Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 69 – 82

doi:10.3906/mat-1608-76

Integral laminations on nonorientable surfaces

Saadet Öykü YURTTAŞ[1,*], **Mehmetcik PAMUK**[2]
[1]Department of Mathematics, Dicle University, Diyarbakır, Turkey
[2]Department of Mathematics, Middle East Technical University, Ankara, Turkey

Received: 21.08.2016 • **Accepted/Published Online:** 16.03.2017 • **Final Version:** 22.01.2018

Abstract: We describe triangle coordinates for integral laminations on a nonorientable surface $N_{k,n}$ of genus k with n punctures and one boundary component, and we give an explicit bijection from the set of integral laminations on $N_{k,n}$ to $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Key words: Nonorientable surfaces, triangle coordinates, Dynnikov coordinates

1. Introduction

Let $N_{k,n}$ be a nonorientable surface of genus k with n punctures and one boundary component. In this paper we shall describe the generalized Dynnikov coordinate system for the set of integral laminations $\mathcal{L}_{k,n}$ and give an explicit bijection between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. To be more specific, we shall first take a particular collection of $3n + 2k - 4$ arcs and k curves embedded in $N_{k,n}$ and describe each integral lamination by an element of $\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k$, its geometric intersection numbers with these arcs and curves. *Generalized Dynnikov coordinates* are certain linear combinations of these integers that provide a one-to-one correspondence between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

The motivation for this paper comes from the recent work of Papadopoulos and Penner [7] where they provided analogs for nonorientable surfaces of several results from the Thurston theory of surfaces, which were studied only for orientable surfaces before [4, 8]. Here we shall give the analogy of the Dynnikov coordinate system [1–3] on a finitely punctured disk that has several useful applications such as giving an efficient method for the solution of the word problem of the n-braid group [1], computing the geometric intersection number of integral laminations [9], and counting the number of components they contain [11].

Throughout the text we shall work on a standard model of $N_{k,n}$ as illustrated in Figure 1, where a disk with a cross drawn within it represents a crosscap; that is, the interior of the disk is removed and the antipodal points on the resulting boundary component are identified (i.e. the boundary component bounds a Möbius band).

The structure of the paper is as follows. In Section 1.1 we give the necessary terminology and background. In Section 2 we describe and study the triangle coordinates for integral laminations on $N_{k,n}$, and we construct

*Correspondence: yurttas@dicle.edu.tr
2010 *AMS Mathematics Subject Classification:* 57N05, 57N16, 57M50

the generalized Dynnikov coordinate system giving the bijection $\rho\colon \mathcal{L}_{k,n} \to (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. An explicit formula for the inverse of this bijection is given in Theorem 2.14.

1.1. Basic terminology and background

A simple closed curve in $N_{k,n}$ is inessential if it bounds an unpunctured disk, a once punctured disk, or an unpunctured annulus. It is called essential otherwise. A simple closed curve is called *2-sided* (respectively *1-sided*) if a regular neighborhood of the curve is an annulus (respectively Möbius band). We say that a 2-sided curve is *nonprimitive* if it bounds a Möbius band [7], and a 1-sided curve is *nonprimitive* if it is a core curve of a Möbius band. They are called *primitive* otherwise.

An integral lamination $\mathcal{L}$ on $N_{k,n}$ is a disjoint union of finitely many essential simple closed curves in $N_{k,n}$ modulo isotopy. Let $\mathcal{A}_{k,n}$ be the set of arcs α_i $(1 \le i \le 2n-2)$, β_i $(1 \le i \le n+k-1)$, γ_i $(1 \le i \le k-1)$, which have each endpoint either on the boundary or at a puncture, and the curves c_i $(1 \le i \le k)$, which are the core curves of Möbius bands in $N_{k,n}$ as illustrated in Figure 1: the arcs α_{2i-3} and α_{2i-2} for $2 \le i \le n$ join the ith puncture to $\partial N_{k,n}$, the arc β_i has both end points on $\partial N_{k,n}$ and passes between the ith and $(i+1)$st punctures for $1 \le i \le n-1$, the nth puncture and the first crosscap for $i = n$, and the $(i-n)$th and $(i+1-n)$th crosscaps for $n+1 \le i \le n+k-1$. The arc γ_i ($1 \le i \le k-1$) has both endpoints on $\partial N_{k,n}$ and surrounds the ith crosscap.

