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Unconditional wavelet bases in Lebesgue spaces

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Abstract: In this paper, we prove that an orthonormal wavelet basis associated with a general isotropic expansive matrix must be an unconditional basis for all $L^p(\mathbb{R}^d)$ with $1 < p < \infty$, provided the wavelet functions satisfy some usual conditions.

Key words: Wavelet basis, unconditional basis, unconditional wavelet basis

1. Introduction

It is a significant event in wavelet analysis that good wavelet bases are unconditional bases in $L^p(\mathbb{R}^d)$ with $1 < p < \infty$. Results of this type are given in some important books on wavelets such as [2, 9, 12, 15], and in some articles such as [7, 8, 13, 16]. However, all those works but [13] consider only the case of the expansive matrix $A = 2I_d$, and most of them are in dimension $d = 1$, where I_d is the $d \times d$ identity matrix. This paper deals with the case of isotropic expansive matrices.

We start with some definitions and notations. Let d be a fixed positive integer. $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^d)$ its dual space, i.e. the so-called tempered distribution space. We denote by $\tilde{f}$ its conjugate reflection, i.e. $\tilde{f}(\cdot) = \overline{f(-\cdot)}$, and define $D_\alpha f(x_1, x_2, \cdots, x_d) = \frac{\partial^{(\alpha_1+\alpha_2+\cdots+\alpha_d)} f(x_1, x_2, \cdots, x_d)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$ for a tempered distribution f and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ with $\alpha_i \in \mathbb{Z}_+$ (the set of nonnegative integers). For two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, we denote by $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ the set of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. For a countable set E, we denote by $l_0(E)$ the set of finitely supported sequences on E. A $d \times d$ matrix A is called an *expansive matrix* if it is an integer matrix with all its eigenvalues being greater than 1 in modulus, and it is called an *isotropic matrix* if it is similar to a $d \times d$ diagonal matrix $diag\,(\lambda_1, \lambda_2, \cdots, \lambda_d)$ with $|\lambda_i| = |\det A|^{\frac{1}{d}}$. Given a $d \times d$ expansive matrix A, we denote by A^* its transpose, and we define the *dilation operator* D and the *shift operator* T_{x_0} with $x_0 \in \mathbb{R}^d$ respectively by

$$Df(\cdot) = |\det A|^{\frac{1}{2}} f(A\cdot) \text{ and } T_{x_0} f(\cdot) = f(\cdot - x_0) \tag{1.1}$$

for a measurable function f. Obviously, they are both unitary operators on $L^2(\mathbb{R}^d)$. For a measurable function

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f, we write

$$f_{j,k}(\cdot) = D^j T_k f(\cdot), \text{ and } f_{A^{-j}}(\cdot) = q^j f(A^j \cdot) \tag{1.2}$$

for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, where $q = |\det A|$, which will be used throughout this paper. The *Fourier transform* is defined by

$$\hat{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle} dx$$

for $f \in L^1(\mathbb{R}^d)$ and naturally extended to $L^2(\mathbb{R}^d)$ and distributions, where $\langle \cdot, \cdot \rangle$ denotes the inner Euclidean product in $\mathbb{R}^d$. We denote by $|\cdot|$ the Euclidean norm, that is

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_d^2\right)^{\frac{1}{2}},$$

for $x \in \mathbb{R}^d$ with x_i being its ith component. Let f, h, and ψ be measurable functions, $\mathcal{A}$ a finite set of measurable functions, and $\beta > 0$. We make the following notations if they make sense:

$$\langle f, h \rangle = \int_{\mathbb{R}^d} f(x)\overline{h(x)} dx, \tag{1.3}$$

$$g_\psi(f)(\cdot) = \left(\sum_{j \in \mathbb{Z}} |\psi_{A^{-j}} * f(\cdot)|^2 \right)^{\frac{1}{2}}, \tag{1.4}$$

$$X(\mathcal{A}) = \left\{ a_{j,k} : a \in \mathcal{A}, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^d \right\}, \tag{1.5}$$

$$\mathcal{W}_\mathcal{A} f(\cdot) = \left\{ \sum_{a \in \mathcal{A}} \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^d} |\langle f, a_{j,k} \rangle|^2 q^j \chi_{\Lambda_{j,k}}(\cdot) \right\}^{\frac{1}{2}}, \tag{1.6}$$

$$\mathcal{T}_{\mathcal{A},\beta} f(\cdot) = \left(\sum_{a \in \mathcal{A}} \sum_{j \in \mathbb{Z}} \left| \sup_{y \in \mathbb{R}^d} \frac{(a_{A^{-j}} * f)(\cdot - y)}{(1 + |A^j y|)^{\beta d}} \right|^2 \right)^{\frac{1}{2}}, \tag{1.7}$$

where

$$\Lambda_{j,k} = A^{-j}(\mathbb{T}^d + k), \ \mathbb{T}^d = [0,1)^d, \tag{1.8}$$

$\chi_{\Lambda_{j,k}}$ denotes the characteristic function of $\Lambda_{j,k}$. We denote by $\mathcal{R}^0(\mathbb{R}^d)$ the set of functions f defined on $\mathbb{R}^d$ satisfying the following: there exist constants $\infty > \gamma \geq \epsilon > 0$ and $0 < C < \infty$ such that

$$\int_{\mathbb{R}^d} f(x) dx = 0, \tag{1.9}$$

$$|f(\cdot)| \leq \frac{C}{(1 + |\cdot|)^{2d+\gamma}}, \tag{1.10}$$

$$|\nabla f(\cdot)| \leq \frac{C}{(1 + |\cdot|)^{d+\epsilon}} \tag{1.11}$$

a.e. on $\mathbb{R}^d$, where ∇f denotes the gradient function of f.

[9, p.287, Theorem 4.15] shows that a one-dimensional dyadic wavelet basis $X(\psi)$ with $\psi \in \mathcal{R}^0(\mathbb{R})$ must be an unconditional basis of $L^p(\mathbb{R})$ for $1 < p < \infty$. Its higher dimensional version with $A = 2I_d$ is obtained in [15, Theorem 8.9]. The goal of this paper is to extend this theorem to the case of general isotropic expansive matrices. To our knowledge, the reference [13] seems to be the first and the only work addressing unconditional wavelet bases for $L^p(\mathbb{R}^d)$, $1 < p < \infty$, associated with general expansive matrices. It provides us with a very general theoretic result. By [13, Theorem 2.4], for a finite subset Ψ of $L^2(\mathbb{R}^d)$, if all elements of Ψ have an A-radial majorant ϕ satisfying

$$\int_{\mathbb{R}^d} \phi(u) \ln(|u| + 1) dx < \infty, \tag{1.12}$$

and $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, then $X(\Psi)$ is an unconditional basis for all $L^p(\mathbb{R}^d)$ with $1 < p < \infty$. Observe that "A-radial" therein is a very technical concept, and it strongly depends on another concept, "A-balanced set" ([13, Definitions 2.2 and 2.3]). [1, Lemma 2.2] proves the existence of the A-balanced set, but its proof is not a direct constructive one. Therefore, it is not easy to find one A-radial (1.12)-majorant ϕ of a function. At least an A-radial majorant ϕ of the function $(1 + |x|)^{-t}$ with $t > 0$ strongly depends on A. It is unknown whether the function $(1 + |x|)^{-t}$ with $t > 0$ has a good A-radial majorant for an arbitrary expansive matrix A, so it is natural for us to ask the following question:

Question. Suppose that A is a $d \times d$ expansive matrix and Ψ is a finite subset of $\mathcal{R}^0(\mathbb{R}^d)$. Is $X(\Psi)$ an unconditional basis for all $L^p(\mathbb{R}^d)$ with $1 < p < \infty$ provided that it is an orthonormal basis for $L^2(\mathbb{R}^d)$?

For isotropic expansive matrices, in this paper we give an affirmative answer to this question. It is unresolved whether it is true for a general expansive matrix. This is because our method strongly depends on a norm associated with the expansive matrix, which is equivalent to the Euclidean norm in $\mathbb{R}^d$. A general expansive matrix need not correspond a quasi-norm equivalent to the Euclidean one in $\mathbb{R}^d$ by [14, Definition 1-8, Proposition 1-9], or [1, Lemma 3.2], while [10, Lemma 1.1] shows that every isotropic expansive matrix corresponds a norm equivalent to the Euclidean one in $\mathbb{R}^d$. Our main result can be stated as follows.

Theorem 1.1 *Let $1 < p < \infty$, let A be a $d \times d$ isotropic expansive matrix, and let Ψ be a finite subset of $\mathcal{R}^0(\mathbb{R}^d)$. Suppose that $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$; then it is an unconditional basis for $L^p(\mathbb{R}^d)$.*

When $A = 2I_d$, Theorem 1.1 reduces to [15, Theorem 8.9]. It was proved by duality and the interpolation between $H_1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$. Therein [15, Proposition 8.8] and

$$|||D^j f|||_{1,q} = |\det A|^{-\frac{j}{2}} |||f|||_{1,q} \tag{1.13}$$

played a key role. Here $||| \cdot |||_{1,q}$ is a norm on $H_1^q(\mathbb{R}^d)$ with $1 < q \le \infty$, but we do not know whether [15, Proposition 8.8] is true for a general expansive matrix or even for a general isotropic expansive matrix. At least, (1.13) need not hold in this case. Therefore, Theorem 1.1 can not be proved similarly to [15, Theorem 8.9]. We should avoid $H_1(\mathbb{R}^d)$-related arguments. With the help of harmonic analysis tools and a suitable norm related to a general isotropic expansive matrix, we prove Theorem 1.1. To prove Theorem 1.1, the following two theorems are needed.

Theorem 1.2 *Let A be a $d\times d$ isotropic expansive matrix, and $\psi\in\mathcal{R}^0(\mathbb{R}^d)$. Then, for every $1<p<\infty$, there exists a positive constant B_p such that*

$$\|g_\psi(f)\|_p\le B_p\|f\|_p$$

for $f\in L^p(\mathbb{R}^d)$.

Theorem 1.3 *Let $1<p<\infty$, let A be a $d\times d$ isotropic expansive matrix, and let Ψ be a finite subset of $\mathcal{R}^0(\mathbb{R}^d)$. Supposing that $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, then there exist constants $0<c\le C<\infty$ such that*

$$c\|f\|_p\le\|\mathcal{W}_\Psi f\|_p\le C\|f\|_p \tag{1.14}$$

for $f\in L^p(\mathbb{R}^d)$.

