure of a semisymmetric space [15].

**Theorem 5** *For every semisymmetric space, there exists an open dense subset $U$ of $M$ such that around every point of $U$ the manifold is locally isometric to a Riemannian product of type*

$$\mathbb{R}^k \times M_1 \times ... \times M_r, \tag{7}$$

*where $k \geq 0$, $r \geq 0$, and each $M_i$ is either a symmetric space, a two-dimensional manifold, a real cone, a Kählerian cone, or a Riemannian space foliated by Euclidean leaves of codimension two.*

He arrived at this result by the study of the nullity distribution for the curvature.

**Definition 1** *The nullity vector space of the curvature tensor at a point $p$ of a Riemannian manifold $(M,g)$ is given by*

$$E_{0p} = \{X \in T_pM \mid R(X,Y)Z = 0 \quad for\ all \quad Y, Z \in T_pM\}.$$

*The index of nullity and conullity at $p$ are the numbers $\nu(p) = dimE_{0p}$ and $u(p) = dimM - \nu(p)$, respectively.*

In the local decomposition theorem, a different irreducible factor corresponds to different possible values for $\nu(p)$ and $u(p)$.

**Theorem 6** *[15] Let $(M,g)$ be an $n$-dimensional locally irreducible semisymmetric space and $p$ a point of a dense open subset $U$ of $M$. Then $M$ is locally isometric to one of the following spaces:*

*(1) a symmetric space when $\nu(p) = 0$ and $u(p) > 2$,*

*(2) a real cone when $\nu(p) = 1$ and $u(p) = n - 1 > 2$,*

*(3) a Kählerian cone when $\nu(p) = 2$ and $u(p) = n - 2 > 2$,*

*(4) a Riemannian manifold foliated by Euclidean leaves of codimension two or a two-dimensional manifold (in the case $n = 2$) when $\nu(p) = n - 2$ and $u(p) = 2$.*

**Lemma 1** *[8] Let $(M,g)$ be a Riemannian manifold, locally isometric to a Riemannian product $M_1 \times ... \times M_r$. Then, at any point $p = (p_1, ..., p_r)$ of $M$, we have*

$$\nu(p) = \nu(p_1) + ... + \nu(p_r).$$

## 3. Irreducible semisymmetric contact metric manifolds of dim $\geq 5$

**Definition 2** *A Riemannian manifold $(M,g)$ is said to be conformally flat if for any point $p \in M$ there exist a neighborhood $U$ of $p$ and a smooth function $f$ defined on $U$ such that $(U, e^{2f}g)$ is flat (i.e. the curvature of $e^{2f}g$ vanishes on $U$). The function $f$ need not be defined on all of $M$.*

Let $(M^m, g)$, $m > 2$, be a Riemannian manifold, $p \in M$ and $\{e_1, ..., e_m\}$ be an orthonormal basis of the tangent space $T_pM$. Let $R_{ijk\ell}$ and $Ric_{ik}$ be the components of $R$ and Ric with respect to $\{e_1, ..., e_m\}$. For a conformally flat Riemannian manifold of dimension $m \geqslant 4$ we have

$$\begin{aligned} R_{ijk\ell} &= \tfrac{1}{m-2}(g_{i\ell}Ric_{jk} + g_{jk}Ric_{i\ell} - g_{ik}Ric_{j\ell} - g_{j\ell}Ric_{ik}) \\ &- \tfrac{\tau}{(m-1)(m-2)}(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}), \end{aligned} \tag{8}$$

where $\tau$ denotes the scalar curvature of $M$. For 3-dimensional conformally flat spaces we have the condition

$$\nabla_i Ric_{jk} - \nabla_j Ric_{ik} = \frac{1}{2(m-1)}(g_{jk}\nabla_i \tau - g_{ik}\nabla_j \tau). \tag{9}$$

Calvaruso proved that the nullity index that appears in conformally flat semisymmetric manifolds can only attain some special values.

**Theorem 7** *[8] Let $(M, g)$ be a Riemannian manifold satisfying (8), of dimension $m \geq 3$ (that is, either $dimM = 3$ or $M$ is conformally flat). Then, at each point $p$ of $M$, the index of nullity is either $\nu(p) = 0, 1$ or $m$.*

If the nullity index is constant and equal to m (respectively, to 0), then the space is flat (respectively, locally symmetric). Now let $\nu(p) = 1$ and $\{e_0, e_1, ..., e_{m-1}\}$ be an orthonormal basis of $T_pM$. If $e_0 \in E_{0p}$, the Ricci tensor at $p$ is described by [8]:

$$\begin{cases} Ric_{ij} = \frac{\tau}{m-1} \quad \text{if} \quad i = j \geq 1 \\ Ric_{ij} = 0 \quad \text{in all the other cases.} \end{cases} \tag{10}$$

We note that every semisymmetric real cone is a conformally flat Riemannian manifold and never locally symmetric [8].

Conformally flat contact metric manifolds were studied by many authors. Bang proved the next important theorem.