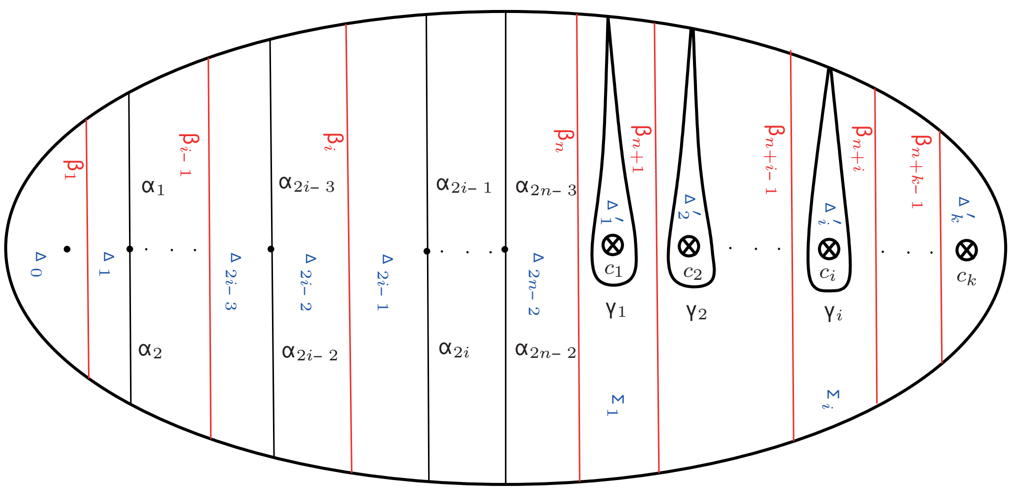

Figure 1. The arcs α_i, β_i, γ_i; the 1-sided curves $c_1, c_2, \ldots, c_k$; and the regions Δ_i and Σ_i.

The surface is divided by these arcs into $2n+2k-2$ regions; $2n+k-3$ of these are triangular since each Δ_i $(1 \le i \le 2n-2)$ and Σ_i $(1 \le i \le k-1)$ is bounded by three arcs when the boundary of the surface is identified to a point. The two triangles Δ_{2i-3} and Δ_{2i-2} on the left-hand and right-hand side of the ith puncture are defined by the arcs $\alpha_{2i-3}, \alpha_{2i-2}, \beta_{i-1}$ and $\alpha_{2i-3}, \alpha_{2i-2}, \beta_i$, respectively. The triangle Σ_i is defined by the arcs $\gamma_i, \beta_{n+i-1}, \beta_{n+i}$. Each Δ'_i $(1 \le i \le k-1)$ is bounded by γ_i, and the two end regions Δ_0 and Δ'_k are bounded by β_1 and β_{n+k-1}, respectively. Given $\mathcal{L} \in \mathcal{L}_{k,n}$, let L be a taut representative of $\mathcal{L}$ with respect to the elements of $\mathcal{A}_{k,n}$. That is, L intersects each of the arcs and curves in $\mathcal{A}_{k,n}$ minimally.

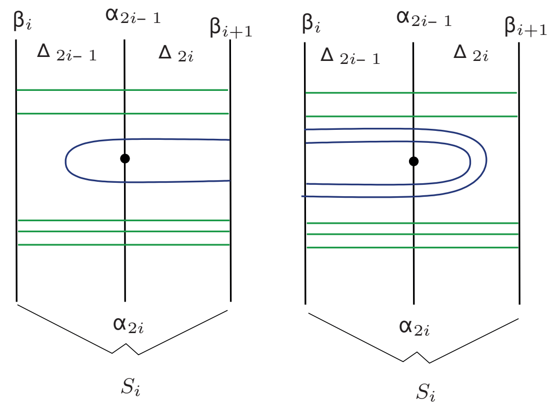

Figure 2. There is 1 left loop component in the first case and 2 right loop components in the second case. There are 2 above and 3 below components in each case.

Definition 1.1 *Set* $S_i = \Delta_{2i-1} \cup \Delta_{2i}$ *for each* i *with* $1 \leq i \leq n-1$*. A* path component *of* L *in* S_i *is a component of* $L \cap S_i$*. There are four types of path components in* S_i *as depicted in Figure 2:*

- *An* above component *has end points on* β_i *and* β_{i+1}*, passing across* α_{2i-1}*;*
- *A* below component *has end points on* β_i *and* β_{i+1}*, passing across* α_{2i}*;*
- *A* left loop component *has both end points on* β_{i+1}*;*
- *A* right loop component *has both end points on* β_i*.*

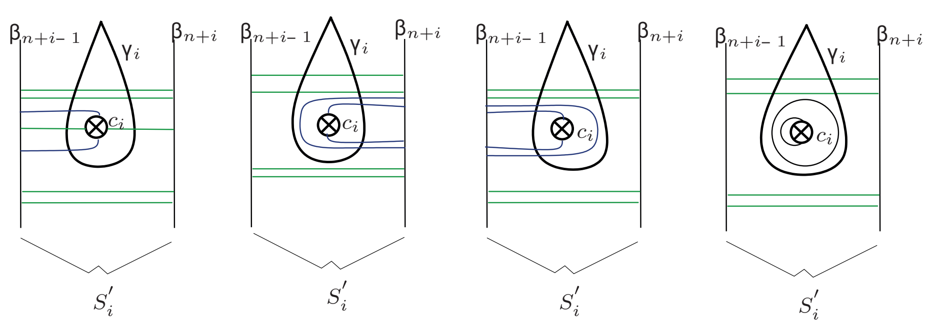

Figure 3. There is 1 right core loop and 1 straight core component in the first case; 1 left loop and 1 left core loop component in the second case; 1 right noncore loop and 1 right core loop component in the third case; and 1 1-sided and 1 2-sided nonprimitive curves in the fourth case. There are 2 above and 2 below components in each case.