The rest of this paper is organized as follows. Section 2 is devoted to proving Theorem 1.2, and Section 3 is devoted to proving Theorems 1.1 and 1.3.

2. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. For this purpose, we introduce some necessary notations and notions. Let $\mathbb{B}$ be a Banach space, and $1\le p\le\infty$. We denote by $L^p(\mathbb{R}^d,\mathbb{B})$ the Banach space consisting of all $\mathbb{B}$-valued measurable functions f defined on $\mathbb{R}^d$ such that

$$\|f(\cdot)\|_{\mathbb{B}}\in L^p(\mathbb{R}^d),$$

where the norm is defined by

$$\|f(\cdot)\|_{L^p(\mathbb{R}^d,\mathbb{B})}=\|\|f(\cdot)\|_{\mathbb{B}}\|_p$$

for $f\in L^p(\mathbb{R}^d,\mathbb{B})$. In particular, $L^p(\mathbb{R}^d,\mathbb{C})=L^p(\mathbb{R}^d)$.

A set $\triangle\subset\mathbb{R}^d$ is said to be an ellipsoid if

$$\triangle=\{x\in\mathbb{R}^d:\,|Ax|<1\}$$

for some real invertible $d\times d$ matrix A. Observe that the transpose of an expansive matrix is still an expansive one. By [1, Lemma 1.1], we have:

Lemma 2.1 *For an arbitrary expansive matrix A, there exist an ellipsoid $\triangle$ and $r>1$ such that*

$$\triangle\subset r\triangle\subset A^*\triangle.$$

Let $\triangle$ be as in Lemma 2.1, and take $S=(A^*\triangle)\backslash\triangle$. Then $\{(A^*)^jS:\,j\in\mathbb{Z}\}$ is a partition of $\mathbb{R}^d$. Without loss of generality, we assume that $|\xi|\le 1$ for $\xi\in S$ later. Indeed, if not, we can do it by scaling.

Observe that the transpose of an isotropic matrix is still an isotropic one, and that the determinant of a matrix equals the one of its transpose. The following lemma is borrowed from [10, Lemma 1.1]:

Lemma 2.2 *Let A be a $d\times d$ isotropic expansive matrix. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that*

$$\|A\cdot\| = \lambda\|\cdot\|, \tag{2.1}$$

where $\lambda = |\det A|^{\frac{1}{d}}$.

Since the norms on $\mathbb{R}^d$ are equivalent to each other, for the norm $\|\cdot\|$ in Lemma 2.2, there exist positive constants λ_1, λ_2 such that

$$\lambda_1|\cdot| \le \|\cdot\| \le \lambda_2|\cdot|. \tag{2.2}$$

Lemma 2.3 *Under the hypotheses of Theorem 1.2,*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{-j}\cdot)|^2 \in L^\infty(\mathbb{R}^d).$$

Proof Without loss of generality, we assume that ψ is a real function. Since $\{(A^*)^{-j}S : j\in\mathbb{Z}\}$ is a partition of $\mathbb{R}^d$, we only need to prove that $\sum\limits_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{-j}\cdot)|^2$ is bounded on S. Suppose $c\le|\xi|\le 1$ for $\xi\in S$ with a positive constant c. Taking $\lambda^{-1}<\delta<1$, then there exists $J_0\in\mathbb{N}$ such that

$$\|(A^*)^{-j}\|^{\frac{1}{j}} < \delta \text{ for } j > J_0. \tag{2.3}$$

It follows that

$$|(A^*)^{-j}\cdot| \le \delta^j|\cdot| \text{ and } |(A^*)^{j}\cdot| \ge \delta^{-j}|\cdot| \text{ on } \mathbb{R}^d \tag{2.4}$$

for $j>J_0$. Since $\psi\in\mathcal{R}^0(\mathbb{R}^d)$, we have $\psi\in L^1(\mathbb{R}^d)$, and thus

$$\sum_{j=-J_0}^{J_0} |\hat{\psi}((A^*)^{-j}\cdot)|^2 \le (2J_0+1)\|\psi\|_1^2. \tag{2.5}$$

Next we estimate $\sum\limits_{|j|>J_0} |\hat{\psi}((A^*)^{-j}\cdot)|^2$ on S. Since $\hat{\psi}(0)=0$, we have

$$\begin{aligned}\hat{\psi}(\xi) &= \int_{\mathbb{R}^d}\psi(x)[e^{-2\pi i\langle x,\,\xi\rangle}-1]dx \\ &= \left(\int_{|x|\le|\xi|^{-\frac{1}{2}}} + \int_{|x|>|\xi|^{-\frac{1}{2}}}\right)\psi(x)[e^{-2\pi i\langle x,\,\xi\rangle}-1]dx \\ &= I_1(\xi)+I_2(\xi)\end{aligned} \tag{2.6}$$

for $\xi\ne 0$. For $I_1(\xi)$, we have

$$\begin{aligned}|I_1(\xi)| &\le \int_{|x|\le|\xi|^{-\frac{1}{2}}}|\psi(x)||e^{-2\pi i\langle x,\,\xi\rangle}-1|dx \\ &\le 2\pi\int_{|x|\le|\xi|^{-\frac{1}{2}}}|\psi(x)||x||\xi|dx \\ &\le 2\pi\|\psi\|_1|\xi|^{\frac{1}{2}}.\end{aligned} \tag{2.7}$$

For $I_2(\xi)$, we have

$$\begin{aligned}|I_2(\xi)| &\leq 2C\int_{|x|>|\xi|^{-\frac{1}{2}}} |x|^{-2d-\gamma}dx \\ &= 2C\int_{|\xi|^{-\frac{1}{2}}}^{\infty} r^{-2d-\gamma}dr \int_{|x|=r} dx \\ &= C'|\xi|^{\frac{d+\gamma}{2}}. \end{aligned} \tag{2.8}$$

Collecting (2.6)–(2.8), we obtain that

$$|\hat{\psi}(\xi)| \leq C''|\xi|^{\frac{1}{2}} \text{ for } 0 < |\xi| \leq 1 \tag{2.9}$$

with $C'' = 2\pi\|\psi\|_1 + C'$. For ξ with $|\xi| > 1$, since

$$\begin{aligned}\hat{\psi}(\xi) &= \int_{\mathbb{R}^d} \psi(x)e^{-2\pi i\langle x,\, \xi\rangle}dx \\ &= -\int_{\mathbb{R}^d} \psi(x+\frac{\xi}{2|\xi|^2})e^{-2\pi i\langle x,\, \xi\rangle}dx, \end{aligned} \tag{2.10}$$

we have

$$\begin{aligned}|\hat{\psi}(\xi)| &= \frac{1}{2}\left|\int_{\mathbb{R}^d}[\psi(x)-\psi(x+\frac{\xi}{2|\xi|^2})]e^{-2\pi i\langle x,\, \xi\rangle}dx\right| \\ &\leq \frac{1}{2}\int_{\mathbb{R}^d}\left|\psi(x)-\psi(x+\frac{\xi}{2|\xi|^2})\right|dx \\ &\leq \frac{1}{4|\xi|}\int_{\mathbb{R}^d}|\nabla\psi(\eta)|dx \\ &\leq \frac{C_1}{|\xi|}\int_{\mathbb{R}^d}\frac{1}{(1+|x|)^{d+\epsilon}}dx \\ &= C_2|\xi|^{-1}, \end{aligned}$$

where $\eta = x + \frac{t\xi}{2|\xi|^2}$, $0 < t < 1$. This implies

$$|\hat{\psi}(\xi)| \leq C_3|\xi|^{-1} \text{ for } \xi \neq 0 \tag{2.11}$$

by (2.9). Collecting (2.3), (2.4), (2.9), and (2.11), we have

$$\begin{aligned}\sum_{|j|>J_0}|\hat{\psi}((A^*)^{-j}\xi)|^2 &= \sum_{j=J_0+1}^{\infty}|\hat{\psi}((A^*)^{-j}\xi)|^2 + \sum_{j=J_0+1}^{\infty}|\hat{\psi}((A^*)^{j}\xi)|^2 \\ &\leq C_4\left(\sum_{j=J_0+1}^{\infty}|(A^*)^{-j}\xi| + \sum_{j=J_0+1}^{\infty}|(A^*)^{j}\xi|^{-2}\right) \\ &\leq C_5\left(\sum_{j=J_0+1}^{\infty}\delta^j + \sum_{j=J_0+1}^{\infty}\delta^{2j}\right) \\ &= M < \infty \end{aligned} \tag{2.12}$$

for $\xi \in S$. This leads to $\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{-j}\xi)|^2 \in L^{\infty}(\mathbb{R}^d)$ by (2.5). The proof is completed. □

The following lemma is partially borrowed from [6, p.492, Theorem 3.4].

Lemma 2.4 *Let* $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *be Banach spaces, and* $T \in \mathcal{L}\left(L^r(\mathbb{R}^d, \mathbb{B}_1), L^r(\mathbb{R}^d, \mathbb{B}_2)\right)$ *for some* $1 \le r \le \infty$. *Assume that for a.e.* $x \in \mathbb{R}^d$, $K(x) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$, *that* $K(x)$ *is measurable and locally integrable away from the origin, that*

$$TF(x) = \int_{\mathbb{R}^d} K(x-y)F(y)dy$$

for compactly supported $F \in L^\infty(\mathbb{R}^d, \mathbb{B}_1)$ *and* $x \notin supp(F)$, *and that* $K(x)$ *satisfies Hörmander's condition: there exists a positive constant* M *such that*

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} dx \le M \text{ for } y \in \mathbb{R}^d. \tag{2.13}$$

Then T *can be extended to an operator defined on* $L^p(\mathbb{R}^d, \mathbb{B}_1)$, $1 \le p < \infty$, *such that*

$$\|TF\|_{L^p(\mathbb{R}^d,\mathbb{B}_2)} \le C_p \|F\|_{L^p(\mathbb{R}^d,\mathbb{B}_1)} \quad (1 < p < \infty),$$
$$|\{x \in \mathbb{R}^d : \|TF(x)\|_{\mathbb{B}_2} > t\}| \le C_1 t^{-1} \|F\|_{L^1(\mathbb{R}^d,\mathbb{B}_1)}.$$

Remark 2.1 *A careful observation to the proof of this lemma shows that (2.13) can be replaced by*

$$\int_{|x|>c|y|} \|K(x-y) - K(x)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} dx \le M \text{ for } y \in \mathbb{R}^d, \tag{2.14}$$

where c *is an arbitrary positive constant.*

Lemma 2.5 *Under the hypothesis of Theorem 1.2, let* λ_1 *and* λ_2 *be as in (2.2), and let* c *be a constant satisfying* $c > \lambda_1^{-2}\lambda_2^2$. *Then there exists a positive constant* M *such that*

$$\int_{|x|>c|y|} \sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)| dx \le M \tag{2.15}$$

for $y \in \mathbb{R}^d$.

