Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 49 – 56
© TÜBİTAK
doi:10.3906/mat-1601-96

Symmetry of numerical range and semigroup generation of infinite dimensional Hamiltonian operators

Jie LIU[1,2], Junjie HUANG[2,*], Alatancang CHEN[3]
[1]Manzhouli College, Inner Mongolia University, Manzhouli, P.R. China
[2]School of Mathematical Sciences, Inner Mongolia University, Hohhot, P.R. China
[3]Department of Mathematics, Hohhot Minzu College, Hohhot, P.R. China

Received: 22.01.2016 • **Accepted/Published Online:** 07.03.2017 • **Final Version:** 22.01.2018

Abstract: This paper deals with the infinite dimensional Hamiltonian operator with unbounded entries. Using the core of its entries, we obtain the conditions under which the numerical range of such an operator is symmetric with respect to the imaginary axis. Based on the symmetry above, a necessary and sufficient condition for generating C_0 semigroups is further given.

Key words: Infinite dimensional Hamiltonian operator, numerical range, C_0 semigroup, symmetry

1. Introduction

The infinite dimensional Hamiltonian operator naturally arises from Hamiltonian systems and plays an important role in lots of areas such as mechanics and optimal control theory (cf. [7, 13]). In these areas, the spectral properties of the corresponding Hamiltonian operator are of great significance on which many authors are attracted to focus (see, e.g., [1, 2, 8, 12]).

As we know, the Hamiltonian matrix H is symplectic symmetric in $\mathcal{X} \oplus \mathcal{X}$, i.e. $(JH)^H = JH$ with $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, where $\mathcal{X}$ is a finite dimensional Hilbert space, I is the identity matrix, and the symbol ^{H} denotes the conjugate transpose operation. Then it is easy to see that the numerical range $W(H)$ of H is symmetric with respect to the imaginary axis, since J is an unitary matrix [4]. However, the same property may not be true for an unbounded infinite dimensional Hamiltonian operator. One of our main purposes is to give certain natural conditions such that the analogous symmetry holds true in the general unbounded case.

The location of the numerical range determines the semigroup generation property for a densely defined closed linear operator. It is well known that the system of linear evolution equations is well posed if and only if the corresponding operator matrix is the generator of a C_0 semigroup on underlying spaces [3]. Therefore, based on the symmetry of the numerical range, we consider the semigroup generation of infinite dimensional Hamiltonian operators. The classical results for operator matrices to generate C_0 semigroups are discussed in the diagonal domain by means of standard perturbation theorems, under the assumption that the diagonal

*Correspondence: huangjunjie@imu.edu.cn
2010 *AMS Mathematics Subject Classification:* 47A10, 47A12, 47D03
This work is supported by the National Natural Science Foundation of China (Nos. 11461049 and 11371185), the Natural Science Foundation of Inner Mongolia (No. 2013ZD01), and the scientific research projects of higher education institutions of Inner Mongolia (No. N1ZY16558).

elements generate semigroups [9, 10]. Note that we are concerned with the Hamiltonian operator with an off-diagonal domain and discuss the core of its off-diagonal elements.

2. Preliminaries

Let T be an (linear) operator between Hilbert spaces; then the domain, nullspace, and range of T are denoted by $\mathcal{D}(T)$, $\mathcal{N}(T)$, and $\mathcal{R}(T)$, respectively. Throughout this paper, $\mathcal{X}$ always denotes an infinite dimensional Hilbert space.

Definition 2.1 Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ be a densely defined closed operator in $\mathcal{X} \oplus \mathcal{X}$. Then H is called an *infinite dimensional Hamiltonian operator*, if A is closed and B, C are self-adjoint, where the symbol * denotes the adjoint operation. In addition, H is said to be *symplectic self-adjoint*, if $(JH)^* = JH$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with I being the identity operator on $\mathcal{X}$.

In what follows, we give a simple example to illustrate that the numerical range of the infinite dimensional Hamiltonian operator is not symmetric with respect to the imaginary axis in general.

Example 2.1 Let $\mathcal{X} = L^2(0,\infty)$. Consider the infinite dimensional Hamiltonian operator $H = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ with $Af = f'$, $\mathcal{D}(A) = \mathcal{H}_0^1(0,\infty)$, and $A^*g = -g'$, $\mathcal{D}(A^*) = \mathcal{H}^1(0,\infty)$. Although H is symplectic self-adjoint, its numerical range $W(H)$ is not symmetric with respect to the imaginary axis.