**Theorem 8** *[1] In dimension $\geq 5$ there are no conformally flat contact metric structures with $R(., \xi)\xi = 0$.*

Now we are ready to prove Theorem 1.

**Proof of Theorem 1** According to Szabó's classification theorem, $M^{2n+1}$ is locally isometric to either a symmetric space, a real cone, a Kählerian cone, or a space foliated by Euclidean leaves of codimension two. We study these possibilities one by one.

*Symmetric spaces* In these cases $(M^{2n+1}, g)$ is locally symmetric and from Theorem 4 it is locally isometric to either $S^{2n+1}(1)$ or $E^{n+1} \times S^n(4)$. However, since $M$ is irreducible, the case $E^{n+1} \times S^n(4)$ is not acceptable.

*Kählerian cones* Since Kählerian cones are even-dimensional [7] and $(M^{2n+1}, g)$ is odd-dimensional, this possibility cannot occur.

*Real cones* In this case $M$ is conformally flat [7, 8] and at each point $p$ of $M$, $\nu(p) = 1$. Let $\{\xi, e_1, \varphi e_1, ..., e_n, \varphi e_n\}$ be an orthonormal basis of smooth eigenvectors of $h$ and $he_i = \lambda_i e_i$, $i = 1, ..., n$, where $\lambda_i$ is a nonvanishing smooth function, which we suppose to be positive. Then the equation $h\varphi = -\varphi h$ yields $h\varphi e_i = -\lambda_i \varphi e_i$ and the spectrum of $h$ is given by the set $\{0, \lambda_1, -\lambda_1, \lambda_2, -\lambda_2, ..., \lambda_n, -\lambda_n\}$. If $\xi \in E_{0p}$ then $R(X, Y)\xi = 0$ for all $X$ and $Y$, and Theorem 3 implies $M^5$ is locally reducible, contrary to the assumption. Now without losing generality, let $e_1 \in E_{0p}$. Then from (4) and $\nabla_\xi h = 0$ we have $0 = R(e_1, \xi)\xi = (1 - \lambda_1^2)e_1$. Since for $i = 1, ..., n$, $\lambda_i > 0$ then $\lambda_1 = 1$ and the spectrum of $h$ reduces to $\{0, +1, -1, \lambda_2, -\lambda_2, ..., \lambda_n, -\lambda_n\}$. Putting $e_j = e_k = \xi$ and $e_i = e_\ell$, $i = 2, ..., n$ in (8), we have

$$1 - \lambda_i^2 = \frac{Ric(\xi, \xi)}{2n-1} + \frac{Ric(e_i, e_i)}{2n-1} - \frac{\tau}{2n(2n-1)}. \tag{11}$$

From (10) and (11), it follows that

$$Ric(\xi, \xi) = (2n-1)(1 - \lambda_i^2). \tag{12}$$

By virtue of (10) and (12) at each point $p$ of $M$, we have

$$\tau = 2n(2n-1)(1 - \lambda_i^2) \text{ for all } i = 2, ..., n. \tag{13}$$

Then $\lambda_2 = ... = \lambda_n$. On the other hand, (6) and (12) imply

$$(2n-1)(1-\lambda_i^2) = 2n - trh^2 = 2n - 2\sum_{j=0}^{n} \lambda_j^2 = 2n - 2(1+\lambda_2^2 + ... + \lambda_n^2) = 2n - 2 - 2(n-1)\lambda_i^2.$$

Hence, for all $i = 2, .., n$, $\lambda_i = 1$ and $R(e_i, \xi)\xi = (1-\lambda_i^2)e_i = 0$, which is impossible by Theorem 8.

*Foliated spaces* In this case $M$ is an irreducible semisymmetric space with nullity index $2n-1$. Then either $\xi \in E_{0p}$ or without losing generality we suppose $e_1, ..., e_n, \varphi e_1, ..., \varphi e_{n-1} \in E_{0p}$. In two cases, $R(X,Y)\xi = 0$ for all $X$ and $Y$. Thus, from Theorem 3, $M^5 \simeq E^{n+1} \times S^n(4)$, contrary to the assumption. □

## 4. Reducible 5-dimensional semisymmetric contact metric manifolds

Let $M^5$ be a semisymmetric contact metric manifold and $\nabla_\xi h = 0$. Let $\{e_0 = \xi, e_1, e_2 = \varphi e_1, e_3, e_4 = \varphi e_3\}$ be a local orthonormal basis of smooth eigenvectors of $h$ and $he_1 = \lambda e_1$, $he_3 = \mu e_3$ where $\lambda$ and $\mu$ are smooth functions, which we suppose to be positive. Then from (2) we get $he_2 = -\lambda e_2$ and $he_4 = -\mu e_4$.