Proof Without loss of generality, we assume that ψ is a real function. The left-hand side of (2.15) vanishes if $y = 0$, so we only treat the case $0 \ne y \in \mathbb{R}^d$. It is obvious that

$$\begin{aligned} \int_{|x|>c|y|} \sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)| dx &= \sum_{k\in\mathbb{Z}} \int_{|A^{-k}x|>c|y|} |\psi(x - A^k y) - \psi(x)| dx \\ &= I_1(y) + I_2(y), \end{aligned} \tag{2.16}$$

where

$$I_1(y) = \sum_{k:\lambda^k|y|\ge 1} \int_{|A^{-k}x|>c|y|} |\psi(x - A^k y) - \psi(x)| dx, \tag{2.17}$$

$$I_2(y) = \sum_{k:\lambda^k|y|<1} \int_{|A^{-k}x|>c|y|} |\psi(x - A^k y) - \psi(x)| dx, \tag{2.18}$$

and λ is as in Lemma 2.2.

Next we prove that $I_1(y)$ and $I_2(y)$ are both bounded on $\mathbb{R}^d\backslash\{0\}$ to finish the proof. We first treat $I_1(y)$. Observing that

$$\begin{aligned}\int_{|A^{-k}x|>c|y|}|\psi(x-A^ky)|dx &= \int_{|A^{-k}x+y|>c|y|}|\psi(x)|dx\\ &\leq \int_{|A^{-k}x|>(c-1)|y|}|\psi(x)|dx,\end{aligned}$$

we have

$$\int_{|A^{-k}x|>c|y|}|\psi(x-A^ky)-\psi(x)|dx \leq 2\int_{|A^{-k}x|>(c-1)|y|}|\psi(x)|dx. \tag{2.19}$$

Also, $|A^{-k}x|\leq \lambda_1^{-1}\|A^{-k}x\|=\lambda_1^{-1}\lambda^{-k}\|x\|\leq \lambda_2\lambda_1^{-1}\lambda^{-k}|x|$ by (2.2) and Lemma 2.2, so

$$\int_{|A^{-k}x|>c|y|}|\psi(x-A^ky)-\psi(x)|dx\leq 2C\int_{|x|>\alpha\lambda^k|y|}\frac{dx}{(1+|x|)^{2d+\gamma}}dx \tag{2.20}$$

by (1.10) and (2.19), where $\alpha=\lambda_2^{-1}\lambda_1(c-1)$. Also observe that

$$\begin{aligned}\int_{|x|>\alpha\lambda^k|y|}\frac{dx}{(1+|x|)^{2d+\gamma}} &= \int_{\alpha\lambda^k|y|}^{\infty}\frac{dr}{(1+r)^{2d+\gamma}}\int_{|x|=r}d\sigma(x)\\ &= C'\int_{\alpha\lambda^k|y|}^{\infty}\frac{r^{d-1}}{(1+r)^{2d+\gamma}}dr\\ &\leq C'\int_{\alpha\lambda^k|y|}^{\infty}\frac{dr}{(1+r)^{d+1+\gamma}}\\ &\leq C''(\lambda^{-d-\gamma})^k|y|^{-d-\gamma}.\end{aligned}$$

It follows that

$$\begin{aligned}I_1(y) &\leq C'''|y|^{-d-\gamma}\sum_{k:k\geq -\log_\lambda|y|}(\lambda^{-d-\gamma})^k\\ &\leq C_1|y|^{-d-\gamma}(\lambda^{-(d+\gamma)})^{-\log_\lambda|y|-1}\\ &= C_1\lambda^{d+\gamma}<\infty\end{aligned}$$

by (2.17) and (2.20). Now we turn to (2.18). By the mean value theorem,

$$|\psi(x-A^ky)-\psi(x)|\leq|\nabla\psi(\xi)||A^ky|\leq\frac{C|A^ky|}{(1+|\xi|)^{d+\epsilon}},$$

where $\xi=x-tA^ky$ for some $0<t<1$. By (2.2) and Lemma 2.2, we have

$$|A^{-k}x|\leq\lambda_1^{-1}\|A^{-k}x\|=\lambda_1^{-1}\lambda^{-k}\|x\|\leq\lambda_1^{-1}\lambda_2\lambda^{-k}|x|,$$

and

$$|A^ky|\leq\lambda_1^{-1}\|A^ky\|=\lambda_1^{-1}\lambda^k\|y\|\leq\lambda_1^{-1}\lambda^k\lambda_2|y|\leq c^{-1}\lambda_1^{-1}\lambda_2\lambda^k|A^{-k}x|$$

if $|A^{-k}x| > c|y|$. This implies that

$$|A^k y| \leq c^{-1}\lambda_1^{-2}\lambda_2^2|x|,$$

and

$$|\xi| \geq |x| - |A^k y| \geq (1 - c^{-1}\lambda_1^{-2}\lambda_2^2)|x| = \tilde{\alpha}|x|$$

for $x \in \mathbb{R}^d$ with $|A^{-k}x| > c|y|$. Thus, we have

$$|\psi(x - A^k y) - \psi(x)| \leq \frac{C_1|A^k y|}{(1+\tilde{\alpha}|x|)^{d+\epsilon}} \leq \frac{C_2|A^k y|}{(1+|x|)^{d+\epsilon}}$$

for $x \in \mathbb{R}^d$ with $|A^{-k}x| > c|y|$, and thus

$$\begin{aligned} I_2(y) &\leq C_2 \sum_{k:\lambda^k|y|<1} |A^k y| \int_{\mathbb{R}^d} \frac{dx}{(1+|x|)^{d+\epsilon}} \\ &\leq C_3 \sum_{k:\lambda^k|y|<1} |A^k y| \\ &\leq C_4|y| \sum_{k:\lambda^k|y|<1} \lambda^k \end{aligned} \tag{2.21}$$

by (2.2) and Lemma 2.2. Let us estimate $\sum\limits_{k:\lambda^k|y|<1} \lambda^k$:

$$\begin{aligned} \sum_{k:\lambda^k|y|<1} \lambda^k &= \sum_{k:k>\log_\lambda|y|} \lambda^{-k} \\ &= \sum_{k=[\log_\lambda|y|]+1} \lambda^{-k} \\ &\leq (1-\lambda^{-1})^{-1}|y|^{-1}. \end{aligned}$$

Therefore,

$$I_2(y) \leq C_5 < \infty$$

by (2.21). The proof is completed. □

Proof of Theorem 1.2. We use the notations in Lemma 2.4. Take $\mathbb{B}_1 = \mathbb{C}$ and $\mathbb{B}_2 = l^2(\mathbb{Z})$. Define

$$Tf(x) = \{\psi_{A^{-k}} * f(x)\}_{k\in\mathbb{Z}} \tag{2.22}$$

for f with (2.22) being well defined. Then

$$g_\psi(f)(x) = \|Tf(x)\|_{l^2(\mathbb{Z})}. \tag{2.23}$$

By the Plancherel theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \|Tf(x)\|_{l^2(\mathbb{Z})}^2 dx &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} |\psi_{A^{-k}} * f(x)|^2 dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{k\in\mathbb{Z}} \left| \hat{\psi}((A^*)^{-k}\xi) \right|^2 \right) |\hat{f}(\xi)|^2 d\xi \\ &\leq \left\| \sum_{k\in\mathbb{Z}} |\hat{\psi}((A^*)^{-k}\cdot)|^2 \right\|_{\infty} \|\hat{f}\|^2 \\ &= \left\| \sum_{k\in\mathbb{Z}} |\hat{\psi}((A^*)^{-k}\cdot)|^2 \right\|_{\infty} \|f\|^2\end{aligned}$$

for $f \in L^2(\mathbb{R}^d)$. Also observing that $\sum_{k\in\mathbb{Z}} |\hat{\psi}((A^*)^{-k}\cdot)|^2 \in L^\infty(\mathbb{R}^d)$ by Lemma 2.3, we have that T is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, l^2(\mathbb{Z}))$. Thus, by (2.23), to prove the theorem, we only need to prove that the kernel

$$K(x) = (\psi_{A^{-k}}(x))_{k\in\mathbb{Z}}$$

satisfies Hörmander's condition; that is,

$$\int_{|x|>2|y|} \left(\sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)|^2 \right)^{\frac{1}{2}} dx$$

is bounded on $\mathbb{R}^d$. By Remark 2.1, it suffices to prove that, for some $c > 0$,

$$\int_{|x|>c|y|} \left(\sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)|^2 \right)^{\frac{1}{2}} dx$$

is bounded on $\mathbb{R}^d$. Also observe that

$$\left(\sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)|^2 \right)^{\frac{1}{2}} \leq \sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)|.$$

We only need to show that, for some $c > 0$,

$$\int_{|x|>c|y|} \sum_{k\in\mathbb{Z}} |\psi_{A^{-k}}(x-y) - \psi_{A^{-k}}(x)| dx$$

is bounded on $\mathbb{R}^d$. Lemma 2.5 tells us this is true. The theorem therefore follows. □

3. Proofs of Theorems 1.3 and 1.1

Lemma 3.1 *([4, Theorem 1]) Let $1 < p, q < \infty$. Then there exists constant $0 < C < \infty$ such that*

$$\left\| \left\{ \sum_{l=1}^{\infty} (\mathcal{M} f_l)^q \right\}^{\frac{1}{q}} \right\|_p \leq C \left\| \left\{ \sum_{l=1}^{\infty} |f_l|^q \right\}^{\frac{1}{q}} \right\|_p \tag{3.1}$$

for any sequence $\{f_l\}_{l=1}^{\infty}$ *of locally integrable function, where* $\mathcal{M}f$ *is defined by*

$$\mathcal{M}f(\cdot) = \sup_{\delta>0} \frac{1}{|B(0,\,\delta)|} \int_{B(0,\,\delta)} |f(\cdot - y)|dy$$

a.e. on $\mathbb{R}^d$ *for a measurable function* f*, where* $B(0,\,\delta) = \{x \in \mathbb{R}^d : |x| < \delta\}$*.*