We claim that

$$W(H) = \overline{W(H)} = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \le 0\},$$

where $\mathrm{Re}\lambda$ is the real part of the complex number λ. Indeed, for $v = (f \quad g)^t \in \mathcal{D}(H)$, we have

$$\begin{aligned}
(Hv, v) &= (Af, f) - (A^*g, g) \\
&= \int_0^\infty f'(x)\overline{f(x)}\mathrm{d}x + \int_0^\infty g'(x)\overline{g(x)}\mathrm{d}x \\
&= \int_0^\infty ((f(x)\overline{f(x)})' - f(x)\overline{f'(x)})\mathrm{d}x + \int_0^\infty ((g(x)\overline{g(x)})' - g(x)\overline{g'(x)})\mathrm{d}x \\
&= -|g(0)|^2 - \int_0^\infty f(x)\overline{f'(x)}\mathrm{d}x - \int_0^\infty g(x)\overline{g'(x)}\mathrm{d}x \\
&= -|g(0)|^2 - (f, Af) + (g, A^*g) \\
&= -|g(0)|^2 - (v, Hv).
\end{aligned}$$

Hence, $\mathrm{Re}(Hv, v) = -\frac{1}{2}|g(0)|^2$, and the claim follows.

Definition 2.2 [6] Let $\mathcal{E}, \mathcal{F}$ be Banach spaces, and let $T : \mathcal{D}(T) \subset \mathcal{E} \to \mathcal{F}$ be a closed operator. If $S : \mathcal{D}(S) \subset \mathcal{E} \to \mathcal{F}$ is closable and $\overline{S} = T$, then $\mathcal{D}(S)$ is called the *core* of T.

Lemma 2.1 [2] Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ be a symplectic self-adjoint infinite dimensional Hamiltonian operator. Then $\sigma(H)$, $\sigma_p(H) \cup \sigma_r(H)$ and $\sigma_c(H)$ are symmetric with respect to the imaginary axis, respectively. Here $\sigma(H)$, $\sigma_p(H)$, $\sigma_r(H)$, and $\sigma_c(H)$ are the spectrum, point spectrum, residual spectrum, and continuous spectrum of H, respectively.

Lemma 2.2 [12] If T is a densely defined closed operator, then $\sigma_{app}(T) \subset \overline{W(T)}$, where $\sigma_{app}(T)$ is the approximate point spectrum of T.

Lemma 2.3 Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \mathcal{D}(H) := \mathcal{D}_1 \oplus \mathcal{D}_2 \to \mathcal{X} \oplus \mathcal{X}$ be an infinite dimensional Hamiltonian operator. Then

(i) $W^2(H) \subset W(H)$,

(ii) $W(\tilde{A}) \cup W(-\tilde{A}^*) \subset W^2(H)$,

where $W(H)$ is the numerical range of H, i.e. $W(H) = \{(Hv, v) : v \in \mathcal{D}(H), \|v\| = 1\}$, $W^2(H)$ is the quadratic numerical range of H [11], and $\tilde{A} := A|_{\mathcal{D}_1}$ and $\tilde{A^*} := A^*|_{\mathcal{D}_2}$, respectively, denote the corresponding restrictions to $\mathcal{D}_1$ and $\mathcal{D}_2$.

Proof According to [11, Theorems 2.5.3, 2.5.4], the results hold immediately since $\mathcal{X}$ is an infinite dimensional Hilbert space. □

Lemma 2.4 Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \mathcal{D}(H) \subset \mathcal{X} \oplus \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$ be an infinite dimensional Hamiltonian operator. Then H is symplectic self-adjoint, if one of the following statements holds:

(i) A is C-bounded with relative bound 0, and $\mathcal{D}(B) \subset \mathcal{D}(A^*)$,

(ii) A^* is B-bounded with relative bound 0, and $\mathcal{D}(C) \subset \mathcal{D}(A)$.

Proof The result follows from [5, Corollary 3.2] immediately. □

3. Main results

To begin with, we consider the symmetry of the numerical range of infinite dimensional Hamiltonian operators with off-diagonal domain.

Theorem 3.1 Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \mathcal{D}(C) \oplus \mathcal{D}(B) \to \mathcal{X} \oplus \mathcal{X}$ be an infinite dimensional Hamiltonian operator. Then $\overline{W(H)}$ is symmetric with respect to the imaginary axis, if $\mathcal{D}(B) \cap \mathcal{D}(C)$ is a core of B and C.