Using (4), (1), and $\nabla_\xi h = 0$ we have

$$R_{X\xi}\xi = X - \eta(X)\xi - h^2 X. \tag{14}$$

**Lemma 2** *The Levi-Civita connection of $M$ satisfies the following relations:*

$$\begin{array}{ll}
\nabla_{e_1}\xi = -(1+\lambda)e_2, & \nabla_{e_2}\xi = (1-\lambda)e_1, \\
\nabla_{e_3}\xi = -(1+\mu)e_4, & \nabla_{e_4}\xi = (1-\mu)e_3, \\
\nabla_\xi e_1 = ae_2 + be_3 + ce_4, & \nabla_\xi e_2 = -ae_1 - ce_3 + be_4, \\
\nabla_\xi e_3 = -be_1 + ce_2 + de_4, & \nabla_\xi e_4 = -ce_1 - be_2 - de_3, \\
\nabla_{e_1} e_1 = a_2 e_2 + a_3 e_3 + a_4 e_4, & \\
\nabla_{e_2} e_1 = (\lambda - 1)\xi + b_2 e_2 + b_3 e_3 + b_4 e_4, & \\
\nabla_{e_1} e_2 = (1+\lambda)\xi - a_2 e_1 + c_3 e_3 + c_4 e_4, & \\
\nabla_{e_3} e_4 = (1+\mu)\xi - f_4 e_1 - u_4 e_2 - p_4 e_3, & \\
\nabla_{e_3} e_1 = f_2 e_2 + f_3 e_3 + f_4 e_4, & \nabla_{e_1} e_3 = -a_3 e_1 - c_3 e_2 + h_4 e_4, \\
\nabla_{e_4} e_1 = k_2 e_2 + k_3 e_3 + k_4 e_4, & \nabla_{e_1} e_4 = -a_4 e_1 - c_4 e_2 - h_4 e_3, \\
\nabla_{e_4} e_2 = -k_2 e_1 + m_3 e_3 + m_4 e_4, & \nabla_{e_2} e_4 = -b_4 e_1 - d_4 e_2 + n_3 e_3, \\
\nabla_{e_3} e_2 = -f_2 e_1 + u_3 e_3 + u_4 e_4, & \nabla_{e_2} e_3 = -b_3 e_1 - d_3 e_2 - n_3 e_4, \\
\nabla_{e_3} e_3 = -f_3 e_1 - u_3 e_2 + p_4 e_4, & \nabla_{e_4} e_3 = (\mu - 1)\xi - k_3 e_1 - m_3 e_2 + q_4 e_4, \\
\nabla_{e_2} e_2 = -b_2 e_1 + d_3 e_3 + d_4 e_4, & \nabla_{e_4} e_4 = -k_4 e_1 - m_4 e_2 - q_4 e_3,
\end{array} \tag{15}$$

*where all coefficients are smooth functions on $M$ and*

$$\begin{array}{l}
a_4 + c_3 - b_3 + d_4 = 0, \\
a_3 - c_4 + b_4 + d_3 = 0, \\
f_4 + u_3 - k_3 + m_4 = 0, \\
f_3 - u_4 + k_4 + m_3 = 0.
\end{array} \tag{16}$$

**Proof** Straightforward computations and using (2) yield (15). Putting $X = Y = e_i$, $i = 1, 3$ in (5) and applying (15), we get (16). □

By direct computations we have

$$R(e_1,\xi)\xi \;= \nabla_{e_1}\nabla_\xi\xi - \nabla_\xi\nabla_{e_1}\xi - \nabla_{[e_1,\xi]}\xi \\ \quad = (1-\lambda^2-2a\lambda)e_1 + \xi(\lambda)e_2 - c(\lambda+\mu)e_3 + b(\lambda-\mu)e_4, \tag{17}$$

$$R(e_3,\xi)\xi \;= \nabla_{e_3}\nabla_\xi\xi - \nabla_\xi\nabla_{e_3}\xi - \nabla_{[e_3,\xi]}\xi \\ \quad = -c(\lambda+\mu)e_1 + b(\lambda-\mu)e_2 + (1-\mu^2-2d\mu)e_3 + \xi(\mu)e_4. \tag{18}$$

On the other hand, (14) gives

$$R(e_i,\xi)\xi = \begin{cases} (1-\lambda^2)e_i & i=1,2 \\ (1-\mu^2)e_i & i=3,4. \end{cases} \tag{19}$$

Then the functions $a,b,c,d,\lambda$, and $\mu$ must satisfy in the system

$$\xi(\lambda)=\xi(\mu)=0, \quad a\lambda=0, \quad d\mu=0, \quad c(\lambda+\mu)=0, \quad b(\lambda-\mu)=0. \tag{20}$$

**Proposition 1** *Let $(M^5,g)$ be a reducible semisymmetric contact metric manifold with $\nabla_\xi h=0$ and at each point $p$ of $M^5$ the index of nullity is $\nu(p)>0$. Then the eigenvalues of the tensor field $h$ cannot be $\pm1$ with multiplicity $1$ and $0$ with multiplicity $3$.*

**Proof** Suppose for contradiction that the spectrum of $h$ is given by the set $\{0,+1,-1\}$ with $\pm1$ as simple eigenvalues and $0$ with multiplicity $3$. Since $\nu(p)>0$, there is $X\in E_{0p}$. If $X=\xi$, then $R(e_i,\xi)\xi=0$ and (19) implies $sp(h)=\{0,+1,-1\}$ where $0$ is a simple eigenvalue, which is a contradiction.