Lemma 3.2 *([9, p.215, Theorem 2.10]) Let* $\mathcal{B} = \{x_j : j \in \mathbb{N}\}$ *be a basis for a Banach space* $\mathbb{B} = (\mathbb{B}, \|\cdot\|)$*. For an arbitrary bounded sequence* $\beta = \{\beta_j\}_{j\in\mathbb{N}}$*, define*

$$S_\beta(x) = \sum_{j\in\mathbb{N}} \beta_j f_j(x) x_j$$

for $x = \sum_{j\in\mathbb{N}} f_j(x)x_j \in \mathbb{B}$*. Then the following statements are equivalent:*

1) $\mathcal{B}$ *is an unconditional basis for* $\mathbb{B}$*;*

2) There exists a constant $C > 0$ *such that* $\|S_\beta(x)\| \le C\|x\|$ *for all* $x \in \mathbb{B}$ *and sequences* $\beta = \{\beta_j\}_{j\in\mathbb{N}}$ *with* $|\beta_j| \le 1$*;*

3) There exists a constant $C > 0$ *such that* $\|S_\beta(x)\| \le C\|x\|$ *for all* $x \in \mathbb{B}$ *and sequences* $\beta = \{\beta_j\}_{j\in\mathbb{N}}$ *with* $\beta_j = \pm 1$*;*

4) There exists a constant $C > 0$ *such that* $\|S_\beta(x)\| \le C\|x\|$ *for all* $x \in \mathbb{B}$ *and* $\beta = \{\beta_j\}_{j\in\mathbb{N}} \in l_0(\mathbb{N})$ *with* $\beta_j = 1$ *or* 0*.*

Lemma 3.3 *([5, Lemma 2.4])For every* $0 < p \le \infty$*, there exists a positive constant* C_p *such that for every* $g \in \mathcal{S}'(\mathbb{R}^d)$ *with supp*$(\hat{g}) \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2^{j+1}\}$ *and* $j \in \mathbb{Z}$*,*

$$\left(\sum_{k\in\mathbb{Z}^d} \sup_{x\in Q_{j,k}} |g(x)|^p\right)^{\frac{1}{p}} \le C_p 2^{\frac{jd}{p}} \|g\|_p, \tag{3.2}$$

where $Q_{j,k} = 2^{-j}([0,\,1)^d + k)$*.*

Lemma 3.4 *Let* A *be a* $d\times d$ *isotropic expansive matrix, and* $\gamma \ge \epsilon > 0$*. Assume that* g *and* h *satisfy*

$$|g(\cdot)|, |\nabla g(\cdot)| \le \frac{B}{(1+|\cdot|)^{d+\epsilon}}, \tag{3.3}$$

$$\int_{\mathbb{R}^d} h(x)dx = 0, \tag{3.4}$$

$$|h(\cdot)| \le \frac{B}{(1+|\cdot|)^{2d+\gamma}}, \tag{3.5}$$

for some positive constant B*. Then there exists a positive constant* C *such that for* $l \ge 0$

$$|g_{0,0} * h_{l,0}(\cdot)| \le \frac{Cq^{-l(\frac{1}{2}+\frac{1}{d})}}{(1+|\cdot|)^{d+\epsilon}} \text{ on } \mathbb{R}^d. \tag{3.6}$$

Proof Without loss of generality, we assume that both g and h are real functions. We use the norm $\|\cdot\|$ in Lemma 2.2. By (2.2), we only need to prove that there exists a positive constant C such that for $l \geq 0$,

$$|g_{0,0} * h_{l,0}(\cdot)| \leq \frac{Cq^{-l(\frac{1}{2}+\frac{1}{d})}}{(1+\|\cdot\|)^{d+\epsilon}} \tag{3.7}$$

a.e. on $\mathbb{R}^d$. By (3.4), we have

$$\begin{aligned} |g_{0,0} * h_{l,0}(x)| &= |\int_{\mathbb{R}^d} (g(y)-g(x))h_{l,0}(x-y)dy| \\ &\leq (\int_{E_1} + \int_{E_2} + \int_{E_3})|g(y)-g(x)||h_{l,0}(x-y)|dy \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.8}$$

where $E_1 = \{y \in \mathbb{R}^d : \|x-y\| \leq 2\}$, $E_2 = \{y \in \mathbb{R}^d : \|x-y\| > 2 \text{ and } \|y\| \leq \frac{1}{2}\|x\|\}$, $E_3 = \{y \in \mathbb{R}^d : \|x-y\| > 2 \text{ and } \|y\| > \frac{1}{2}\|x\|\}$.

We first deal with I_1. By (3.3) and (2.2), we have

$$|g(y)-g(x)| = |\langle \nabla g(\xi),\, y-x\rangle| \leq \frac{B|x-y|}{(1+|\xi|)^{d+\epsilon}} \leq \frac{C'\|x-y\|}{(1+\|\xi\|)^{d+\epsilon}},$$

where $\xi = x + \eta(y-x)$ with $0<\eta<1$. For $y \in E_1$, it leads to

$$1+\|x\| \leq 1+\|x-\xi\|+\|\xi\| \leq 1+\|x-y\|+\|\xi\| \leq 3(1+\|\xi\|),$$

so

$$I_1 \leq \frac{C''q^{\frac{l}{2}}}{(1+\|x\|)^{d+\epsilon}} \int_{E_1} \frac{\|x-y\|}{(1+\lambda^l\|x-y\|)^{2d+\gamma}}dy$$

by (3.5), (2.2), and Lemma 2.2. Substituting $\lambda^l(x-y) = y'$ in the above formula, we have

$$\begin{aligned} I_1 &\leq \frac{C''q^{\frac{l}{2}}}{(1+\|x\|)^{d+\epsilon}} \int_{\{y\in\mathbb{R}^d:\|y\|\leq 2\lambda^l\}} \frac{\lambda^{-ld}\lambda^{-l}\|y\|}{(1+\|y\|)^{2d+\gamma}}dy \\ &= \frac{C''q^{-\frac{l}{2}}\lambda^{-l}}{(1+\|x\|)^{d+\epsilon}} \int_0^{2\lambda^l} \frac{r}{(1+r)^{2d+\gamma}}dr \int_{\|y\|=r} d\sigma(y) \\ &= \frac{C''q^{-\frac{l}{2}}\lambda^{-l}}{(1+\|x\|)^{d+\epsilon}} \int_0^{2\lambda^l} \frac{r^d}{(1+r)^{2d+\gamma}}dr \int_{\|y\|=1} d\sigma(y) \\ &\leq \frac{C''q^{-\frac{l}{2}}\lambda^{-l}}{(1+\|x\|)^{d+\epsilon}} \int_0^{\infty} \frac{1}{(1+r)^{d+\gamma}}dr \int_{\|y\|=1} d\sigma(y) \\ &= \frac{\tilde{C}q^{-l(\frac{1}{2}+\frac{1}{d})}}{(1+\|x\|)^{d+\epsilon}}. \end{aligned} \tag{3.9}$$

Next we turn to I_2. Fix $x \in \mathbb{R}^d$ and $l \in \mathbb{Z}$. From (2.2), we have

$$\|A^l(x-y)\| = \lambda^l\|x-y\| \geq \lambda^l(1+\frac{1}{2}\|x-y\|) \geq \lambda^l(1+\frac{\|x\|}{4}) \geq C_1\lambda^l(1+\|x\|) \tag{3.10}$$

for $y \in E_2$, which implies that

$$\begin{aligned}
I_2 &\le C' q^{\frac{l}{2}} \int_{E_2} \left(\frac{1}{(1+\|y\|)^{d+\epsilon}} + \frac{1}{(1+\|x\|)^{d+\epsilon}} \right) \frac{1}{(1+\|A^l(x-y)\|)^{2d+\gamma}} dy \\
&\le C'' q^{\frac{l}{2}} \int_{E_2} \left(\frac{1}{(1+\|y\|)^{d+\epsilon}} + \frac{1}{(1+\|x\|)^{d+\epsilon}} \right) \frac{\lambda^{-l(2d+\gamma)}}{(1+\|x\|)^{2d+\gamma}} dy \\
&\le C'' \frac{\lambda^{-l(2d+\gamma)} q^{\frac{l}{2}}}{(1+\|x\|)^{2d+\gamma}} \int_{\{y \in \mathbb{R}^d : \|y\| \le \frac{1}{2}\|x\|\}} \left(\frac{1}{(1+\|y\|)^{d+\epsilon}} + \frac{1}{(1+\|x\|)^{d+\epsilon}} \right) dy \\
&\le C'' \frac{\lambda^{-l(2d+\gamma)} q^{\frac{l}{2}}}{(1+\|x\|)^{2d+\gamma}} \left(\int_0^{\frac{\|x\|}{2}} \frac{1}{(1+r)^{d+\epsilon}} dr \int_{\|y\|=r} d\sigma(y) + \frac{\|x\|^d}{(1+\|x\|)^{d+\epsilon}} \right) \\
&\le C''' \frac{\lambda^{-l(2d+\gamma)} q^{\frac{l}{2}}}{(1+\|x\|)^{2d+\gamma}} \left(\int_0^{\frac{\|x\|}{2}} \frac{dr}{(1+r)^{1+\epsilon}} + \|x\|^d \right) \\
&\le \frac{C_1 \lambda^{-l(2d+\gamma)} q^{\frac{l}{2}}}{(1+\|x\|)^{2d+\gamma}} (1+\|x\|^d) \\
&\le \frac{C_1' q^{-l(\frac{3}{2}+\frac{\gamma}{d})}}{(1+\|x\|)^{d+\gamma}} \\
&\le \frac{\tilde{C}_1 q^{-l(\frac{1}{2}+\frac{1}{d})}}{(1+\|x\|)^{d+\epsilon}}
\end{aligned} \tag{3.11}$$

by (3.3), (3.5), and the fact that $\gamma \ge \epsilon$.