Definition 1.2 *Set* $S_i' = \Delta_i' \cup \Sigma_i$ *for each* $1 \leq i \leq k-1$*. A* path component *of* L *in* S_i' *is a component of* $L \cap S_i'$*. There are* 7 *types of path components in* S_i' *as depicted in Figure 3.*

- *An* above component *has end points on* β_{n+i-1} *and* β_{n+i}*, and passes across* γ_i *without intersecting* c_i*;*
- *A* below component *has end points on* β_{n+i-1} *and* β_{n+i}*, and does not pass across* γ_i*;*
- *A* left loop component *has both end points on* β_{n+i}*;*

- *A* right loop component *has both end points on* β_{n+i-1}*:*

  If a loop component intersects c_i*, it is called* core loop component*, otherwise it is called noncore loop component;*

- *A* straight core component *has end points on* β_{n+i-1} *and* β_{n+i}*, and intersects* c_i*;*

- *A non-primitive* 1*-sided curve:*

  If L *contains a nonprimitive* 1*-sided curve* c_i *we depict it with a ring with end points on the* i*th crosscap as shown in the fourth case in Figure 3;*

- *A nonprimitive* 2*-sided curve.*

2. Triangle coordinates

Let L be a taut representative of $\mathcal{L}$. Write $\alpha_i, \beta_i, \gamma_i$, and c_i for the geometric intersection number of L with the arc $\alpha_i, \beta_i, \gamma_i$ and the core curve c_i, respectively. It will always be clear from the context whether we mean the arc or the geometric intersection number assigned on the arc.

Definition 2.1 *The triangle coordinate function* $\tau\colon \mathcal{L}_{k,n} \to (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ *is defined by*

$$\tau(\mathcal{L}) = (\alpha_1, \ldots, \alpha_{2n-2}; \beta_1, \ldots, \beta_{n+k-1}; \gamma_1, \ldots, \gamma_{k-1}; c_1, \ldots, c_k),$$

where $c_i = -1$ *if* L *contains the* i*th core curve,* $c_i = -2m$ *if it contains* $m \in \mathbb{Z}^+$ *disjoint copies 2-sided nonprimitive curves around the* i*th crosscap, and* $c_i = -2m-1$ *if it contains* m *disjoint copies of 2-sided nonprimitive curves around the* i*th crosscap plus the* i*th core curve.*

Remark 2.2 *Let* $b_i = \frac{\beta_i - \beta_{i+1}}{2}$ *for* $1 \leq i \leq n+k-2$*. Then in each* S_i $(1 \leq i \leq n-1)$ *and* S'_i $(n \leq i \leq n+k-2)$ *there are* $|b_i|$ *loop components. Furthermore, if* $b_i < 0$*, these loop components are left, and if* $b_i > 0$ *they are right.*

The proof of the next lemma is obvious from Figure 2.

Lemma 2.3 *Let* $1 \leq i \leq n-1$*. The number of above and below components in* S_i *are given by* $a_{S_i} = \alpha_{2i-1} - |b_i|$ *and* $b_{S_i} = \alpha_{2i} - |b_i|$*, respectively.*

Let λ_i and λ_{c_i} denote the number of noncore and core loop components, ψ_i the number of straight core components, and $a_{S'_i}$ and $b_{S'_i}$ the number of above and below components in S'_i.

Lemma 2.4 *Let* L *be a taut representative of* $\mathcal{L} \in \mathcal{L}_{k,n}$*, and set* $c_i^+ = \max(c_i, 0)$*. Then, for each* $1 \leq i \leq k-1$*, we have*

$$\lambda_i = \max(|b_{n+i-1}| - c_i^+, 0), \qquad \lambda_{c_i} = \min(|b_{n+i-1}|, c_i^+),$$

$$\psi_i = \max(c_i^+ - |b_{n+i-1}|, 0).$$

Proof Assume that L does not contain any nonprimitive curve in S_i'. Since c_i gives the sum of straight core and core loop components and $|b_{n+i-1}|$ gives the sum of noncore loop and core loop components in S_i' (see Figure 3), we have

$$c_i = \psi_i + \lambda_{c_i} \quad \text{and} \quad |b_{n+i-1}| = \lambda_i + \lambda_{c_i}. \tag{1}$$

If $c_i > |b_{n+i-1}|$, then clearly there exists a straight core component in S_i' and hence no noncore loop component in S_i'; that is, $\lambda_i = 0$. Therefore, in this case, $\lambda_{c_i} = |b_{n+i-1}|$ and hence $\psi_i = c_i - |b_{n+i-1}|$ by Equation 1.

If $c_i < |b_{n+i-1}|$, there exists a noncore loop component in S_i' and hence no straight core components in S_i'; that is, $\psi_i = 0$. Therefore, $c_i = \lambda_{c_i}$ and hence $\lambda_i = |b_{n+i-1}| - c_i$ by Equation 1. We get:

$$\lambda_i = \max(|b_{n+i-1}| - c_i, 0)$$
$$\psi_i = \max(c_i - |b_{n+i-1}|, 0).$$

Also, if $|b_{n+i-1}| < c_i, \lambda_i = 0$ and hence $\lambda_{c_i} = |b_{n+i-1}|$; if $|b_{n+i-1}| > c_i, \psi_i = 0$ and hence $\lambda_{c_i} = c_i$ by Equation 1. Therefore, we get $\lambda_{c_i} = \min(|b_{n+i-1}|, c_i)$.