Without losing generality suppose $X=e_1$. Then $\lambda=1$, $\mu=0$, and system (20) implies $a=b=c=0$. From $R(e_1,e_i)\xi=0$ for $i=2,3,4$, using (15), we have

$$a_2=b_2=0, \qquad 2d_3-c_4+b_4=0, \quad 2d_4+c_3-b_3=0, \tag{21}$$

$$a_4=2f_2, \quad c_4=2a_3, \quad 2u_3=-f_4, \quad 2u_4=f_3, \tag{22}$$

$$a_3=-2k_2, \quad c_3=-2a_4, \quad 2m_3=-k_4, \quad 2m_4=k_3. \tag{23}$$

By virtue of (16), (21), (22), and (23), it follows that

$$a_3=d_3, \quad b_4=0, \quad a_4=d_4, \quad b_3=0. \tag{24}$$

Then (24) and (16) give

$$u_3=-m_4, \quad f_4=k_3, \quad u_4=m_3, \quad f_3=-k_4. \tag{25}$$

Applying the above equations in $R(e_1,\xi)e_i=0$, $i=1,2$ implies

$$\xi(d_3)=da_4, \quad \xi(d_4)=-da_3, \tag{26}$$

$$\xi(c_3)=2d_3+dc_4, \quad \xi(c_4)=2d_4-dc_3. \tag{27}$$

By the second Bianchi identity

$$(\nabla_\xi R)(e_1,e_2)\xi+(\nabla_{e_1}R)(e_2,\xi)\xi+(\nabla_{e_2}R)(\xi,e_1)\xi=0, \tag{28}$$

(22), and (24), we get

$$\xi(d_4)=-(1+d)a_3. \tag{29}$$

Comparing (26) and (29) and using the above equations, we have

$$a_3=c_4=d_3=k_2=0. \tag{30}$$

Hence, by (26), $0=\xi(d_3)=da_4$. In view of (23), (24), and (27), it follows that

$$0=\xi(c_4)=2d_4-dc_3=2a_4+2da_4=2a_4.$$

Then from (22), (23), and (24), we obtain

$$d_4=c_3=f_2=0. \tag{31}$$

Equation $R(e_1,e_3)e_i=0$ for $i=1,2$ together with (22), (23), and (25) yields

$$e_1(m_3)-2m_4h_4+2m_4^2+2m_3^2=0,$$

$$e_1(m_3)-2m_4h_4+2+2m_3^2+2m_4^2=0.$$

Subtracting the two last equations gives $2=0$, which is a contradiction. This completes the proof. □

**Proposition 2** *Let $(M^5,g)$ be a reducible semisymmetric contact metric manifold with $\nabla_\xi h=0$ and at each point $p$ of $M^5$ the index of nullity is $\nu(p)>0$. Then the eigenvalues of the tensor field $h$ are $\pm1$ with multiplicity $2$ and $0$ with multiplicity $1$.*

**Proof** Since $\nu(p)>0$, there is $X\in E_{0p}$. If $X=\xi$ then $R(e_i,\xi)\xi=0$ and from (19) one can easily get the result. Now, without losing generality, let $\xi\neq X=e_1$. Then $\lambda=1$. Suppose for contradiction $\mu\neq1$. Then the system (20) provides $a=b=c=d=0$, $\xi(\mu)=0$. From $R(e_1,e_i)\xi=0$ for $i=2,3,4$ and (15), we have

$$a_2=b_2=0,\qquad 2d_3-(1-\mu)(c_4-b_4)=0, \tag{32}$$

$$2d_4+(1+\mu)(c_3-b_3)=0, \tag{33}$$

$$a_4=\frac{2f_2}{1+\mu},\qquad 2u_3+2\mu h_4+(1-\mu)f_4=0, \tag{34}$$

$$e_1(\mu)=2u_4-(1+\mu)f_3,\qquad c_4=\frac{2a_3}{1+\mu}, \tag{35}$$

$$a_3=\frac{-2k_2}{1-\mu},\qquad 2m_4-2\mu h_4-(1+\mu)k_3=0, \tag{36}$$

$$e_1(\mu) = 2m_3 + (1-\mu)k_4, \quad c_3 = \frac{-2a_4}{1-\mu}. \tag{37}$$

Using (16) in (32) and (33), we get

$$a_3 = \frac{1+\mu}{1-\mu}d_3, \tag{38}$$

$$a_4 = \frac{1-\mu}{1+\mu}d_4, \tag{39}$$

respectively. Applying (35) and (38) in (32) and (37) and (39) in (33) gives

$$b_3 = b_4 = 0. \tag{40}$$

By $R(e_1,\xi)e_1 = 0$ we have $\xi(a_i) = 0,\ i = 3 = 4$. Differentiating (38) and (39) with respect to $\xi$, using $\xi(\mu) = 0$, shows that $\xi(d_i) = 0,\ i = 3 = 4$. On the other hand, (28) implies