Proof Write $\mathcal{D}_0 = \mathcal{D}(B) \cap \mathcal{D}(C)$. We first claim

$$\overline{W(H)} = \overline{W(H|_{\mathcal{D}})}, \tag{3.1}$$

where $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$. It suffices to show $W(H) \subset \overline{W(H|_{\mathcal{D}})}$. To this end, let $\lambda \in W(H)$. Then there exists $v = (f \quad g)^t \in \mathcal{D}(H) = \mathcal{D}(C) \oplus \mathcal{D}(B)$ with $\|v\| = 1$ such that $\lambda = (Hv, v)$. Since $\mathcal{D}_0$ is a core of B and C, there are two sequences $(\hat{f}_n)_{n=1}^{\infty}$ and $(\hat{g}_n)_{n=1}^{\infty}$ in $\mathcal{D}_0$ such that

$$\hat{f}_n \to f,\, n \to \infty, \quad C\hat{f}_n \to Cf,\, n \to \infty,$$
$$\hat{g}_n \to g,\, n \to \infty, \quad B\hat{g}_n \to Bg,\, n \to \infty.$$

We write $\hat{v}_n = (\hat{f}_n \quad \hat{g}_n)^t$; then $\hat{v}_n \to v, n \to \infty$. Obviously, $\|\hat{v}_n\| \to 1$, $n \to \infty$. We may assume $\hat{v}_n \neq 0$, and let $v_n = (f_n \quad g_n)^t$ with $f_n = \frac{\hat{f}_n}{\|\hat{v}_n\|}$, $g_n = \frac{\hat{g}_n}{\|\hat{v}_n\|}$, then $v_n \in \mathcal{D}$, $\|v_n\|^2 = \|f_n\|^2 + \|g_n\|^2 = 1$, and

$$f_n \to f,\, n \to \infty, \quad Cf_n \to Cf,\, n \to \infty,$$
$$g_n \to g,\, n \to \infty, \quad Bg_n \to Bg,\, n \to \infty.$$

The closedness of A, B, and C implies A is C-bounded and A^* is B-bounded. It follows that $(Af_n)_{n=1}^{\infty}$ and $(A^*g_n)_{n=1}^{\infty}$ are both Cauchy sequences, and hence

$$Af_n \to Af,\, n \to \infty, \quad A^*g_n \to A^*g,\, n \to \infty.$$

Hence

$$(Af_n, f_n) \to (Af, f),\, n \to \infty, \quad (Bg_n, f_n) \to (Bg, f),\, n \to \infty,$$
$$(Cf_n, g_n) \to (Cf, g),\, n \to \infty, \quad (-A^*g_n, g_n) \to (-A^*g, g),\, n \to \infty.$$

It follows from

$$\begin{aligned}(Hv_n, v_n) &= (\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \begin{pmatrix} f_n \\ g_n \end{pmatrix}) \\ &= (Af_n, f_n) + (Bg_n, f_n) + (Cf_n, g_n) + (-A^*g_n, g_n)\end{aligned}$$

that

$$(Hv_n, v_n) \to (Hv, v),\, n \to \infty.$$

Thus $\lambda \in \overline{W(H|_{\mathcal{D}})}$. This proves (3.1).

In view of (3.1), it remains to prove

$$\lambda \in W(H|_{\mathcal{D}}) \iff -\overline{\lambda} \in W(H|_{\mathcal{D}}). \tag{3.2}$$

We decompose H as $H = S + T$, where

$$S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad T = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}.$$

Then it is easy to verify

$$\begin{pmatrix} A^* & C \\ B & -A \end{pmatrix} = S^* + T^* \subset (S+T)^* = H^*,$$

which implies $JHJ \subset H^*$, and hence $J\mathcal{D}(H) \subset \mathcal{D}(H^*)$. It follows from $\mathcal{D} \subset J\mathcal{D}(H) \subset \mathcal{D}(H^*)$ that $H^*|_{\mathcal{D}} = \left(\begin{smallmatrix} A^* & C \\ B & -A \end{smallmatrix}\right)|_{\mathcal{D}}$. For each $v = (f \quad g)^t \in \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$ with $\|v\| = 1$, we note $\hat{v} = (-g \quad f)^t$, then $\|\hat{v}\| = 1$ and

$$\begin{aligned}(H^*|_{\mathcal{D}}v, v) &= (\begin{pmatrix} A^* & C \\ B & -A \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}) \\ &= (A^*f, f) + (Cg, f) + (Bf, g) + (-Ag, g) \\ &= (\begin{pmatrix} -A & -B \\ -C & A^* \end{pmatrix} \begin{pmatrix} -g \\ f \end{pmatrix}, \begin{pmatrix} -g \\ f \end{pmatrix}) = -(H\hat{v}, \hat{v}),\end{aligned}$$

which implies

$$\lambda \in W(H|_{\mathcal{D}}) \iff -\lambda \in W(H^*|_{\mathcal{D}}).$$

Then it follows from $W(H|_{\mathcal{D}})^* = W(H^*|_{\mathcal{D}})$ that (3.2) holds. □

Similarly, we have the following result in the case of the diagonal domain.