Finally, if L contains a nonprimitive curve in S_i', there can be no straight core and core loop component in S_i'; that is, $\psi_i = \lambda_{c_i} = 0$, and hence $\lambda_i = |b_{n+i-1}|$. Since $c_i < 0$ by definition, setting $c_i^+ = \max(c_i, 0)$, we can write:

$$\lambda_i = \max(|b_{n+i-1}| - c_i^+, 0), \qquad \lambda_{c_i} = \min(|b_{n+i-1}|, c_i^+),$$
$$\psi_i = \max(c_i^+ - |b_{n+i-1}|, 0).$$

□

Lemma 2.5 *Let L be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$. For each $1 \le i \le k-1$ we have:*

$$a_{S_i'} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i$$
$$b_{S_i'} = \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}.$$

Proof

To compute the number of above and below components in S_i' we observe that each path component other than a below component in S_i' intersects γ_i twice; that is, $\gamma_i = 2(a_{S_i'} + |b_{n+i-1}| + \psi_i)$. Therefore, we get

$$a_{S_i'} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i.$$

To compute the number of below components, we note that the sum of all path components in S_i' is given by $\beta = \max(\beta_{n+i-1}, \beta_{n+i})$. Then $b_{S_i'}$ is β minus the number of above, straight core components and twice the number loop components in S_i' (each loop component intersects β twice). We get

$$b_{S_i'} = \max(\beta_{n+i-1}, \beta_{n+i}) - a_{S_i'} - 2|b_{n+i-1}| - \psi_i$$
$$= \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}.$$

□

Another way of expressing $a_{S'_i}$ and $b_{S'_i}$ is given in item P4. in Properties 2.12.

Remark 2.6 *Observe that the loop components in* Δ_0 *are always left and the number of them is given by* $\frac{\beta_1}{2}$. *Similarly, the loop components in* Δ'_k *are always right and the numbers of core and noncore loop components in* Δ'_k *are given by* c_k *and* $\lambda_k = \frac{\beta_{n+k-1}}{2} - c_k$.

Lemma 2.7 and Lemma 2.8 are obvious from Figures 2 and 3.

Lemma 2.7 *There are equalities for each* S_i:

- *When there are left loop components* ($b_i < 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_{i+1}$$
$$\alpha_{2i} + \alpha_{2i-1} - \beta_i = 2|b_i|;$$

- *When there are right loop components* ($b_i > 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_i$$
$$\alpha_{2i} + \alpha_{2i-1} - \beta_{i+1} = 2|b_i|;$$

- *When there are no loop components* ($b_i = 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_i = \beta_{i+1}.$$

Lemma 2.8 *There are equalities for each* S'_i:

- *When there are left loop components* ($b_{n+i-1} < 0$),

$$a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i}$$
$$a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i-1};$$

- *When there are right loop components* ($b_{n+i-1} > 0$)

$$a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| = \beta_{n+i-1}.$$
$$a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i};$$

- *When there are no loop components* $b_{n+i-1} = 0$

$$a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i} = \beta_{n+i-1}.$$

Example 2.9 *Let* $\tau(\mathcal{L}) = (4,2,2,6;2,6,8,4;8;1,1)$ *be the triangle coordinates of an integral lamination* $\mathcal{L} \in \mathcal{L}_{2,3}$. *We shall show how we draw* $\mathcal{L}$ *from its given triangle coordinates. First, we compute the loop components in the two end regions* Δ_0 *and* Δ'_2 *using Remark 2.6. Since* $\beta_1 = 2$ *there is one loop component in* Δ_0. *Similarly, since* $\beta_4 = 4$ *and* $c_2 = 1$, *we get* $\lambda_2 = \frac{\beta_4}{2} - c_2 = 1$.

Next we compute loop components in S_1, S_2, *and* S_1'. *Since* $b_i = \frac{\beta_i - \beta_{i+1}}{2}$ *for each* $1 \le i \le 3$, *we have* $b_1 = -2, b_2 = -1$. *Hence, there are two left loop components in* S_1 *and one left component in* S_2. *Similarly, since* $b_3 = 2$, *there are* 2 *right loop components in* S_1', *and by Lemma 2.4,* $\lambda_1 = \max(|b_3| - c_1, 0) = 1$ *(hence* $\psi_1 = 0$*) and* $\lambda_{c_1} = \min(|b_3|, c_1) = 1$. *Using Lemma 2.3 and Lemma 2.5 we compute the number of above and below components. We get* $a_{S_1} = \alpha_1 - |b_1| = 2$, $b_{S_1} = \alpha_2 - |b_1| = 0$, $a_{S_2} = \alpha_3 - |b_2| = 1$, $b_{S_2} = \alpha_4 - |b_2| = 5$, *and*

$$a_{S_1'} = \frac{\gamma_1}{2} - |b_3| - \psi_1 = 2,$$

$$b_{S_1'} = \max(\beta_3, \beta_4) - |b_3| - \frac{\gamma_1}{2} = 2.$$

Connecting the path components in each Δ_0, Δ_2', S_1, S_2, *and* S_1' *we draw the integral lamination as shown in Figure 4.*

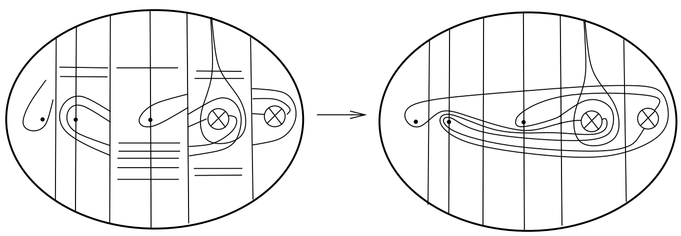

Figure 4. $\tau(L) = (4, 2, 2, 6; 2, 6, 8, 4; 8; 1, 1)$.