$$\xi(d_i) = \frac{-1}{2}(1-\mu^2)c_i,\ i = 3, 4. \tag{41}$$

Then we get

$$c_3 = c_4 = a_3 = a_4 = d_3 = d_4 = f_2 = k_2 = 0.$$

From $R(e_i,\xi)e_1 = 0$ for $i = 3, 4$ we obtain

$$\xi(f_i) = (1+\mu)k_i, \quad \xi(k_i) = (\mu-1)f_i. \tag{42}$$

Subtracting (35) and (37) and using (16) yields

$$f_3 = \frac{1+\mu}{\mu-1}k_4. \tag{43}$$

Taking the derivative of (43) with respect to $\xi$ and using $\xi(\mu) = 0$, (42), and (16), it follows that

$$f_4 = k_3, \quad u_3 = -m_4. \tag{44}$$

Applying (44) in (34) and summing the resulting equation by (36), one can get $f_4 = k_3 = 0$. Then (42) provides $f_3 = k_4 = 0$ and from (16) $m_3 = u_4$.

Equation $R(e_1,e_i)e_2 = 0,\ i = 3, 4$ together with the above equations implies

$$e_1(m_3) - 2m_4h_4 - 2(1-\mu) = 0, \tag{45}$$

$$e_1(m_3) - 2m_4h_4 + 2(1+\mu) = 0. \tag{46}$$

Subtracting the two last equations gives $2 = 0$, which is a contradiction. □

**Proposition 3** *The eigenvector of the tensor field $h$ with eigenvalue $+1$ cannot be a member of the nullity vector space.*

**Proof** Assume for contradiction $e_1 \in E_{0p}$. Since $\lambda = \mu = 1$, (20) implies $a = d = c = 0$. From $R(e_1, e_i)\xi = 0$ for $i = 2, 3, 4$ and (16) we have

$$a_2 = b_2 = d_3 = a_4 = 0, \tag{47}$$

$$f_2 = 0, \quad c_4 = a_3, \quad h_4 = -u_3, \quad u_4 = f_3, \tag{48}$$

$$k_2 = 0, \quad m_3 = 0, \quad m_4 - h_4 - k_3 = 0. \tag{49}$$

Applying (47), (48), and (49) in (16) gives

$$f_4 = 0, \quad b_4 = 0, \quad k_4 = 0. \tag{50}$$

Using the above equations in $R(e_1, \xi)e_i = 0$, $i = 1, 2$, yields

$$bc_3 = 0, \quad bh_4 = 0, \quad e_1(b) - \xi(a_3) + 2b_3 + bf_3 = 0, \tag{51}$$

$$2bh_4 + \xi(c_3) = 0, \quad e_1(b) - \xi(c_4) + 2d_4 + bu_4 = 0. \tag{52}$$

Subtracting (51) and (52) and using (48) and (16), it follows that

$$b_3 = d_4, \quad c_3 = 0. \tag{53}$$

Also from $R(e_1, e_3)e_i = 0$, $i = 1, 2$, one can see

$$e_1(f_3) - e_3(a_3) + a_3^2 - h_4 k_3 + f_3^2 = 0, \tag{54}$$

$$f_3 h_4 - a_3 p_4 = 0, \quad a_3 u_3 = 0, \tag{55}$$

$$e_1(u_3) + c_4 p_4 + f_3 u_3 - u_4 h_4 = 0, \tag{56}$$

$$u_3 h_4 + 4 - e_3(c_4) + a_3 c_4 - h_4 m_4 + f_3 u_4 + e_1(u_4) = 0. \tag{57}$$

Subtracting (54) and (57) and using (48) and (49) implies

$$h_4^2 = 2. \tag{58}$$

Then in view of (51), (54), (55), (48), and (49), we get

$$b = 0, \quad a_3 = c_4 = 0, \quad f_3 = u_4 = 0, \quad b_3 = d_4 = 0, \quad k_3 = 0, \quad m_4 = h_4. \tag{59}$$

Using the above equations in $R(e_1, e_4)e_2 = 0$ gives $h_4^2 = 0$, which is a contradiction.