Now we estimate I_3. From (3.3), (3.5), and (2.2), it follows that

$$\begin{aligned}
I_3 &\le C' q^{\frac{l}{2}} \int_{E_3} \left(\frac{1}{(1+\|y\|)^{d+\epsilon}} + \frac{1}{(1+\|x\|)^{d+\epsilon}} \right) \frac{1}{(1+\|A^l(x-y)\|)^{2d+\gamma}} dy \\
&\le \frac{C'' q^{\frac{l}{2}}}{(1+\|x\|)^{d+\epsilon}} \int_{\{y \in \mathbb{R}^d : \|x-y\| \ge 2\}} \frac{1}{(1+\lambda^l \|x-y\|)^{2d+\gamma}} dy \\
&\le \frac{C'' q^{\frac{l}{2}}}{(1+\|x\|)^{d+\epsilon}} \int_2^{\infty} \frac{r^{d-1}}{(1+\lambda^l r)^{2d+\gamma}} dr \int_{\|y\|=1} d\sigma(y) \\
&\le \frac{C''' q^{\frac{l}{2}}}{(1+\|x\|)^{d+\epsilon}} \int_2^{\infty} \frac{r^{d-1}}{(1+\lambda^l r)^{2d+\gamma}} dr \\
&\le \frac{C_1 q^{\frac{l}{2}} \lambda^{-ld}}{(1+\|x\|)^{d+\epsilon}} \int_{2\lambda^l}^{\infty} \frac{1}{(1+r)^{d+\gamma+1}} dr \\
&\le \frac{\tilde{C}_1 q^{-l(\frac{1}{2}+\frac{1}{d})}}{(1+\|x\|)^{d+\epsilon}}.
\end{aligned} \tag{3.12}$$

Collecting (3.8), (3.9), (3.11), and (3.12), we have (3.6). The proof is completed. □

Observe that

$$|\langle \psi_{j,k}, \phi_{m,n} \rangle| = |(\phi_{0,0} * \tilde{\psi}_{j-m,0})(A^{m-j}k - n)| \text{ if } m \le j, \tag{3.13}$$

and

$$|\langle \psi_{j,k},\, \phi_{m,n}\rangle| = |(\psi_{0,0} * \tilde{\phi}_{m-j,0})(k - A^{j-m}n)| \text{ if } m > j. \tag{3.14}$$

As an immediate consequence of Lemma 3.4, we have the following lemma, for which related results can be found in [3] and [11]:

Lemma 3.5 *Let A be a $d\times d$ isotropic expansive matrix, and $\psi,\, \phi \in \mathcal{R}^0(\mathbb{R}^d)$. Then there exists a positive constant C such that for $j,\, m\in\mathbb{Z}$ and $k,\, n\in\mathbb{Z}^d$*

1)

$$|\langle \psi_{j,k},\, \phi_{m,n}\rangle| \le \frac{Cq^{(m-j)(\frac{1}{2}+\frac{1}{d})}}{(1+|A^{m-j}k-n|)^{d+\epsilon}}$$

if $m\le j$, and

2)

$$|\langle \psi_{j,k},\, \phi_{m,n}\rangle| \le \frac{Cq^{(j-m)(\frac{1}{2}+\frac{1}{d})}}{(1+|k-A^{j-m}n|)^{d+\epsilon}}$$

if $m>j$.

Lemma 3.6 *Let A be a $d\times d$ isotropic expansive matrix, and $\epsilon>0$. Then there exists a positive constant C such that for all sequences $\{s_{j,k}:(j,k)\in\mathbb{Z}\times\mathbb{Z}^d\}$ of complex numbers and all $x\in\Lambda_{j,k}$ with $(j,k)\in\mathbb{Z}\times\mathbb{Z}^d$,*

$$\sum_{m\in\mathbb{Z}^d}\frac{|s_{l,m}|}{(1+|A^{l-j}k-m|)^{d+\epsilon}} \le C\mathcal{M}\left(\sum_{m\in\mathbb{Z}^d}|s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \tag{3.15}$$

if $l\le j$, and

$$\sum_{m\in\mathbb{Z}^d}\frac{|s_{l,m}|}{(1+|A^{j-l}m-k|)^{d+\epsilon}} \le Cq^{l-j}\mathcal{M}\left(\sum_{m\in\mathbb{Z}^d}|s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \tag{3.16}$$

if $l\ge j$.

Proof By (2.2), we only need to prove that there exists a positive constant C such that $x\in\Lambda_{j,k}$ with $(j,k)\in\mathbb{Z}\times\mathbb{Z}^d$,

$$\sum_{m\in\mathbb{Z}^d}\frac{|s_{l,m}|}{(1+\|A^{l-j}k-m\|)^{d+\epsilon}} \le C\mathcal{M}\left(\sum_{m\in\mathbb{Z}^d}|s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \text{ if } l\le j, \tag{3.17}$$

and

$$\sum_{m\in\mathbb{Z}^d}\frac{|s_{l,m}|}{(1+\|A^{j-l}m-k\|)^{d+\epsilon}} \le Cq^{l-j}\mathcal{M}\left(\sum_{m\in\mathbb{Z}^d}|s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \text{ if } l\ge j. \tag{3.18}$$

Next we prove (3.17) and (3.18). We first consider (3.17). Fix $x \in \Lambda_{j,k}$. For $l \leq j$, write

$$E_0 = \{m \in \mathbb{Z}^d : \|A^{l-j}k - m\| \leq 1\} \text{ and } E_n = \{m \in \mathbb{Z}^d : \lambda^{n-1} < \|A^{l-j}k - m\| \leq \lambda^n\}$$

with $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \sum_{m\in\mathbb{Z}^d} \frac{|s_{l,m}|}{(1+\|A^{l-j}k-m\|)^{d+\epsilon}} &= \sum_{n=0}^{\infty} \sum_{m\in E_n} \frac{|s_{l,m}|}{(1+\|A^{l-j}k-m\|)^{d+\epsilon}} \\ &\leq C \sum_{n=0}^{\infty} \lambda^{-(n-1)(d+\epsilon)} \sum_{m\in E_n} |s_{l,m}|. \end{aligned} \tag{3.19}$$

For $m \in E_n$, $y \in \Lambda_{l,m}$, we have $y - x \in A^{-l}(m - A^{l-j}k + \mathbb{T}^d - A^{l-j}\mathbb{T}^d)$. Also observe that $l \leq j$. It follows that

$$\|y - x\| \leq \lambda^{-l}(\lambda^n + C_1) \leq C_2\lambda^{n-l}$$

for some constant C_2 independent of y and x, so

$$\begin{aligned} \sum_{m\in E_n} |s_{l,m}| &= q^l \int_{\{y\in\mathbb{R}^d:\|y-x\|\leq C_2\lambda^{n-l}\}} \left(\sum_{m\in E_n} |s_{l,m}|\chi_{\Lambda_{l,m}}(y)\right) dy \\ &\leq q^l \int_{\{y\in\mathbb{R}^d:|y-x|\leq C_3\lambda^{n-l}\}} \left(\sum_{m\in E_n} |s_{l,m}|\chi_{\Lambda_{l,m}}(y)\right) dy \\ &\leq C'\lambda^{ld}(\lambda^{n-l})^d \mathcal{M}\left(\sum_{m\in E_n} |s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \\ &= C'\lambda^{nd}\mathcal{M}\left(\sum_{m\in E_n} |s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \end{aligned} \tag{3.20}$$

by (2.2). Combining (3.19) with (3.20) leads to

$$\begin{aligned} \sum_{m\in\mathbb{Z}^d} \frac{|s_{l,m}|}{(1+\|A^{l-j}k-m\|)^{d+\epsilon}} &\leq C'' \sum_{n=0}^{\infty} \lambda^{-n\epsilon}\mathcal{M}\left(\sum_{m\in E_n} |s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x) \\ &\leq C'''\mathcal{M}\left(\sum_{m\in\mathbb{Z}^d} |s_{l,m}|\chi_{\Lambda_{l,m}}\right)(x). \end{aligned}$$

For $l \geq j$, let $F_0 = \{m \in \mathbb{Z}^d : \|A^{j-l}m - k\| \leq 1\}$ and $F_n = \{m \in \mathbb{Z}^d : \lambda^{n-1} < \|A^{j-l}m - k\| \leq \lambda^n|\}$ with $n \in \mathbb{N}$. Similarly, we have

$$\sum_{m\in\mathbb{Z}^d} \frac{|s_{l,m}|}{(1+\|A^{j-l}m-k\|)^{d+\epsilon}} \leq C\sum_{n=0}^{\infty} \lambda^{-(n-1)(d+\epsilon)} \sum_{m\in F_n} |s_{l,m}|, \tag{3.21}$$

and $\Lambda_{l,m} \subseteq \{y \in \mathbb{R}^d : \|y - x\| \leq C\lambda^{n-j}\}$ for some constant C related with λ_1, λ_2 in (2.2). Then we can prove (**??**) by the same procedure as in the proof of (3.17). This completes the proof. □

Lemma 3.7 *Let A be a $d\times d$ isotropic expansive matrix, Ψ and Φ two finite subsets of $\mathcal{R}^0(\mathbb{R}^d)$ with the same cardinality, and $X(\Phi)$ an orthonormal basis for $L^2(\mathbb{R}^d)$. Then for $1<p<\infty$, there exists a positive constant C_p such that*

$$\|\mathcal{W}_\Psi f\|_p \le C_p \|\mathcal{W}_\Phi f\|_p \tag{3.22}$$

for $f\in L^p(\mathbb{R}^d)$.