Corollary 3.1 Let $H = \left(\begin{smallmatrix} A & B \\ C & -A^* \end{smallmatrix}\right) : \mathcal{D}(A) \oplus \mathcal{D}(A^*) \to \mathcal{X} \oplus \mathcal{X}$ be an infinite dimensional Hamiltonian operator. Then $\overline{W(H)}$ is symmetric with respect to the imaginary axis, if $\mathcal{D}(A) \cap \mathcal{D}(A^*)$ is a core of A and A^*.

Proof The proof is completely similar to that of Theorem 3.1. □

Example 3.1 Let $\mathcal{X} = L^2(0,1)$. Consider the infinite dimensional Hamiltonian operator

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} = \begin{pmatrix} \mathrm{i}I & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} & \mathrm{i}I \end{pmatrix} : \mathcal{D}(C) \oplus \mathcal{D}(B) \to \mathcal{X} \oplus \mathcal{X},$$

and

$$\mathcal{D}(B) = \mathcal{D}(C) = \{f \in \mathcal{X} : f', f'' \in \mathcal{X}, f' \text{ is absolutely continuous, and } f(0) = f(1) = 0\}.$$

It is not hard to verify that H satisfies the conditions of Theorem 3.1. For each $v = (f \quad g)^t \in \mathcal{D}(H)$ with $\|v\| = 1$, we have

$$\begin{aligned}(Hv, v) &= (\begin{pmatrix} \mathrm{i}I & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} & \mathrm{i}I \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}) \\ &= (\mathrm{i}f, f) + (g'', f) + (f'', g) + (\mathrm{i}g, g) \\ &= \mathrm{i} + 2\mathrm{Re}(f'', g).\end{aligned}$$

Obviously, $\overline{W(H)}$ is symmetric with respect to the imaginary axis.

Next, we discuss the semigroup generation property of the infinite dimensional Hamiltonian operator $H = \left(\begin{smallmatrix} A & B \\ C & -A^* \end{smallmatrix}\right) : \mathcal{D}(C) \oplus \mathcal{D}(B) \to \mathcal{X} \oplus \mathcal{X}$. The following are some useful constants defined for H:

$$\begin{aligned}\alpha_0 &= \inf\{\mathrm{Re}\lambda : \lambda = (-Af, f), \|f\| = 1, f \in \mathcal{D}(C)\}, \\ \beta_0 &= \inf\{\mathrm{Re}\lambda : \lambda = (A^*g, g), \|g\| = 1, g \in \mathcal{D}(B)\}, \\ \delta_0 &= \min\{\alpha_0, \beta_0\}, \\ \gamma_0 &= \sup\{\mathrm{Re}\lambda : \lambda = \frac{(Cf, g) + (Bg, f)}{\|f\|^2 + \|g\|^2}, (f \quad g)^t \in \mathcal{D}(H)\}.\end{aligned}$$

Theorem 3.2 Let $H = \left(\begin{smallmatrix} A & B \\ C & -A^* \end{smallmatrix}\right) : \mathcal{D}(C) \oplus \mathcal{D}(B) \to \mathcal{X} \oplus \mathcal{X}$ be an infinite dimensional Hamiltonian operator. Assume that one of the following statements is satisfied:

(a) The C-bound of A is 0,

(b) The B-bound of A^* is 0.

