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A Taylor operation method for solutions of generalized pantograph type delay differential equations

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Abstract: In this paper, a new operational matrix method based on the Taylor polynomials is presented to solve generalized pantograph type delay differential equations. The method is based on operational matrices of integration and product for Taylor polynomials. These matrices are obtained by using the best approximation of function by the Taylor polynomials. The advantage of the method is that the method does not require collocation points. By using the proposed method, the generalized pantograph equation problem is reduced to a system of linear algebraic equations. The solving of this system gives the coefficients of our solution. Numerical examples are given to demonstrate the accuracy of the technique and the numerical results show that the error of the present method is superior to that of other methods. The residual correction technique is used to improve the accuracy of the approximate solution and estimation of absolute error. The estimation aspect of the residual method is useful when the exact solution of the considered equation is unknown and this feature of the method is shown in numerical examples.

Key words: Pantograph equation, delay dif ferential equations, Taylor operation method, inner product, error estimation, residual correction.

1. Introduction

Delay differential equations describe the models of phenomenons with delay effects such as stress-strain states of materials, population dynamics, and so on [9].

The studying of numerical aspects of functional equations constitutes a large and very important branch of modern mathematics. Developing computer technologies give the opportunity to consider various functional equations, obtain numerical results, and visualize graphics of approximate solutions. In recent years, many techniques have been applied to a large class of problems. In particular, generalized pantograph equations are numerically solved by using the Adomian decomposition method [7], the Taylor method is utilized in [14], and the Bessel matrix method based on collocation points is applied in [19]. Hermite polynomials have been used to find the approximate solutions of pantograph equations [16]. In [8], Keskin et al. applied the differential transform method to get the Taylor expansion of exact solutions. Additionally, the approximate solutions of generalized pantograph equations were obtained by using the homotopy method and variational iteration method in [12, 18], respectively. Brunner et al. [4] used the Galerkin methods for solutions of delayed differential equations of pantograph type. In [1, 13, 15, 20], the collocation methods were developed for functional equations with delays.

Operational matrices of integration, derivation, and product for Bernstein, Jacobi, and Chebyshev

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polynomials were presented in [2, 6, 10, 17]. Our aim is to derive operational matrices of integration and product for Taylor polynomials and utilize them for solving pantograph type differential equations. To determine these matrices we use the best approximation by Taylor polynomials and our product matrix is defined, unlike in the above papers.

In this paper, we propose a method based on operational matrices for Taylor polynomials to numerically solve the generalized pantograph equations

$$u^{(n)}(x) = f(x)\, u(x) + \sum_{i=0}^{n-1} h_i(x)\, u^{(i)}(\alpha_i x + \beta_i) + g(x) \quad (x \in [0,1]), \tag{1}$$

with initial conditions

$$u^{(i)}(0) = \lambda_i \;\; (i = 0, 1, ..., n-1). \tag{2}$$

Here, α_i, β_i are positive constants and $h_i(t)$ and $g(x)$ are continuous functions on $[0,1]$.

This paper is organized as follows. In Section 2.1, the operational matrices of Taylor polynomials are constructed. In Section 2.2, we obtain the operational matrix of integration for Taylor polynomials. We form the operational matrix of production based on the Taylor polynomials in Section 2.3. In Section 3, a numerical scheme is presented for the solution of generalized pantograph type delay differential equations by using relations in Section 2. In Section 4, the residual error estimation and correction method are discussed. In Section 5, the proposed method is applied to examples, and comparisons are made with the existing numerical solutions reported in other published works in the literature. Finally, conclusions are given in Section 6.

2. Operational matrices based on Taylor polynomials

In this section, we derive operational matrices of the Taylor polynomials and operational matrices of integration and product by using the best approximation by Taylor polynomials.