Proof Since $X(\Phi)$ is an othonormal basis for $L^2(\mathbb{R}^d)$, we have

$$\psi_{j,k}(\cdot)=\sum_{\phi\in\Phi}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^d}\langle\psi_{j,k},\,\phi_{m,n}\rangle\phi_{m,n}(\cdot)$$

for $(\psi,\,j,\,k)\in\Psi\times\mathbb{Z}\times\mathbb{Z}^d$, so

$$\mathcal{W}_\Psi f(\cdot) \;=\; \left\{\sum_{\psi\in\Psi}\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d}\left|\sum_{\phi\in\Phi}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^d}\overline{\langle\psi_{j,k},\,\phi_{m,n}\rangle}\langle f,\,\phi_{m,n}\rangle\right|^2 q^j\chi_{\Lambda_{j,k}}(\cdot)\right\}^{\frac12}. \tag{3.23}$$

Fix $(\psi,j,k)\in\Psi\times\mathbb{Z}\times\mathbb{Z}^d$. Write

$$A_1(\psi,j,k)=\sum_{\phi\in\Phi}\sum_{m\le j}\sum_{n\in\mathbb{Z}^d}\langle f,\,\phi_{m,n}\rangle\overline{\langle\psi_{j,k},\,\phi_{m,n}\rangle},$$

and

$$A_2(\psi,j,k)=\sum_{\phi\in\Phi}\sum_{m>j}\sum_{n\in\mathbb{Z}^d}\langle f,\,\phi_{m,n}\rangle\overline{\langle\psi_{j,k},\,\phi_{m,n}\rangle}.$$

Then

$$\begin{aligned}\mathcal{W}_\Psi f(\cdot) &\le \|A_1(\psi,j,k)q^{\frac{j}{2}}\chi_{\Lambda_{j,k}}(\cdot)\|_{l^2(\Psi\times\mathbb{Z}\times\mathbb{Z}^d)}+\|A_2(\psi,j,k)q^{\frac{j}{2}}\chi_{\Lambda_{j,k}}(\cdot)\|_{l^2(\Psi\times\mathbb{Z}\times\mathbb{Z}^d)}\\ &= A_1(\cdot)+A_2(\cdot),\end{aligned} \tag{3.24}$$

and thus

$$\|\mathcal{W}_\Psi f(\cdot)\|_p\le\|A_1(\cdot)\|_p+\|A_2(\cdot)\|_p. \tag{3.25}$$

For $x\in\Lambda_{j,k}$, $(j,k)\in\mathbb{Z}\times\mathbb{Z}^d$, we have

$$\begin{aligned}|A_1(\psi,j,k)| &\le \sum_{\phi\in\Phi}\sum_{m\le j}\sum_{n\in\mathbb{Z}^d}|\langle f,\,\phi_{m,n}\rangle||\langle\psi_{j,k},\,\phi_{m,n}\rangle|\\ &\le C\sum_{m\le j}q^{(m-j)(\frac12+\frac1d)}\sum_{n\in\mathbb{Z}^d}\frac{\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|}{(1+|A^{m-j}k-n|)^{d+\epsilon}}\\ &\le C'\sum_{m\le j}q^{(m-j)(\frac12+\frac1d)}\mathcal{M}\left(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|\,\chi_{\Lambda_{m,n}}\right)(x)\end{aligned}$$

by Lemmas 3.5 and 3.6. This implies that

$$\begin{aligned}\|A_1(\cdot)\|_p &\leq C''\left\|\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{m\leq j}q^{(m-j)\frac{1}{d}}\mathcal{M}(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}})(\cdot)\right)^2\right\}^{\frac{1}{2}}\right\|_p\\ &= C''\left\|\|\{a_m\}*\{b_m(\cdot)\}\|_{l^2(\mathbb{Z})}\right\|_p,\end{aligned}$$

where

$$a_j=\begin{cases}q^{-\frac{j}{d}} & j\geq 0,\\ 0 & j<0,\end{cases}\quad \text{and } b_m(x)=\mathcal{M}\left(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}}\right)(x)$$

for $m\in\mathbb{Z}$ and a.e. $x\in\mathbb{R}^d$. It follows that

$$\begin{aligned}\|A_1(\cdot)\|_p &\leq C'\,\|\|\{a_m\}\|_{l_1}\|\{b_m(\cdot)\}\|_{l^2}\|_p\\ &= C'\left\|\left(\sum_{j\geq 0}q^{-\frac{j}{d}}\right)\left\{\sum_{m\in\mathbb{Z}}\left(\mathcal{M}(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}})(\cdot)\right)^2\right\}^{\frac{1}{2}}\right\|_p\\ &= C''\left\|\left\{\sum_{m\in\mathbb{Z}}\left(\mathcal{M}(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}})(\cdot)\right)^2\right\}^{\frac{1}{2}}\right\|_p.\end{aligned}$$

Applying Lemma 3.1 with $q=2$, we have

$$\begin{aligned}\|A_1(\cdot)\|_p &\leq C\left\|\left\{\sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}^d}\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}}(\cdot)\right)^2\right\}^{\frac{1}{2}}\right\|_p\\ &= C\left\|\left\{\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\left(\sum_{\phi\in\Phi}|\langle f,\,\phi_{m,n}\rangle|q^{\frac{m}{2}}\chi_{\Lambda_{m,n}}(\cdot)\right)^2\right\}^{\frac{1}{2}}\right\|_p\\ &\leq C\mathrm{card}(\Phi)^{\frac{1}{2}}\|\mathcal{W}_\Phi f\|_p, \qquad (3.26)\end{aligned}$$

where we use the fact that $\{\Lambda_{m,n}:\,n\in\mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$ for each $m\in\mathbb{Z}$, and $\mathrm{card}(\Phi)$ denotes the cardinality of Φ. Similarly, we can also prove that there exists a positive constant $\tilde{C}$ such that

$$\|A_2(\cdot)\|_p\leq\tilde{C}\|\mathcal{W}_\Phi f\|_p$$

for $f\in L^p(\mathbb{R}^d)$. The lemma therefore follows. □

Lemma 3.8 *Let A be a $d\times d$ isotropic expansive matrix, $1<p<\infty$, λ as in Lemma 2.2, and Ψ a finite function set of $\mathcal{R}^0(\mathbb{R}^d)$ whose elements are all band-limited. Then there exists a positive constant C such that*

$$\mathcal{W}_\Psi f(\cdot)\le C\mathcal{T}_{\tilde{\Psi},\lambda}f(\cdot)$$

for $f\in L^p(\mathbb{R}^d)$, where $\tilde{\Psi}=\{\tilde{\psi}:\psi\in\Psi\}$.

Proof Observe that $\widetilde{\psi}_{A^{-j}}*f$ is band-limited for $f\in L^p(\mathbb{R}^d)$, $j\in\mathbb{Z}$ due to the fact of $\psi\in\Psi$ being band-limited, and thus it is differentiable by the Paley–Wiener theorem. Therefore, considering its point-wise values makes sense. Fix $f\in L^p(\mathbb{R}^d)$. It is easy to check that

$$\begin{aligned}|\langle f,\,\psi_{j,k}\rangle| &= q^{-\frac{j}{2}}|(\widetilde{\psi}_{A^{-j}}*f)(A^{-j}k)|\\ &\le q^{-\frac{j}{2}}\sup_{y\in\Lambda_{j,k}}|(\widetilde{\psi}_{A^{-j}}*f)(y)|\end{aligned}$$

for $(\psi,\,j,\,k)\in\Psi\times\mathbb{Z}\times\mathbb{Z}^d$. For an arbitrarily fixed $x_0\in\mathbb{R}^d$ and $j_0\in\mathbb{Z}$, there exists a unique $k_0\in\mathbb{Z}^d$ such that $x_0\in\Lambda_{j_0,k_0}$. It follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\psi_{j_0,k}\rangle|^2q^{j_0}\chi_{\Lambda_{j_0,k}}(x_0) &\le \sup_{y\in\Lambda_{j_0,k_0}}|\tilde{\psi}_{A^{-j_0}}*f(y)|^2\\ &= \sup_{z\in-\Lambda_{j_0,k_0}+x_0}\frac{|(\widetilde{\psi}_{A^{-j_0}}*f)(x_0-z)|^2}{(1+\|A^{j_0}z\|)^{2\lambda d}}(1+\|A^{j_0}z\|)^{2\lambda d}\\ &\le \sup_{z\in\mathbb{R}^d}\frac{|(\widetilde{\psi}_{A^{-j_0}}*f)(x_0-z)|^2}{(1+\|A^{j_0}z\|)^{2\lambda d}}\sup_{z\in-\Lambda_{j_0,k_0}+x_0}(1+\lambda^{j_0}\|z\|)^{2\lambda d}.\end{aligned}$$

Observe that $z\in-\Lambda_{j_0,k_0}+x_0\subseteq A^{-j_0}([-1,1)^d)$. It follows that $\|z\|\le C\lambda^{-j_0}$, and thus

$$\sum_{k\in\mathbb{Z}^d}|\langle f,\,\psi_{j_0,k}\rangle|^2q^{j_0}\chi_{\Lambda_{j_0,k}}(x_0)\le C'\sup_{z\in\mathbb{R}^d}\frac{|\tilde{\psi}_{A^{-j_0}}*f(x_0-z)|^2}{(1+|A^{j_0}z|)^{2\lambda d}}$$

by (2.2). This leads to $\mathcal{W}_\Psi f(x_0)\le C\mathcal{T}_{\tilde{\Psi},\lambda}f(x_0)$. This finishes the proof by the arbitrariness of x_0. □

Given $\beta>0$ and a function f defined on $\mathbb{R}^d$, define

$$f^*_\beta(\cdot)=\sup_{y\in\mathbb{R}^d}\frac{|f(\cdot-y)|}{(1+|y|)^{\beta d}} \tag{3.27}$$

a.e. on $\mathbb{R}^d$.

Lemma 3.9 *Let $\beta>0$ and $g\in L^1(\mathbb{R}^d)$ with $\hat{g}$ being compactly supported. Then, for every $\alpha\in\mathbb{Z}^d$ with $\alpha_i\ge0$, $1\le i\le d$, there exists a positive constant C_α such that*

$$(D_\alpha g)^*_\beta(\cdot)\le C_\alpha g^*_\beta(\cdot)$$

on $\mathbb{R}^d$.

Proof Take $\gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{\gamma}(\cdot) = 1$ on $\text{supp}(\hat{g})$. This can be done since $\text{supp}(\hat{g})$ is compact. Then $\hat{g} = \hat{g}\hat{\gamma}$, and thus $g = \gamma * g$. It follows that $D_\alpha g = D_\alpha \gamma * g$. Next we estimate $D_\alpha g(x-y)$ with $x, y \in \mathbb{R}^d$.

$$\begin{aligned} |D_\alpha g(x-y)| &= \left| \int_{\mathbb{R}^d} D_\alpha \gamma(x-y-z) g(z) dz \right| \\ &= \left| \int_{\mathbb{R}^d} D_\alpha \gamma(t-y) g(x-t) dt \right| \\ &\leq \int_{\mathbb{R}^d} |D_\alpha \gamma(t-y)| \, (1+|t|)^{\beta d} \frac{|g(x-t)|}{(1+|t|)^{\beta d}} dt \\ &\leq (1+|y|)^{\beta d} \int_{\mathbb{R}^d} |D_\alpha \gamma(t-y)| \, (1+|t-y|)^{\beta d} \frac{|g(x-t)|}{(1+|t|)^{\beta d}} dt \\ &\leq C_\alpha (1+|y|)^{\beta d} g^*_\beta(x), \end{aligned} \tag{3.28}$$

where $C_\alpha = \int_{\mathbb{R}^d} |D_\alpha \gamma(t)|(1+|t|)^{\beta d} dt$. It follows that

$$(D_\alpha g)^*_\beta(\cdot) \leq C_\alpha g^*_\beta(\cdot)$$

by the arbitrariness of x and y. □

Applying Lemma 3.3 and by the same procedure as in [9, p.271, Corollary 3.9], we have:

Lemma 3.10 *Let $\beta > 0$, and let g a band-limited function with $g \in L^p(\mathbb{R}^d)$, $0 < p \leq \infty$. Then we have*

$$g^*_\beta(\cdot) < \infty \text{ on } \mathbb{R}^d.$$

Lemma 3.11 *For $\beta > 0$, there exists a positive constant C_β such that*

$$g^*_\beta(\cdot) \leq C_\beta \left(\mathcal{M}(|g|^{\frac{1}{\beta}})(\cdot) \right)^\beta$$

on $\mathbb{R}^d$ for an arbitrary band-limited function g satisfying $g^*_\beta(\cdot) < \infty$ on $\mathbb{R}^d$.