If $\gamma_0 \leq \delta_0$ and $\mathcal{D}(C) \cap \mathcal{D}(B)$ is a core of B and C, then H generates a C_0 semigroup $(T(t))_{t \geq 0}$ with $\|T(t)\| \leq e^{\beta t}$ for some $\beta \geq 0$ if and only if

$$W(A|_{\mathcal{D}_0}) \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}, \tag{3.3}$$

where $\mathcal{D}_0 = \mathcal{D}(B) \cap \mathcal{D}(C)$. In addition, as (3.3) holds, we have

$$\sigma(H) \subset \overline{W(H)} \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}.$$

Proof The relation (3.3) implies

$$W(A|_{\mathcal{D}_0}) \cup W(-A^*|_{\mathcal{D}_0}) \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}$$

since $W(A|_{\mathcal{D}_0})^* = W(A^*|_{\mathcal{D}_0})$. Hence α_0, β_0 are well defined. For each $v = (f \quad g)^t \in \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$ with $f \neq 0$ and $g \neq 0$, let $\tau = \frac{\|f\|^2}{\|f\|^2 + \|g\|^2}$. It follows from $\gamma_0 \leq \delta_0$ that

$$\begin{aligned}
&\frac{\mathrm{Re}(Cf, g) + \mathrm{Re}(Bg, f)}{\|f\|^2 + \|g\|^2} \\
&\leq \gamma_0 \leq \delta_0 \leq \tau\alpha_0 + (1-\tau)\beta_0 \\
&\leq \tau(\alpha_0 + \beta) + (1-\tau)(\beta_0 + \beta) \\
&\leq \frac{\|f\|^2}{\|f\|^2 + \|g\|^2} \frac{\mathrm{Re}(-Af, f) + \beta(f, f)}{\|f\|^2} + \frac{\|g\|^2}{\|f\|^2 + \|g\|^2} \frac{\mathrm{Re}(A^*g, g) + \beta(g, g)}{\|g\|^2} \\
&= \frac{\mathrm{Re}(-Af, f) + \beta(f, f)}{\|f\|^2 + \|g\|^2} + \frac{\mathrm{Re}(A^*g, g) + \beta(g, g)}{\|f\|^2 + \|g\|^2}.
\end{aligned}$$

Hence

$$\mathrm{Re}(Cf, g) + \mathrm{Re}(Bg, f) + \mathrm{Re}(Af, f) + \mathrm{Re}(-A^*g, g) \leq \beta(f, f) + \beta(g, g),$$

and

$$\begin{aligned}
\mathrm{Re}(Hv, v) &= \mathrm{Re}(\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}) \\
&= \mathrm{Re}(Af, f) + \mathrm{Re}(Bg, f) + \mathrm{Re}(Cf, g) + \mathrm{Re}(-A^*g, g) \\
&\leq \beta(v, v).
\end{aligned}$$

If $f = 0$ or $g = 0$, it is easy to prove $\mathrm{Re}(Hv, v) \leq \beta(v, v)$. Hence,

$$W(H|_{\mathcal{D}}) \subset \{z \in \mathbb{C} : \mathrm{Re}z \leq \beta\}.$$

Since $\mathcal{D}_0$ is a core of B and C, by Theorem 3.1, $\overline{W(H)}$ is symmetric with respect to the imaginary axis. Combining with (3.1), we have

$$W(H) \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}. \tag{3.4}$$

Since $\mathcal{D}(H) = \mathcal{D}(C) \oplus \mathcal{D}(B)$, the closedness of A, B, and C implies that A is C-bounded and A^* is B-bounded. Hence, either of the conditions (a) and (b) demonstrates that H is symplectic self-adjoint by Lemma 2.4. Thus $\sigma_p(H) \cup \sigma_r(H)$ is symmetric with respect to the imaginary axis by Lemma 2.1. By Lemma 2.2, we know $\sigma_p(H) \subset \sigma_{app}(H) \subset \overline{W(H)}$, and hence $\sigma_p(H) \cup \sigma_r(H) \subset \overline{W(H)}$. In view of $\sigma_c(H) \subset \sigma_{app}(H)$, we conclude

$$\sigma(H) \subset \overline{W(H)} \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}.$$

Obviously,

$$\{z \in \mathbb{C} : \mathrm{Re}z < -\beta\} \cup \{z \in \mathbb{C} : \mathrm{Re}z > \beta\} \subset \rho(H).$$

Therefore, H generates a C_0 semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq e^{\beta t}$ for some $\beta \geq 0$.

Conversely, if H generates a C_0 semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq e^{\beta t}$ for some $\beta \geq 0$, then $W(H) \subset \{z \in \mathbb{C} : \mathrm{Re}z \leq \beta\}$ and $\mathcal{R}(H-\lambda) = \mathcal{X} \oplus \mathcal{X}$ for $\lambda > \beta$. Since $\mathcal{D}_0$ is a core of B and C, $\overline{W(H)}$ is symmetric with respect to the imaginary axis by Theorem 3.1. It follows that (3.4) holds, and hence (3.3) is true, according to (3.1) and Lemma 2.3. □

We end this section with an illustrating example.