2.1. Operational matrix of Taylor polynomials

In this section, we construct the operational matrices of Taylor polynomials. Let $H = L^2[0,1]$ and suppose that any given $f \in H$ is to be approximated by $y \in Y$, where $Y = Span\{1, x, x^2, ..., x^N\}$. Since Y is a finite dimensional vector space, f has the unique best approximation out of $y \in Y$. Now, any element $y_0 \in Y$ is uniquely determined by coefficients $\mathbf{A} = [a_0, a_1, ..., a_N]$ such that

$$y_0 = \sum_{i=0}^{N} a_i x^i = \mathbf{A}\mathbf{X}(x),$$

where $\mathbf{X}(x) = [1\ x\ x^2 ...\ x^N]^T$. Let y_0 be the best approximation of f; that is, $\|f - y_0\|_2 \le \|f - y\|_2$ for all $y \in Y,$ where $\|f\|_2^2 = \langle f, f\rangle$. Then $\mathbf{A}$ can be obtained by

$$\mathbf{A}\langle \mathbf{X}, \mathbf{X}\rangle = \langle f, \mathbf{X}^T\rangle,$$

where

$$\langle f, \mathbf{X}\rangle = \int_0^1 f(x)\, \mathbf{X}(x)^T dx = [\langle f, 1\rangle,\ \langle f, x\rangle,\ ...,\ \langle f, x^N\rangle]$$

and $\langle \mathbf{X}, \mathbf{X} \rangle$ is an $(N+1) \times (N+1)$ matrix. Let

$$\mathbf{Q} = \langle \mathbf{X}, \mathbf{X} \rangle = \int_0^1 \mathbf{X}(x)\, \mathbf{X}(x)^T \, dx,$$

then

$$\mathbf{A} = \left(\int_0^1 f(x)\, \mathbf{X}(x)^T \, dx \right) \mathbf{Q}^{-1}. \tag{3}$$

Now let us determine elements of matrix $\mathbf{Q}$:

$$\begin{aligned} \mathbf{Q} &= [Q_{i,j}] = \left[\int_0^1 x^{i-1} x^{j-1} dx \right] \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1} \end{bmatrix} \end{aligned}$$

where $i, j = 1, ..., N+1$, which is well known as the Hilbert matrix.

2.2. Operational matrix of integration for Taylor polynomials

In this subsection, we obtain the operational matrix of integration for Taylor polynomials. Suppose that $\mathbf{P}$ is an $(N+1) \times (N+1)$ operational matrix of integration; then $\int_0^x \mathbf{X}(t)\, dt \simeq \mathbf{P}\mathbf{X}(x)$ where $0 \leq x \leq 1$.

Integrating components of $\mathbf{X}(t)$, we have following product of matrices:

$$\int_0^x \mathbf{X}(t)\, dt = \mathbf{\Lambda} \mathbf{X}_1(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N+1} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{N+1} \end{bmatrix}.$$

Components of $\mathbf{X}_1(x)$ can be expressed in terms of the basis set $\mathbf{X}(x)$. We can write $\mathbf{X}(x) = \mathbf{I}\mathbf{X}(x)$, where $\mathbf{I}$ is an $(N+1) \times (N+1)$ identity matrix. Then $x^{i-1} = \mathbf{I}_i \mathbf{X}(x)$ $(i = 1, 2, ..., N+1)$, where $\mathbf{I}_i$ is the ith row of $\mathbf{I}$. For $f(x) = x^{N+1}$, by (2), we have $x^{N+1} \simeq \mathbf{A}_{N+1} \mathbf{X}(x)$. Now, substituting the expressions for x^i $(i = 1, 2, ..., N+1)$ into $\mathbf{X}_1(x)$, we have $\int_0^x \mathbf{X}(t)\, dt \simeq \mathbf{\Lambda} \mathbf{L} \mathbf{X}(x)$, where $\mathbf{L} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{I}_3 \\ \vdots \\ \mathbf{I}_{N+1} \\ \mathbf{A}_{N+1} \end{bmatrix}$ is an $(N+1) \times (N+1)$ matrix. Finally, we get the operational matrix of integration in the following form:

$$\mathbf{P} = \mathbf{\Lambda} \mathbf{L}. \tag{4}$$

2.3. Operational matrix of product for Taylor polynomials

In this subsection, we get the operational matrix of product, which is very useful when the considered functional equation has variable coefficients. Letting $\alpha(x) \in L^2[0,1]$ be any given function, then $\mathbf{C}$ is an $(N+1)\times(N+1)$ operational matrix of product and the relation

$$\alpha(x)\mathbf{X}(x) \simeq \mathbf{C}\mathbf{X}(x)$$

holds.

Now we want to approximate elements of product $\alpha(x)\mathbf{X}(x)$ by the scheme given at the beginning this section.