Proof Without loss of generality, we assume that g is a real function. Since g is band-limited, it is differentiable by the Paley–Wiener theorem, so considering its point-wise values makes sense. Fix $x, y \in \mathbb{R}^d$, $0 < \delta < 1$. Choose $z \in \mathbb{R}^d$ such that $z \in B(x-y, \delta) = \{x : |z-x+y| \leq \delta\}$. Then we have $|g(x-y) - g(z)| = |\langle \nabla g(\xi), z-x+y \rangle|$ with $\xi = (x-y) + t(z-x+y)$, $0 < t < 1$. It follows that

$$|g(x-y)| \leq |g(z)| + |z-x+y||\nabla g(\xi)| \leq |g(z)| + C_d \delta \sup_{\{\xi : \xi \in B(x-y, \delta), |\alpha|=1\}} |D_\alpha g(\xi)|,$$

and thus

$$|g(x-y)|^{\frac{1}{\lambda}} \leq C \left(|g(z)|^{\frac{1}{\lambda}} + \delta^{\frac{1}{\lambda}} \sup_{\{\xi : \xi \in B(x-y, \delta), |\alpha|=1\}} |D_\alpha g(\xi)|^{\frac{1}{\lambda}} \right) \tag{3.29}$$

for some constant C related to λ and d. Integrating the above formula on $B(x-y, \delta)$, we have

$$\int_{B(x-y, \delta)} |g(x-y)|^{\frac{1}{\lambda}} dz \leq C \left(\int_{B(x-y, \delta)} |g(z)|^{\frac{1}{\lambda}} dz + \delta^{\frac{1}{\lambda}} |B(x-y, \delta)| \sup_{\{\xi : \xi \in B(x-y, \delta), |\alpha|=1\}} |D_\alpha g(\xi)|^{\frac{1}{\lambda}} \right),$$

which leads to

$$|g(x-y)|^{\frac{1}{\lambda}} \le C'\delta^{-d}\left(\int_{B(x-y,\,\delta)}|g(z)|^{\frac{1}{\lambda}}dz+\delta^{\frac{1}{\lambda}+d}\sup_{\{\xi:\xi\in B(x-y,\,\delta),|\alpha|=1\}}|D_\alpha g(\xi)|^{\frac{1}{\lambda}}\right). \tag{3.30}$$

Also observe that $B(x-y,\,\delta)\subseteq B(x,\,|y|+\delta)$. Then we have

$$\begin{aligned}\int_{B(x-y,\,\delta)}|g(z)|^{\frac{1}{\lambda}}dz &\le \int_{B(x,\,|y|+\delta)}|g(z)|^{\frac{1}{\lambda}}dz\\ &\le C(|y|+\delta)^d\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\sup_{\{\xi:\xi\in B(x-y,\,\delta),|\alpha|=1\}}|D_\alpha g(\xi)|^{\frac{1}{\lambda}} &\le \sup_{\{\xi:\xi\in B(x,\,|y|+\delta),|\alpha|=1\}}|D_\alpha g(\xi)|^{\frac{1}{\lambda}}\\ &= \sup_{\{\xi:|\xi|\le|y|+\delta,|\alpha|=1\}}|D_\alpha g(x-\xi)|^{\frac{1}{\lambda}}\\ &= \sup_{\{\xi:|\xi|\le|y|+\delta,|\alpha|=1\}}\left(\frac{|D_\alpha g(x-\xi)|}{(1+|\xi|)^{\lambda d}}\right)^{\frac{1}{\lambda}}(1+|\xi|)^d\\ &\le ((D_\alpha g)^*_\lambda(x))^{\frac{1}{\lambda}}(1+|y|+\delta)^d\end{aligned} \tag{3.32}$$

with $|\alpha|=1$. Combining (3.30) and (3.31) with (3.32) leads to

$$\begin{aligned}|g(x-y)|^{\frac{1}{\lambda}} &\le C'\left\{\frac{(|y|+\delta)^d}{\delta^d}\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)+\delta^{\frac{1}{\lambda}}((D_\alpha g)^*_\lambda)(x))^{\frac{1}{\lambda}}(1+|y|+\delta)^d\right\}\\ &\le C'\left\{\frac{1}{\delta^d}\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)+\delta^{\frac{1}{\lambda}}((D_\alpha g)^*_\lambda)(x))^{\frac{1}{\lambda}}\right\}(1+|y|+\delta)^d\\ &\le C'\left\{\frac{1}{\delta^d}\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)+\delta^{\frac{1}{\lambda}}((D_\alpha g)^*_\lambda)(x))^{\frac{1}{\lambda}}\right\}2^d(1+|y|)^d\end{aligned}$$

by $0<\delta<1$. This implies that

$$\begin{aligned}|g(x-y)| &\le C''\left\{\frac{1}{\delta^d}\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)+\delta^{\frac{1}{\lambda}}((D_\alpha g)^*_\lambda)(x))^{\frac{1}{\lambda}}\right\}^\lambda 2^{\lambda d}(1+|y|)^{\lambda d}\\ &\le C'''\left\{\frac{1}{\delta^{\lambda d}}\left(\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)\right)^\lambda+\delta(D_\alpha g)^*_\lambda)(x)\right\}(1+|y|)^{\lambda d},\end{aligned} \tag{3.33}$$

and thus

$$\frac{|g(x-y)|}{(1+|y|)^{\lambda d}}\le C'''\left\{\frac{1}{\delta^{\lambda d}}\left(\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)\right)^\lambda+\delta(D_\alpha g)^*_\lambda(x)\right\}.$$

Therefore, we have

$$g^*_\lambda(x)\le C'''\left\{\frac{1}{\delta^{\lambda d}}\left(\mathcal{M}(|g|^{\frac{1}{\lambda}})(x)\right)^\lambda+C_\alpha\delta g^*_\lambda(x)\right\} \tag{3.34}$$

by Lemma 3.9. Taking δ small enough such that $C'''C_\alpha\delta < \frac{1}{2}$ in the above formula, we get

$$g_\lambda^*(\cdot) \le C\left(\mathcal{M}(|g|^{\frac{1}{\lambda}})(\cdot)\right)^\lambda$$

a.e. on $\mathbb{R}^d$ for some constant due to the fact $g_\lambda^* < \infty$. This completes the proof. □

Lemma 3.12 *Let A be a $d \times d$ isotropic expansive matrix, $0 < p \le \infty$, and ϕ a band-limited function on $\mathbb{R}^d$. Suppose $f \in \mathcal{S}'(\mathbb{R}^d)$, and $\phi_{A^{-j}} * f \in L^p(\mathbb{R}^d)$. Then there exists a positive constant C such that*

$$\sup_{y\in\mathbb{R}^d} \frac{|(\phi_{A^{-j}} * f)(\cdot - y)|}{(1+|A^j y|)^{\lambda d}} \le C\left\{\mathcal{M}(|\phi_{A^{-j}} * f|^{\frac{1}{\lambda}})(\cdot)\right\}^\lambda \tag{3.35}$$

on $\mathbb{R}^d$ for $j \in \mathbb{Z}$, where λ is as in Lemma 2.2.

Proof For $j \in \mathbb{Z}$, write $g(\cdot) = (\phi_{A^{-j}} * f)(A^{-j}\cdot)$ a.e. on $\mathbb{R}^d$. Then g is band-limited and $g \in L^p(\mathbb{R}^d)$. Arbitrarily fix $x \in \mathbb{R}^d$. Observing that

$$\begin{aligned}\sup_{y\in\mathbb{R}^d} \frac{|(\phi_{A^{-j}} * f)(x - y)|}{(1+|A^j y|)^{\lambda d}} &= \sup_{y\in\mathbb{R}^d} \frac{|(\phi_{A^{-j}} * f)(x - A^{-j}y)|}{(1+|y|)^{\lambda d}} \\ &= g_\lambda^*(A^j x),\end{aligned} \tag{3.36}$$

we have

$$\sup_{y\in\mathbb{R}^d} \frac{|(\phi_{A^{-j}} * f)(x - y)|}{(1+|A^j y|)^{\lambda d}} \le C\left\{\mathcal{M}(|g|^{\frac{1}{\lambda}})(A^j x)\right\}^\lambda \tag{3.37}$$

by Lemmas 3.10 and 3.11. We only need to prove

$$\mathcal{M}(|g|^{\frac{1}{\lambda}})(A^j x) \le C\mathcal{M}(|\phi_{A^{-j}} * f|^{\frac{1}{\lambda}})(x) \tag{3.38}$$

to finish the proof. Next we prove (3.38). For any y satisfying $|y - A^j x| < \delta$, we have

$$\begin{aligned}|A^{-j}y - x| &= |A^{-j}(y - A^j x)| \\ &\le \lambda_1^{-1}\|A^{-j}(y - A^j x)\| \\ &= \lambda_1^{-1}\lambda^{-j}\|y - A^j x\| \\ &\le \lambda_2\lambda_1^{-1}\lambda^{-j}|y - A^j x| \\ &< \lambda_2\lambda_1^{-1}\lambda^{-j}\delta\end{aligned}$$

by (2.2) and Lemma 2.2. It follows that $A^{-j}\left(B(A^j x, \delta)\right) \subset B(x, \lambda_2\lambda_1^{-1}\lambda^{-j}\delta)$, and thus

$$\begin{aligned}\mathcal{M}(|g|^{\frac{1}{\lambda}})(A^j x) &= \sup_{\delta>0} \frac{1}{|B(0,\delta)|}\int_{B(A^j x,\delta)} |(\phi_{A^{-j}} * f)(A^{-j}y)|^{\frac{1}{\lambda}}dy \\ &\le \frac{1}{q^{-j}|B(0,\delta)|}\int_{B(x,\lambda_2\lambda_1^{-1}\lambda^{-j}\delta)} |(\phi_{A^{-j}} * f)(y)|^{\frac{1}{\lambda}}dy \\ &\le C\mathcal{M}\left(|\phi_{A^{-j}} * f|^{\frac{1}{\lambda}}\right)(x).\end{aligned} \tag{3.39}$$