Lemma 2.10 *The triangle coordinate function* $\tau\colon \mathcal{L}_{k,n} \to (\mathbb{Z}_{\ge 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ *is injective.*

Proof We can determine the number of loop, above, and below components in each S_i by Remark 2.2 and Lemma 2.3 and core and noncore loop, straight core, above, and below components in each S_i' by Lemma 2.4 and Lemma 2.5 as illustrated in Example 2.9. The components in each S_i and S_i' are glued together in a unique way up to isotopy, and hence $\mathcal{L}$ is constructed uniquely. □

Remark 2.11 *The triangle coordinate function* $\tau\colon \mathcal{L}_{k,n} \to (\mathbb{Z}_{\ge 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ *is not surjective: an integral lamination must satisfy the triangle inequality in each* S_i *and* S_i', *and some additional conditions such as the equalities in Lemma 2.7 and Lemma 2.8.*

Next we give a list of properties that an integral lamination $\mathcal{L} \in \mathcal{L}_{k,n}$ satisfies in terms of its triangle coordinates as in [9], and then we construct a new coordinate system from the triangle coordinates that describes integral laminations in a unique way. In particular, we shall generalize the Dynnikov coordinate system [1–3, 5, 9–11] for $N_{k,n}$.

Properties 2.12 *Let* L *be a taut representative of* $\mathcal{L} \in \mathcal{L}_{k,n}$.

P1. *Every component of* L *intersects each* β_i *an even number of times. Recall from Remark 2.2 that the number of loop components is given by* $|b_i|$ *where* $b_i = \frac{\beta_i - \beta_{i+1}}{2}$.

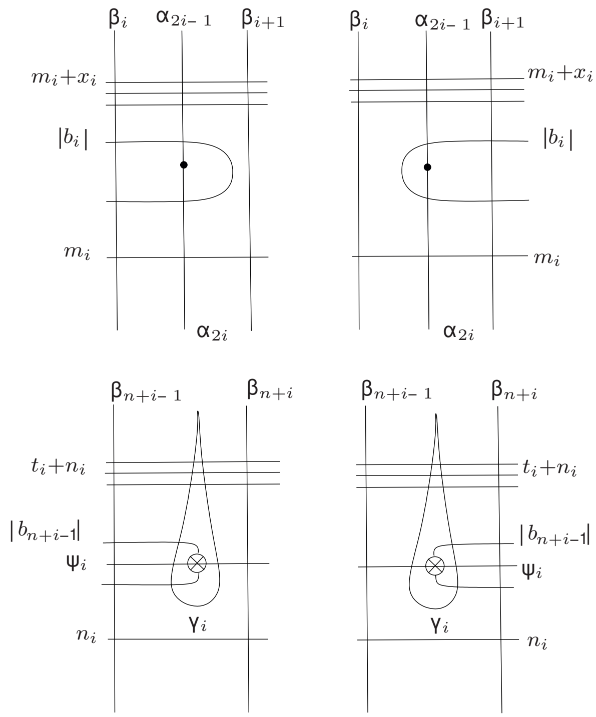

Figure 5. m_i and n_i denote the smaller of the above and below components in S_i and S'_i, respectively.

P2. *Set* $x_i = |\alpha_{2i} - \alpha_{2i-1}|$ *and* $t_i = |a_{S'_i} - b_{S'_i}|$. *Then* x_i *and* t_i *give the difference between the number of above and below components in* S_i *and* S'_i*, respectively. Set* $m_i = \min\{\alpha_{2i} - |b_i|, \alpha_{2i-1} - |b_i|\}$*;* $1 \leq i \leq n-1$ *and* $n_i = \min\{a_{S'_i}, b_{S'_i}\}$*;* $1 \leq i \leq k-1$*. See Figure 5. Note that* x_i *is even since* L *intersects* $\alpha_{2i} \cup \alpha_{2i-1}$ *an even number of times. Clearly, this may not hold for* t_i *since when* ψ_i *is odd the sum of above and below components (and hence their difference) is odd. See Lemma 2.8.*

P3. *Set* $2a_i = \alpha_{2i} - \alpha_{2i-1}$ $(|a_i| = x_i/2)$. *Then, by Lemma 2.7, we get:*

- *If* $b_i \geq 0$, *then* $\beta_i = \alpha_{2i} + \alpha_{2i-1}$ *and hence*

$$\alpha_{2i} = a_i + \frac{\beta_i}{2} \text{ and } \alpha_{2i-1} = -a_i + \frac{\beta_i}{2};$$

- *If* $b_i \leq 0$, *then* $\beta_{i+1} = \alpha_{2i} + \alpha_{2i-1}$ *and hence*

$$\alpha_{2i} = a_i + \frac{\beta_{i+1}}{2} \text{ and } \alpha_{2i-1} = -a_i + \frac{\beta_{i+1}}{2}.$$

That is,

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0, \end{cases}$$

where $\lceil i/2 \rceil$ *denotes the smallest integer that is not less than* $i/2$.