Example 3.2 Let $\mathcal{X} = L^2[a,b]$. Consider the infinite dimensional Hamiltonian operator $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ with $Af = p_1 f' + p_2 f$, $Bf = f''$, and $Cf = -f''$, where p_1, p_1', and p_2 are real valued continuous on $[a,b]$, and

$$\mathcal{D}(A) = \{f \in \mathcal{X} : f' \text{ is absolutely continuous, } f(a) = f(b)\},$$

$$\mathcal{D}(B) = \mathcal{D}(C) = \{f \in \mathcal{X} : f' \text{ is absolutely continuous, } f(a) = f(b) = 0\}.$$

We claim that H generates a C_0 semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq e^{\beta t}$ for some $\beta \geq 0$, and

$$\sigma(H) \subset \overline{W(H)} \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}.$$

Indeed, it is easy to verify that $\gamma_0 \leq \delta_0$, $\mathcal{D}_0 = \mathcal{D}(B) \cap \mathcal{D}(C)$ is a core of B and C, and A is C-bounded with relative bound 0. On the other hand, for $v \in \mathcal{D}_0$, we have

$$\begin{aligned}(Av, v) &= \int_a^b (p_1(x)v'(x) + p_2(x)v(x))\overline{v(x)}\mathrm{d}x \\ &= -\int_a^b p_1'(x)|v(x)|^2\mathrm{d}x - \int_a^b p_1(x)v(x)\overline{v'(x)}\mathrm{d}x + \int_a^b p_2(x)|v(x)|^2\mathrm{d}x,\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Re}(Av, v) &= -\frac{1}{2}\int_a^b p_1'(x)|v(x)|^2\mathrm{d}x + \int_a^b p_2(x)|v(x)|^2\mathrm{d}x \\ &= \int_a^b (-\frac{1}{2}p_1'(x) + p_2(x))|v(x)|^2\mathrm{d}x.\end{aligned}$$

Since $|\frac{1}{2}p_1'(x) - p_2(x)| \leq \beta$ for some constant $\beta \geq 0$,

$$-\beta \int_a^b |v(x)|^2\mathrm{d}x \leq \mathrm{Re}(Av, v) \leq \beta \int_a^b |v(x)|^2\mathrm{d}x.$$

Hence,

$$W(A|_{\mathcal{D}_0}) \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}.$$

Therefore, according to Theorem 3.2, H generates a C_0 semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq e^{\beta t}$ and

$$\sigma(H) \subset \overline{W(H)} \subset \{z \in \mathbb{C} : -\beta \leq \mathrm{Re}z \leq \beta\}.$$

References

[1] Azizov TY, Dijksma AA. On the boundness of Hamiltonian operators. Proc Amer Math Soc 2002; 131: 563-576.

[2] Alatancang, Huang JJ, Fan XY. Structure of the spectrum of infinite dimensional Hamiltonian operators. Science in China Series A: Mathematics 2008; 51: 915-924.

[3] Engel KJ, Nagel R. One Parameter Semigroups for Linear Evolution Equations. New York, NY, USA: Springer-Verlag, 2000.

[4] Gustafson K, Rao DKM. Numerical Range: the Field of Values of Linear Operators and Matrices. New York, NY, USA: Springer-Verlag, 1997.

[5] Jin GH, Chen A. Some basic properties of block operator matrices. arXiv: 1403.7732vl [math.FA] 30 Mar 2014.

[6] Kato T. Perturbation Theory for Linear Operators. Berlin, Germany: Springer-Verlag, 1995.

[7] Kurina GA. Invertibility of an operator appearing in the control theory for linear systems. Math Notes 2001; 70: 206-212.

[8] Kurina GA. Invertibility of nonnegatively Hamiltonian operators in a Hilbert space. Differential Equations 2001; 37: 880-882.

[9] Mugnolo D. Asymptotics of semigroups generated by operator matrices. Arab J Math 2014; 3: 419-435.

[10] Nagel R. Towards a "matrix theory" for unbounded operator matrices. Math Z 1989; 201: 57-68.

[11] Tretter C. Spectral Theory of Block Operator Matrices and Applications. London, UK: Imperial College Press, 2008.

[12] Wu DY, Chen A. Spectral inclusion properties of the numerical range in a space with an indefinite metric. Linear Algebra and its Applications 2011; 435: 1131-1136.

[13] Zhong WX. A New Systematic Methodology for Theory of Elasticity. Dalian, LN, China: Dalian University of Technology Press, 1995 (in Chinese).