This completes the proof. □

Lemma 3.13 *Let* $1 < p \leq \infty$*, let* A *be a* $d \times d$ *isotropic expansive matrix, and let* Ψ *be a finite subset of* $\mathcal{R}^0(\mathbb{R}^d)$*. Suppose that* $X(\Psi)$ *is an orthonormal basis for* $L^2(\mathbb{R}^d)$*. Then there exists a positive constant* C *such that*

$$\|\mathcal{T}_{\Psi,\lambda} f\|_p \leq C\|f\|_p$$

for $f \in L^p(\mathbb{R}^d)$*, where* λ *is as in Lemma 2.2.*

Proof Observing that $\Psi \subset L^1(\mathbb{R}^d)$, we have $\psi_{A^{-j}} * f \in L^p(\mathbb{R}^d)$ for $f \in L^p(\mathbb{R}^d)$, $j \in \mathbb{Z}$ and $\psi \in \Psi$. Thus, $\mathcal{T}_{\Psi,\lambda} f$ is well defined for $f \in L^p(\mathbb{R}^d)$, and

$$\begin{aligned}
\|\mathcal{T}_{\Psi,\lambda} f(\cdot)\|_p &= \left\| \left(\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} \sup_{y\in\mathbb{R}^d} \frac{|(\psi_{A^{-j}} * f)(\cdot - y)|^2}{(1+|A^j y|)^{2\lambda d}} \right)^{\frac{1}{2}} \right\|_p \\
&\leq C \left\| \left(\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} [\mathcal{M}(|\psi_{A^{-j}} * f|^{\frac{1}{\lambda}})(\cdot)]^{2\lambda} \right)^{\frac{1}{2}} \right\|_p \\
&= C \left\| \left(\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} [\mathcal{M}(|\psi_{A^{-j}} * f|^{\frac{1}{\lambda}})(\cdot)]^{2\lambda} \right)^{\frac{1}{2\lambda}} \right\|_{p\lambda}^{\lambda}
\end{aligned}$$

by Lemma 3.12. Then applying Lemma 3.1 and Theorem 1.2, we obtain

$$\begin{aligned}
\|\mathcal{T}_{\Psi,\lambda} f\|_p &\leq C' \left\| \left\{ \sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} |\psi_{A^{-j}} * f|^2 \right\}^{\frac{1}{2\lambda}} \right\|_{p\lambda}^{\lambda} \\
&\leq C'' \left\| \sum_{\psi\in\Psi} g_\psi f \right\|_p \\
&\leq C''' \|f\|_p.
\end{aligned}$$

This completes the proof. □

Combining Lemmas 3.8 and 3.13, we have:

Lemma 3.14 *Let* $1 < p < \infty$*, let* A *be a* $d \times d$ *isotropic expansive matrix, and let* Ψ *be a finite subset of* $\mathcal{R}^0(\mathbb{R}^d)$ *whose every element is band-limited. Suppose that* $X(\Psi)$ *is an orthonormal basis for* $L^2(\mathbb{R}^d)$*. Then there exist constants* $0 < c \leq C < \infty$ *such that*

$$c\|f\|_p \leq \|\mathcal{W}_\Psi f\|_p \leq C\|f\|_p \tag{3.40}$$

for $f \in L^p(\mathbb{R}^d)$*.*

Proof Since $\Psi \subset \mathcal{R}^0(\mathbb{R}^d)$ implies $\tilde{\Psi} \subset \mathcal{R}^0(\mathbb{R}^d)$, the right-hand side inequality in (3.40) is an immediate consequence of Lemmas 3.8 and 3.13. Next we prove the left-hand side inequality. Observe that $\|\mathcal{W}_\Psi f\|_2 = \|f\|_2$ for $f \in L^2(\mathbb{R}^d)$. Write

$$\Omega_\Psi f(x) = \{\langle f, \psi_{j,k}\rangle q^{\frac{j}{2}} \chi_{\Lambda_{j,k}}(x) : (\psi, j, k) \in \Psi \times \mathbb{Z} \times \mathbb{Z}^d\}$$

for a.e. $x \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$. Then by the polarization identity and a density argument, we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^d} \langle \Omega_\Psi f(x), \, \Omega_\Psi g(x)\rangle_{l^2(\Psi\times\mathbb{Z}\times\mathbb{Z}^d)}dx$$

for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $\frac{1}{p}+\frac{1}{q}=1$. This implies that

$$\begin{aligned} \|f\|_p &= \sup\left\{\left|\int_{\mathbb{R}^d} \langle \Omega_\Psi f(x), \, \Omega_\Psi g(x)\rangle_{l^2(\Psi\times\mathbb{Z}\times\mathbb{Z}^d)}dx\right| : \, \|g\|_q \le 1\right\} \\ &\le \sup\left\{\int_{\mathbb{R}^d} \mathcal{W}_\Psi f(x)\mathcal{W}_\Psi g(x)dx : \, \|g\|_q \le 1\right\} \\ &\le \sup\left\{\|\mathcal{W}_\Psi f\|_p \|\mathcal{W}_\Psi g\|_q : \, \|g\|_q \le 1\right\} \\ &\le C\|\mathcal{W}_\Psi f\|_p \end{aligned}$$

by applying Lemma 3.14 to g. This completes the proof. □

Proof of Theorem 1.3 Choose a finite subset Φ of $\mathcal{R}^0(\mathbb{R}^d)$ with the same cardinality to Ψ such that the elements of Φ are all band-limited, and $X(\Phi)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Then there exist positive constants C_1 and C_2 such that

$$C_1\|\mathcal{W}_\Phi f\|_p \le \|\mathcal{W}_\Psi f\|_p \le C_2\|\mathcal{W}_\Phi f\|_p$$

for $f \in L^p(\mathbb{R}^d)$ by Lemma 3.7. It leads to the theorem by Lemma 3.14.

Proof of Theorem 1.1 We first show that $X(\Psi)$ is a basis for $L^p(\mathbb{R}^d)$. Arbitrarily fix $f \in L^p(\mathbb{R}^d)$. Define

$$\mathcal{S}_{N,M}f(\cdot) = \sum_{\psi\in\Psi}\sum_{|j|\le N, |k|\le M} \langle f, \, \psi_{j,k}\rangle \psi_{j,k}(\cdot)$$

for $N, \, M \in \mathbb{N}$. Observe that

$$\langle \mathcal{S}_{N,M}f, \, \psi_{j_0,k_0}\rangle = \begin{cases} \langle f, \, \psi_{j_0,k_0}\rangle & \text{if } |j_0| \le N, |k_0| \le M, \\ 0 & \text{otherwise,} \end{cases}$$

for each $(\psi, \, j_0, \, k_0) \in \Psi\times\mathbb{Z}\times\mathbb{Z}^d$. It follows that

$$\mathcal{W}_\Psi \mathcal{S}_{N,M}f(\cdot) = \left\{\sum_{\psi\in\Psi}\sum_{|j|\le N, |k|\le M} |\langle f, \, \psi_{j,k}\rangle|^2 q^j \chi_{\Lambda_{j,k}}(\cdot)\right\}^{\frac{1}{2}}, \tag{3.41}$$

and thus $\|\mathcal{W}_\Psi f - \mathcal{W}_\Psi S_{N,M}f\|_p \to 0$ as $N, M \to \infty$ by the Lebesgue dominated convergence theorem and the fact that $\mathcal{W}_\Psi f \in L^p(\mathbb{R}^d)$, which is derived from Theorem 1.3. Also, we have

$$\|f - \mathcal{S}_{N,M}f\|_p \le C\|\mathcal{W}_\Psi f - \mathcal{W}_\Psi \mathcal{S}_{N,M}(f))\|_p$$

by Theorem 1.3. Therefore, $\lim_{N,M\to\infty} \|f - \mathcal{S}_{N,M}(f)\|_p = 0$, i.e.

$$f = \sum_{\psi\in\Psi}\sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d} \langle f, \, \psi_{j,k}\rangle \psi_{j,k} \tag{3.42}$$

in $L^p(\mathbb{R}^d)$. Now suppose f has another expression:

$$f = \sum_{\psi\in\Psi} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d} c_{\psi,j,k}\psi_{j,k}$$

in $L^p(\mathbb{R}^d)$. Also observing that $\psi \in L^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$ for $\psi \in \Psi$, we have $\langle f, \psi_{j,k}\rangle = c_{\psi,j,k}$ for $(\psi, j, k) \in \Psi \times \mathbb{Z} \times \mathbb{Z}^d$; that is, the expression (3.42) is unique for every $f \in L^p(\mathbb{R}^d)$. Therefore, $X(\Psi)$ is a basis for $L^p(\mathbb{R}^d)$.

Next we prove that the basis $X(\Psi)$ is an unconditional one. By Lemma 3.2, we only need to prove the existence of a constant C such that

$$\|\mathcal{S}_\beta f\|_p \le C\|f\|_p \tag{3.43}$$

for $f \in L^p(\mathbb{R}^d)$ and $\beta \in l_0(\Psi \times \mathbb{Z} \times \mathbb{Z}^d)$ satisfying $\beta_{\psi,j,k} = 1$ on its support, where

$$\mathcal{S}_\beta f = \sum_{\psi\in\Psi} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d} \beta_{\psi,j,k}\langle f, \psi_{j,k}\rangle\psi_{j,k}.$$

It is easy to check that

$$0 \le \mathcal{W}_\Psi\mathcal{S}_\beta f = \left(\sum_{\psi\in\Psi} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^d} \beta_{\psi,j,k}|\langle f, \psi_{j,k}\rangle|^2 q^j \chi_{\Lambda_{j,k}}\right)^{\frac{1}{2}} \le \mathcal{W}_\Psi f.$$

Thus,

$$\|\mathcal{S}_\beta f\|_p \le c^{-1}\|\mathcal{W}_\Psi\mathcal{S}_\beta f\|_p \le c^{-1}\|\mathcal{W}_\Psi f\|_p \le c^{-1}C\|f\|_p$$

by Theorem 1.3; that is, (3.43) holds. The proof is completed.

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