By (2), $x^{i-1}\alpha(x) \simeq \mathbf{E}_i\mathbf{X}(x)$ $(i=1,2,...,N+1)$, and then

$$\alpha(x)\mathbf{X}(x) = \left[\alpha(x)\ \ x\alpha(x)\ \ ...\ x^N\alpha(x)\right]^T = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_{N+1} \end{bmatrix} \mathbf{X}(x).$$

Therefore, we have

$$\mathbf{C} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_{N+1} \end{bmatrix}. \tag{5}$$

In the next section, we utilize operational matrices for solving generalized pantograph type DDEs.

3. Solution method

In this section, the present method takes advantage of operational matrices. Let equation (1) have a unique solution that can be expressed by the power series expansion in terms of the Taylor polynomials

$$u(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Our aim is to find an approximate solution of equation (1) expressed in the Taylor polynomial form. Let us approximate the derivative as

$$u^{(n)}(x) \simeq u_N^{(n)}(x) = \mathbf{A}\mathbf{X}(x), \tag{6}$$

and then by using the operational matrix of integration $\mathbf{P}$ and initial conditions λ_k we have

$$u(x) \simeq u_N(x) = \mathbf{A}\mathbf{P}^n\mathbf{X}(x) + \sum_{k=0}^{n-1} \frac{\lambda_k x^k}{k!}, \tag{7}$$

where $\mathbf{A} = [a_0\ a_1\ ...\ a_N]$ is a matrix of unknown coefficients. Unknown functions with delays are also expressed in the following form:

$$u^{(i)}(\alpha_i x + \beta_i) \simeq u_N^{(i)}(\alpha_i x + \beta_i) = \mathbf{A}\mathbf{P}^{n-i}\mathbf{Q}_i\mathbf{X}(x) + \sum_{k=0}^{n-1} \frac{\lambda_k (\alpha_i x + \beta_i)^k}{k!},$$

so that

$$\mathbf{Q}_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \beta_i & \alpha_i & 0 & \cdots & 0 \\ \beta_i^2 & 2\alpha_i\beta_i & \alpha_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_N^N\beta_i^N & C_N^{N-1}\beta_i^{N-1}\alpha_i & C_N^{N-2}\beta_i^{N-2}\alpha_i^2 & \cdots & C_N^0\alpha_i^N \end{bmatrix},$$

where C_N^k $(k = 0, ..., N)$ are binomial coefficients. The given functions $h_i(x)$ in equation (1) are approximated according to Section 2 such as $h_i(x) \simeq \mathbf{H}_i\mathbf{X}(x)$. Substituting the obtained expressions into equation (1), we get:

$$\mathbf{AX}(x) = f(x)\left[\mathbf{AP}^n\mathbf{X}(x) + \sum_{k=0}^{n-1}\frac{\lambda_k x^k}{k!}\right] + \sum_{i=0}^{n-1}h_i(x)\left[\mathbf{AP}^{n-i}\mathbf{Q}_i\mathbf{X}(x) + \sum_{k=0}^{n-1}\frac{\lambda_k(\alpha_i x+\beta_i)^k}{k!}\right] + g(x)$$

or

$$\begin{aligned} \mathbf{AX}(x) &= \mathbf{AP}^n f(x)\mathbf{X}(x) + \sum_{i=0}^{n-1}\mathbf{AP}^{n-i}\mathbf{Q}_i h_i(x)\mathbf{X}(x) \\ &\quad + \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}h_i(x)\frac{\lambda_k(\alpha_i x+\beta_i)^k}{k!} + \sum_{k=0}^{n-1}f(x)\frac{\lambda_k x^k}{k!} + g(x). \end{aligned} \tag{1}$$

Now, using the product matrix for $f(x)\mathbf{X}(x) = \mathbf{CX}(x)$ and $h_i(x)\mathbf{X}(x) = \mathbf{C}_i\mathbf{X}(x)$ and then the approximate function $f_1(x) = \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}h_i(x)\frac{\lambda_k(\alpha_i x+\beta_i)^k}{k!} + \sum_{k=0}^{n-1}f(x)\frac{\lambda_k x^k}{k!} + g(x) = \mathbf{F}_1\mathbf{X}(x)$ and substituting into (8), we get

$$\mathbf{AX}(x) = \mathbf{AP}^n\mathbf{CX}(x) + \sum_{i=0}^{n-1}\mathbf{AP}^{n-i}\mathbf{Q}_i\mathbf{C}_i\mathbf{X}(x) + \mathbf{F}_1\mathbf{X}(x).$$

Removing $\mathbf{X}(x)$ from the last equation, we can find unknown coefficients as

$$\mathbf{A} = \mathbf{F}_1\left(\mathbf{I} - \mathbf{P}^n\mathbf{C} - \sum_{i=0}^{n-1}\mathbf{P}^{n-i}\mathbf{Q}_i\mathbf{C}_i\right),$$

and using (7) we obtain the approximate solution.