Now let $e_3 \in E_{0p}$. Equation $R(e_3, e_i)\xi = 0$ for $i = 1, 2, 4$ yields

$$f_2 = a_4, \quad a_3 = c_4, \quad u_3 = -h_4, \quad u_4 = f_3, \tag{60}$$

$$b_4 = 0, \quad n_3 = 0, \quad u_3 = 0, \quad d_4 - b_3 + f_2 = 0, \tag{61}$$

$$k_4 = 0, \quad q_4 = 0, \quad p_4 = 0, \quad m_4 + f_4 - k_3 = 0. \tag{62}$$

By virtue of the above equations and (16), we have

$$b_4 = -d_3, \quad m_3 = 0, \quad h_4 = 0, \quad c_3 = 0. \tag{63}$$

Using the above equations in $R(e_3, \xi)e_i = 0$ for $i = 3, 4$ implies

$$-e_3(b) + \xi(f_3) - 2k_3 + ba_3 = 0, \quad bf_2 = 0, \quad bf_4 = 0, \tag{64}$$

$$-e_3(b) + \xi(u_4) - 2m_4 + bc_4 = 0, \quad \xi(f_4) + ba_4 = 0. \tag{65}$$

Subtracting the two last equations and using (60) and (16) gives $k_3 = m_4$ and $f_4 = 0$. Equation $R(e_3, e_1)e_i = 0$ for $i = 3, 4$ provides

$$-e_3(a_3) + e_1(f_3) + f_3^2 + a_3^2 + b_3 f_2 = 0, \tag{66}$$

$$a_2 f_3 - a_3 f_2 = 0, \quad f_3 a_4 = 0, \tag{67}$$

$$-e_3(c_4) + 4 + e_1(u_4) + f_3 u_4 + a_3 c_4 + f_2 d_4 - a_4 f_2 = 0, \tag{68}$$

$$-e_3(a_4) + c_4 f_2 - u_4 a_2 + a_3 a_4 = 0, \quad a_3 f_3 = 0. \tag{69}$$

Subtracting (66) and (69), using (60) and (61), gives

$$a_4^2 = f_2^2 = 2. \tag{70}$$

Then in view of (64), (67), (65), and (66) we obtain

$$b = 0, \quad f_3 = u_4 = 0, \quad a_3 = c_4 = 0, \quad m_4 = 0, \quad k_3 = 0, \quad b_3 = 0.$$

Using the above equations in $R(e_3, e_2)e_4 = 0$ yields $a_4^2 = 0$, which is a contradiction. This completes the proof. □

**Proposition 4** *Let $(M^5, g)$ be a reducible semisymmetric contact metric manifold and the tensor $h$ is $\xi$-parallel. Then $\nu(p) \neq 1$*

**Proof** Suppose for contradiction that $M^5$ is a semisymmetric contact metric manifold with $\nu(p) = 1$. Then there is $X \in E_{0p}$. If $X = \xi$, for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and from Theorem 3, $M^5 \simeq E^3 \times S^2(4)$. Then $\nu(p) = 3$, which is a contradiction. Since from Proposition 3 for $i = 1, 3$, $e_i \notin E_{0p}$ and then either $X = e_2$ or $X = e_4$. Without losing generality let $e_2 \in E_{0p}$.

Using (15), (16), and $R(e_2, e_i)\xi = 0$, $i = 1, 3, 4$ we get

$$a_2 = a_4 = 0, \quad b_2 = 0, \quad d_3 = 0, \tag{71}$$

$$b_4 = 0, \quad n_3 = 0, \quad u_3 = 0, \quad c_3 = f_2 \quad a_3 = c_4, \tag{72}$$

$$k_2 = 0, \quad m_3 = 0. \tag{73}$$

Applying (72) in $R(e_2, e_1)e_2 = 0$ gives

$$c_3 = f_2 = 0, \quad d_4 = b_3, \quad d_4 h_4 = 0, \quad e_2(c_4) - e_1(d_4) - u_4 b_3 + c_4 m_4 + 2b = 0. \tag{74}$$

By $R(e_2, \xi)e_i = 0$ and $R(e_j, \xi)e_2 = 0$ for $i = 1, 2, 3,\ j = 1, 3$ we have

$$bk_4 = 0, \quad bk_3 = bm_4, \quad bq_4 = 0, \tag{75}$$

$$bh_4 = 0, \quad e_1(b) - \xi(c_4) + 2d_4 + bu_4 = 0, \tag{76}$$

$$bp_4 = 0, \quad e_3(b) - \xi(u_4) + 2m_4 - bc_4 = 0, \quad bf_4 = 0. \tag{77}$$

The proof proceeds via the following steps:

Step 1: The smooth function $b$ on $M$ is zero.

**Proof** Let $b \neq 0$. A direct computation of $R(e_i, e_j)\xi$, using (16) gives

$$\begin{aligned} &R(e_1, e_3)\xi = 2h_4 e_3 + 2(u_4 - f_3)e_4, \\ &R(e_1, e_4)\xi = 2(m_4 - h_4 - k_3)e_4, \\ &R(e_3, e_4)\xi = -2k_4 e_1 - 2q_4 e_3 - 2p_4 e_4. \end{aligned} \tag{78}$$

In view of (75), (76), (77), and (16), it follows that

$$k_4 = q_4 = h_4 = p_4 = f_4 = 0, \quad k_3 = m_4, \quad f_3 = u_4.$$

Then for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and from Theorem 3, $\nu(p) = 3$; that is a contradiction. □

Step 2: The smooth functions $b_3$ and $d_4$ on $M$ are zero.