P4. *Since* $t_i = a_{S'_i} - b_{S'_i}$ *for* $1 \leq i \leq k-1$, *from Lemma 2.8 we get:*

- *If* $b_{n+i-1} \geq 0$ *then* $a_{S'_i} + b_{S'_i} + \psi_i + 2b_{n+i-1} = \beta_{n+i-1}$, *and*

$$a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i-1} - 2b_{n+i-1}}{2};$$

- *If* $b_{n+i-1} \leq 0$ *then* $a_{S'_i} + b_{S'_i} + \psi_i - 2b_{n+i-1} = \beta_{n+i}$, *and*

$$a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i} + 2b_{n+i-1}}{2},$$

  and hence

$$a_{S'_i} = \frac{t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}.$$

  Similarly we compute

$$b_{S'_i} = \frac{-t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}.$$

P5. *It is easy to observe from Figure 5 that*

$$\beta_i = 2\left[|a_i| + \max(b_i, 0) + m_i\right] \quad \text{for} \quad 1 \leq i \leq n-1$$
$$\beta_{n+i} = |t_i| + 2\max(b_{n+i-1}, 0) + \psi_i + 2n_i \quad \text{for} \quad 1 \leq i \leq k-1.$$

Therefore, since $b_i = \frac{\beta_i - \beta_{i+1}}{2}$; $1 \leq i \leq n+k-2$ *we can compute* β_1 *using one of the two equations below:*

$$\beta_1 = 2\left[|a_i| + \max(b_i, 0) + m_i + \sum_{j=1}^{i-1} b_j\right] \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$\beta_1 = |t_i| + 2\max(b_{n+i-1}, 0) + \psi_i + 2n_i + 2\sum_{j=1}^{n+i-2} b_j \quad \text{for} \quad 1 \leq i \leq k-1.$$

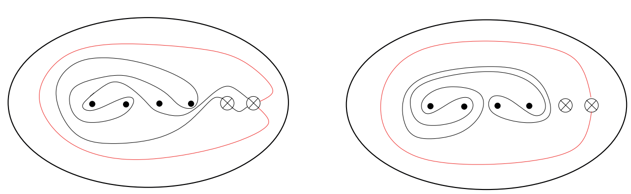

Figure 6. L^* is a simple closed curve on the right but it is not on the left.

P6. *Some integral laminations contain R-components: an R-component of L has geometric intersection numbers $i(R,\alpha_j)=1$ for each $1\leq j\leq 2n-2$, $i(R,\beta_j)=2$ for each $1\leq j\leq n+k-1$, and $i(R,\gamma_j)=2$ for each $1\leq j\leq k-1$, which has its end points on the kth crosscap (denoted red in Figure 6). Set $L^*=L\setminus R$. Note that L^* is a component of L, which is not necessarily a simple closed curve (the two possible cases are depicted in Figure 6). Let α_i^*,β_i^*, and γ_i^* denote the number of intersections of L^* with the arcs α_i,β_i, and γ_i, respectively. Define a_i^*,b_i^*,t_i^* and λ_i^*, $\lambda_{c_i}^*$, $a_{S'}^*$, $b_{S'}^*$ and ψ_i^* similarly as above. We therefore have*

$$\beta_1^*=2\left[|a_i^*|+\max(b_i^*,0)+m_i^*+\sum_{j=1}^{i-1}b_j^*\right]\quad \textit{for}\quad 1\leq i\leq n-1,$$

$$\beta_1^*=|t_i^*|+2\max(b_{n+i-1}^*,0)+\psi_i^*+2n_i^*+2\sum_{j=1}^{n+i-2}b_j^*\quad \textit{for}\quad 1\leq i\leq k-1,$$

where $m_i^*=\min\left\{\alpha_{2i}^*-|b_i^*|,\alpha_{2i-1}^*-|b_i^*|\right\}$; $1\leq i\leq n-1$ and $n_i^*=\min\left\{a_{S_i'}^*,b_{S_i'}^*\right\}$; $1\leq i\leq k-1$. Furthermore, there is some $m_i^*=0$, or some $n_i^*=0$ since otherwise L^* would have above and below components in each S_i and S_i', which would yield curves parallel to $\partial N_{k,n}$, or L^* would contain R-components, which is impossible by definition. Write $a_i^*=a_i,b_i^*=b_i,t_i^*=t_i$ since deleting R-components does not change the a,b,t values. Set

$$X_i=2\left[|a_i|+\max(b_i,0)+\sum_{j=1}^{i-1}b_j\right]\quad \textit{for}\quad 1\leq i\leq n-1,$$

$$Y_i=|t_i|+2\max(b_{n+i-1},0)+\psi_i+2\sum_{j=1}^{n+i-2}b_j\quad \textit{for}\quad 1\leq i\leq k-1.$$

Then one of the three following cases hold for L^*:

I. *If $m_i^*>0$ for all $1\leq i\leq n-1$, then there is some j with $1\leq j\leq k-1$ such that $n_j^*=0$. Therefore, $\beta_1^*>X_i$ and $\beta_1^*=Y_j$.*

II. *If $n_i^*>0$ for all $1\leq i\leq k-1$, then there is some j with $1\leq j\leq n-1$ such that $m_j^*=0$. Therefore, $\beta_1^*>Y_i$ and $\beta_1^*=X_j$.*