**Proof** By virtue of $R(e_2, e_1)e_1 = 0$, $R(e_2, e_i)e_1 = 0$, $i = 3, 4$ and $R(e_2, e_i)e_2 = 0$, $i = 3, 4$ we have

$$e_2(a_3) - e_1(b_3) - b_3 f_3 + c_4 k_3 = 0, \quad - b_3 h_4 - b_3 f_4 + c_4 k_4 = 0, \tag{79}$$

$$f_4 d_4 = 0, \quad e_2(f_3) - e_3(b_3) + b_3 a_3 + u_4 k_3 = 0, \quad e_2(f_4) - b_3 p_4 + u_4 k_4 = 0, \tag{80}$$

$$d_4 k_4 = 0, \quad e_2(k_3) - e_4(b_3) + d_4^2 + m_4 k_3 = 0, \quad e_2(k_4) - b_3 q_4 + m_4 k_4 = 0, \tag{81}$$

$$d_4 p_4 = 0, \quad e_2(u_4) - e_3(d_4) + b_3 c_4 + u_4 m_4 = 0, \tag{82}$$

$$d_4 q_4 = 0, \quad e_2(m_4) - e_4(d_4) + d_4^2 + m_4^2 = 0. \tag{83}$$

If $d_4 = b_3 \neq 0$, the above equations and (16) yield

$$k_4 = q_4 = h_4 = p_4 = f_4 = 0, \quad k_3 = m_4, \quad f_3 = u_4.$$

Hence, for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and $M^5 \simeq E^{n+1} \times S^n(4)$. Then $\nu(p) = 3$, a contradiction. □

Equations (15), (16), (77), and (78) and the second Bianchi identity

$$(\nabla_\xi R)(e_1, e_3)\xi + (\nabla_{e_1} R)(e_3, \xi)\xi + (\nabla_{e_3} R)(\xi, e_1)\xi = 0,$$

imply

$$\xi(f_3) = 2(m_4 + h_4 + f_4). \tag{84}$$

Also from

$$(\nabla_{e_1} R)(e_1, e_3)\xi + (\nabla_{e_2} R)(e_3, e_1)\xi + (\nabla_{e_3} R)(e_1, e_2)\xi = 0,$$

(15), (80), and (82) we have

$$-p_4 c_4 + u_4 h_4 = 0. \tag{85}$$

Using $R(e_1, e_4)e_2 = 0$ and (15), it follows that

$$e_1(m_4) - e_4(c_4) + u_4 h_4 + u_4 k_3 + k_4 m_4 = 0, \tag{86}$$

$$-m_4 h_4 + c_4 q_4 = 0. \tag{87}$$

Step 3: The smooth functions $a_3$ and $c_4$ on $M$ are zero.

**Proof** Let $a_3 = c_4 \neq 0$. Subtracting (74) and (79) using (16) we obtain

$$k_4 = 0, \quad f_4 = 0, \quad m_4 = k_3, \quad f_3 = u_4. \tag{88}$$

From (88) and (84), it follows that

$$\xi(u_4) = 2(m_4 + h_4).$$

Also $b = 0$ and (77) give $\xi(u_4) = 2m_4$. Comparing the two last equations yields $h_4 = 0$. Using the above equations in $R(e_2, e_1)e_4 = 0$ and (85), we get $q_4 = 0$ and $p_4 = 0$, respectively. Then, in view of (78) for all vector fields $X$ and $Y$, $R(X,Y)\xi = 0$ and $M^5 \simeq E^3 \times S^2(4)$. Thus, $\nu(p) = 3$, which is a contradiction. □

A direct computation of $R(e_3, e_i)e_2 = 0$, $i = 1, 4$ shows that

$$\begin{aligned} &e_3(m_4) - e_4(u_4) + p_4 u_4 + q_4 m_4 = 0,\\ &u_4 k_4 - m_4 f_4 = 0,\\ &u_4 q_4 - m_4 p_4 = 0, \end{aligned} \tag{89}$$

$$e_1(u_4) + 4 - m_4 h_4 + m_4 f_4 + f_3 u_4 = 0, \quad u_4 h_4 = 0. \tag{90}$$

Step 4: The smooth function $h_4$ on $M$ is zero.

**Proof** Equation (87) gives $m_4 h_4 = 0$ and then $h_4 = 0$. If $m_4 = 0$, equation (90) reduces to

$$e_1(u_4) + 4 + f_3 u_4 = 0, \quad u_4 h_4 = 0,$$

but $u_4 \neq 0$, because otherwise the above equation yields $4 = 0$, which is a contradiction. Then $h_4 = 0$. □

Step 5: $u_4 \neq 0$.

**Proof** By virtue of (89) and $h = 0$, (90) reduces to $e_1(u_4) + 4 + u_4 k_4 + f_3 u_4 = 0$. If $u_4 = 0$ we obtain $4 = 0$, which is a contradiction. □

Step 6: $m_4k_4 = f_4u_4$.