III. *There is some i with $1\leq i\leq n-1$ such that $m_i^*=0$ and some j with $1\leq j\leq k-1$ such that $n_j^*=0$. Therefore, $\beta_1^*=X_i=Y_j$.*

We therefore have

$$\beta_i^*=\max(X,Y)-2\sum_{j=1}^{i-1}b_j$$

where

$$X = 2 \max_{1 \le r \le n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\}$$

and

$$Y = \max_{1 \le s \le k-1} \left\{ |t_s| + 2\max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}.$$

P7. *If* L *does not have an* R*-component, that is if* $L^* = L$, *then* $2c_k \le \beta^*_{n+k-1} = \beta_{n+k-1}$ *since* $\beta_{n+k-1} = 2c_k + 2\lambda_k$. *If* L *has an* R*-component then* $2c_k > \beta^*_{n+k-1}$ *and* $\lambda_k = 0$. *See Figure 6. Hence, the number of* R*-components of* L *is given by*

$$R = \max(0, 2c_k - \beta^*_{n+k-1})/2.$$

For example, the integral laminations in Figure 6 (from left to right) have $c_1 = 2, \beta^*_5 = 2$, *and hence* $R = 1$; *and* $c_1 = 1, \beta^*_5 = 0$, *and hence* $R = 1$. *Then* L *is constructed by identifying the two end points of an* R *component with the pieces of* L^* *on the* k*th crosscap. Since* R*-components intersect each* β_i *twice, we get*

$$\beta_i = \beta^*_i + 2R; 1 \le i \le n + k - 1.$$

Then

$$\beta_i = \max(X, Y) - 2\sum_{j=1}^{i-1} b_j + 2R.$$

Also, from item P3., we have

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \ge 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \le 0, \end{cases}$$

Finally, it is easy to observe from Figure 3 that

$$\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i).$$

Making use of the properties above, we shall define the generalized Dynnikov coordinate system, which coordinatizes $\mathcal{L}_{k,n}$ bijectively and with the least number of coordinates.

Definition 2.13 *The generalized Dynnikov coordinate function*

$$\rho\colon \mathcal{L}_{k,n} \to (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$$

is defined by

$$\rho(\mathcal{L}) = (a;b;t;c) := (a_1,\ldots,a_{n-1};b_1,\ldots,b_{n+k-2};t_1,\ldots,t_{k-1};c_1,\ldots,c_k)$$

where

$$a_i = \frac{\alpha_{2i}-\alpha_{2i-1}}{2} \qquad \text{for } 1 \le i \le n-1,$$

$$b_i = \frac{\beta_i-\beta_{i+1}}{2} \qquad \text{for } 1 \le i \le n+k-2,$$

$$t_i = a_{S'_i} - b_{S'_i} \qquad \text{for } 1 \le i \le k-1,$$

where $a_{S'_i}$ *and* $b_{S'_i}$ *are as given in Lemma 2.5.*

Theorem 2.14 gives the inverse of $\rho\colon \mathcal{L}_{k,n} \to (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Theorem 2.14 *Let* $(a;b;t;c) \in (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. *Set*

$$X = 2 \max_{1\le r\le n-1} \left\{ |a_r| + \max(b_r,0) + \sum_{j=1}^{r-1} b_j \right\}$$

$$Y = \max_{1\le s\le k-1} \left\{ |t_s| + 2\max(b_{n+s-1},0) + \psi_s + 2\sum_{j=1}^{n+s-2} b_j \right\}.$$

Then $(a;b;t;c)$ *is the Dynnikov coordinate of exactly one element* $\mathcal{L} \in \mathcal{L}_{k,n}$, *which has*

$$\beta_i = \max(X,Y) - 2\sum_{j=1}^{i-1} b_j + 2R, \tag{2}$$

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2\rceil} + \frac{\beta_{\lceil i/2\rceil}}{2} & \text{if } b_{\lceil i/2\rceil} \ge 0, \\ (-1)^i a_{\lceil i/2\rceil} + \frac{\beta_{1+\lceil i/2\rceil}}{2} & \text{if } b_{\lceil i/2\rceil} \le 0, \end{cases} \tag{3}$$

$$\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i), \tag{4}$$

where $a_{S'_i}$ *is defined as in item P4. in Properties 2.12.*

Proof

Given $L \in \mathcal{L}_{k,n}$ with $\tau(L) = (\alpha,\beta,\gamma,c)$ and $\rho(L) = (a,b,t,c)$, Properties 2.12 show that α,β, and γ must be given by (2), (3), and (4), respectively, and hence L is unique by Lemma 2.10. Therefore, ρ is injective. By Properties 2.12 we can draw nonintersecting path components in each S_i $(1 \le i \le n-1)$, S'_i $(1 \le i \le k-1)$, Δ_0, and Δ'_k, which intersect each element of $\mathcal{A}_{k,n}$ the number of times given by (α,β,γ,c). Gluing together these path components gives a disjoint union of simple closed curves in $N_{k,n}$. There are no curves that bound a puncture or parallel to the boundary by construction, and hence (α,β,γ,c) where α,β, and γ are defined by (2), (3), and (4), respectively, correspond to some L with $\rho(L) = (a,b,t,c)$. Therefore, ρ is surjective. □

Example 2.15 *Let* $\rho(\mathcal{L}) = (a_1;b_1,b_2;t_1;c_1,c_2) = (-1;2,0;1;1,0)$ *be the generalized Dynnikov coordinates of an integral lamination* $\mathcal{L}$ *on* $N_{2,2}$. *We shall use Theorem 2.14 to compute the triangle coordinates of* $\mathcal{L}$ *from* *which we determine the number of path components in* S_1 *and* S_1'*, and hence draw* $\mathcal{L}$ *as illustrated in Example 2.9. By Lemma 2.4,* $\psi_1 = \max(c_1^+ - |b_2|, 0) = 1$*, and so we have*

$$X = 2(|a_1| + \max(b_1, 0)) = 6 \quad \textit{and} \quad Y = |t_1| + 2\max(b_2, 0) + \psi_1 + 2b_1 = 6.$$

Therefore,

$$\beta_1 = \max(6,6) = 6,\ \beta_2 = \max(6,6) - 2b_1 = 2,\ \beta_3 = \max(6,6) - 2(b_1 + b_2) = 2,$$

$$\alpha_1 = -a_1 + \frac{\beta_1}{2} = 4,\ \alpha_2 = a_1 + \frac{\beta_1}{2} = 2.$$

Since $0 = 2c_2 < \beta_3^* = 2$*, there are no* R*-components by item P8. of Properties 2.12. Since* $\beta_1 = 6$ *there are 3 loop components in* Δ_0*, and since* $\beta_3 = 2$ *and* $c_2 = 0$*, there is one noncore loop component in* Δ_2'*, i.e.* $\lambda_2 = 1$*. By Remarks 2.2,* $b_1 = 2$ *and* $b_2 = 0$*, and hence there are 2 right loop components in* S_1 *and no loop components in* S_1'*. By Lemma 2.3 we compute that* $a_{S_1} = \alpha_1 - |b_1| = 2$ *and* $b_{S_1} = \alpha_2 - |b_1| = 0$*. Finally, by item P4. of Properties 2.12,*

$$a_{S_1'} = \frac{t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 1,$$

$$b_{S_1'} = \frac{-t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 0.$$

Gluing together the path components in S_1 *and* S_1'*, we construct the integral lamination depicted in Figure 7.*

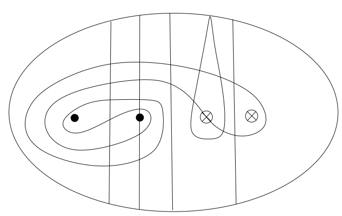

Figure 7. $\rho(L) = (-1; 2, 0; 1; 1, 0)$.

Remark 2.16 *Generalized Dynnikov coordinates for integral laminations can be extended in a natural way to generalized Dynnikov coordinates of measured foliations [5]: the transverse measure on the foliation [4, 7, 8] assigns to each element in* $\mathcal{A}_{k,n}$ *a nonnegative real number, and hence each measured foliation is described by an element of* $(\mathbb{R}_{\geq 0}^{3n+2k-4} \times \mathbb{R}^k) \setminus \{0\}$*, the associated measures of the arcs and curves of* $\mathcal{A}_{k,n}$*. Therefore, the* generalized Dynnikov coordinate system *for measured foliations is defined similarly (see Definition 2.13) and provides a one-to-one correspondence between the set of measured foliations (up to isotopy and Whitehead equivalence) on* $N_{k,n}$ *and* $(\mathbb{R}^{2(n+k-2)} \times \mathbb{R}^k) \setminus \{0\}$.

References

[1] Dehornoy P, Dynnikov I, Rolfsen D, Wiest B. Ordering Braids. Providence, RI, USA: American Mathematical Society, 2008.

[2] Dynnikov I. On a Yang-Baxter mapping and the Dehornoy ordering. Uspekhi Mat Nauk 2002; 57: 151-152.

[3] Dynnikov I, Wiest B. On the complexity of braids. J Eur Math Soc 2007; 9: 801-840.

[4] Fathi A, Laudenbach F. Poenaru V. Travaux de Thurston sur les surfaces. Paris, France: Astérisque, Société Mathématique de France, 1979 (in French).

[5] Hall T, Yurttaş SÖ. On the topological entropy of families of braids, Topology Appl 2009; 156: 1554-1564.

[6] Moussafir JO. On computing the entropy of braids. Funct Anal Other Math 2006; 1: 37-46.

[7] Papadopoulos A, Penner RC. Hyperbolic metrics, measured foliations and pants decompositions for non-orientable surfaces. Asian J Math 2016; 20: 157-182.

[8] Thurston WP. On the geometry and dynamics of diffeomorphisms of surfaces. B Am Math Soc 1988; 19: 417-431.

[9] Yurttaş SÖ. Geometric intersection of curves on punctured disks. J Math Soc Jpn 2013; 65: 1554-1564.

[10] Yurttaş SÖ. Dynnikov and train track transition matrices of pseudo-Anosov braids. Discrete Contin Dyn Syst 2016; 36: 541-570.

[11] Yurttaş SÖ, Hall T. Counting components of an integral lamination. Manuscripta Math 2017; 153: 263-278. doi:10.1007/s00229-016-0885-4.