**Proof** From $R(e_4,\xi)e_2 = 0$ we have $\xi(m_4) = 0$. By the second Bianchi identity

$$(\nabla_\xi R)(e_1, e_4)\xi + (\nabla_{e_1} R)(e_4, \xi)\xi + (\nabla_{e_4} R)(\xi, e_1)\xi = 0,$$

one can see $\xi(k_3) = 0$. Taking the derivative of (16) with respect to $\xi$ and using (77) and (84) gives

$$\xi(f_4) = 0, \quad \xi(k_4) = -2f_4. \tag{91}$$

Applying (16), (81), and (91) in

$$(\nabla_\xi R)(e_2, e_3)e_1 + (\nabla_{e_2} R)(e_3, \xi)e_1 + (\nabla_{e_3} R)(\xi, e_2)e_1 = 0$$

implies

$$m_4k_4 - f_4u_4 = 0. \tag{92}$$

□

Step 7: The smooth functions $f_4$ and $k_4$ on $M$ are zero and $f_3 = u_4, m_4 = k_3$.

**Proof** Differentiating (92) with respect to $\xi$ and using (91), (77), and $\xi(m_4) = 0$ we get $m_4f_4 = 0$. Thus, from (89) and $u_4 \neq 0$ it follows that $k_4 = 0$. Hence, (92) and $u_4 \neq 0$ yield $f_4 = 0$. From (16) one can easily get $f_3 = u_4$, $m_4 = k_3$. □

The second Bianchi identity,

$$(\nabla_Y R)(e_3, e_4)\xi + (\nabla_{e_3} R)(e_4, Y)\xi + (\nabla_{e_4} R)(Y, e_3)\xi = 0,$$

for $Y = \xi, e_1$ together with (78) and (86) gives

$$\xi(p_4) = 0, \tag{93}$$

$$e_1(p_4) = -f_3p_4. \tag{94}$$

Step 8: The smooth functions $p_4$ and $q_4$ on $M$ are zero.

**Proof** Applying (84), (86), (89), (90), (93), and (94) in the second Bianchi identity

$$(\nabla_{e_1} R)(e_2, e_3)e_3 + (\nabla_{e_2} R)(e_3, e_1)e_3 + (\nabla_{e_3} R)(e_1, e_2)e_3 = 0,$$

we get

$$e_1(q_4) = \frac{4q_4 - u_4^2q_4}{u_4}. \tag{95}$$

Using (86), (90), (94), and (95) in

$$(\nabla_{e_1} R)(e_3, e_4)e_2 + (\nabla_{e_3} R)(e_4, e_1)e_2 + (\nabla_{e_4} R)(e_1, e_3)e_2 = 0$$

provides $p_4 = q_4 = 0$. □

In view of these eight steps and (78) for all vector fields $X$ and $Y$, $R(X,Y)\xi = 0$. Then $M^5 \simeq E^3 \times S^2(4)$ and $\nu(p) = 3$, which is a contradiction, and this complete the proof. □

**Proposition 5** *Let* $(M^5, g)$ *be a reducible semisymmetric contact metric manifold and the tensor* $h$ *is* $\xi$*-parallel. Then* $M$ *is locally isometric to* $E^3 \times S^2(4)$.

**Proof** Let $M^5$ is a reducible semisymmetric contact metric manifold. Then, from Theorem 5, there exists an open dense subset $U$ of $M$ such that around every point $p$ of $U$ the manifold is locally isometric to a Riemannian product of type (7) and from Lemma 1, $\nu(p) = \nu(p_1) + ... + \nu(p_r)$. According to Propositions 3 and 4, $\nu(p) = 0, 2$, or $3$.

If $\nu(p) = 0$ then for all $i = 1, ..., r$, $\nu(p_i) = 0$ and all $M_i$ in (7) are locally symmetric. Since the Riemannian product of locally symmetric manifolds is locally symmetric then from Theorem 4, $M^5 \simeq E^3 \times S^2(4)$. Hence, $\nu(p) = 3$, which is a contradiction.

Let $\nu(p) = 2$. If $\xi \in E_{0p}$ then Theorem 4 implies $M^5 \simeq E^3 \times S^2(4)$, i.e. $\nu(p) = 3$, which is a contradiction. Then in view of Proposition 3, $e_2, e_4 \in E_{0p}$. According to the proof of Proposition 4 from $e_2 \in E_{0p}$ we have $M^5 \simeq E^3 \times S^2(4)$. Then $\nu(p) = 3$, which is a contradiction.

If $\nu(p) = 3$, since for $i = 1, 3$, $e_i \notin E_{0p}$ and then $\xi, e_2, e_4 \in E_{0p}$. Hence, for all vector fields $X$ and $Y$, $R(X, Y)\xi = 0$ and from Theorem 3, $M^5 \simeq E^3 \times S^2(4)$. □

**Proof of Theorem 2** It follows from Theorem 1 and Proposition 5. □

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