4. Error estimation and residual correction

This section is devoted to estimation of error function $e_N(x) = u(x) - u_N(x)$ and employing the residual correction function for increased accuracy of the approximate solution of equation (1). The residual correction method was used in [5, 11, 21]. Consider the residual correction function

$$R(x) = u^{(n)}(x) - f(x)u(x) - \sum_{i=0}^{n-1}h_i(x)u^{(i)}(\alpha_i x+\beta_i) - g(x) = 0 \tag{9}$$

of equation (1). In the last equation, replacing $u(x)$ by approximate solution $u_N(x)$, we obtain

$$R_N(x) = u_N^{(n)}(x) - f(x)\,u_N(x) - \sum_{i=0}^{n-1} h_i(x)\,u_N^{(i)}(\alpha_i x + \beta_i) - g(x). \tag{10}$$

Now, subtracting (10) from (9), we have the following equation:

$$e_N^{(n)}(x) = f(x)\,e_N(x) + \sum_{i=0}^{n-1} h_i(x)\,e_N^{(i)}(\alpha_i x + \beta_i) - R_N(x), \tag{11}$$

with initial condition $e_N^{(i)}(0) = u^{(i)}(0) - u_N^{(i)}(0)$ $(i = 0, ..., n-1)$. That is a generalized pantograph equation, too. Then, applying the present method for equation (11), we get the approximate solution $e_{N,M}(x)$ for some M. Finally, we obtain the corrected solution as

$$u_{NM}(x) = u_N(x) + e_{N,M}(x). \tag{12}$$

Corrected actual error $E_{N,M}(x)$ is defined by $E_{N,M}(x) = u(x) - u_{N,M}(x)$.

5. Numerical examples

In this section, we apply the present method and residual correction technique for some pantograph type delay differential equations and numerical results obtained by the proposed method compared with the data of other methods.

Example 1 *Let us consider the following problem [6]:*

$$u'''(x) = -u(x) - u(x - 0.3) + \exp(-x + 0.3) \tag{13}$$

with initial conditions $u''(0) = 1,\ u'(0) = -1,\ u(0) = 1.$

The exact solution of the problem is $u(x) = \exp(-x)$. According to the present method, let us approximate the third derivative of the unknown function as $u_4'''(x) = \mathbf{AX}(x)$. Employing the operational matrix of integration and initial conditions, we have

$$\begin{aligned} u_4''(x) &= \mathbf{APX}(x) + 1, \\ u_4'(x) &= \mathbf{AP}^2\mathbf{X}(x) + x - 1, \\ u_4(x) &= \mathbf{AP}^3\mathbf{X}(x) + \frac{x^2}{2} - x + 1, \end{aligned} \tag{2}$$

and for the function with delay argument

$$u_4(x - 0.3) = \mathbf{AP}^3\mathbf{Q}_1\mathbf{X}(x) + \frac{(x-0.3)^2}{2} - (x - 0.3) + 1.$$

Here,

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 & 0.25 \\ 7.936E-4 & -2.38E-2 & 0.166 & -0.444 & 0.499 \end{bmatrix}$$

is a matrix of integration,

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.3 & 1 & 0 & 0 & 0 \\ 0.09 & -0.6 & 1 & 0 & 0 \\ 0.027 & 0.27 & -0.899 & 1 & 0 \\ 0.0081 & -0.108 & 0.54 & -1.2 & 1 \end{bmatrix}$$

appears in the function with delay argument, and $g(x) = \exp(-x+0.3) - \left[\frac{(x-0.3)^2}{2} - (x-0.3) + 1\right] - \left[\frac{x^2}{2} - x + 1\right]$ is approximated as $g(x) = \mathbf{G}\mathbf{X}(x)$, where

$$\mathbf{G} = [-9.951E-1,\ 9.51E-1,\ -3.313E-1,\ -2.073E-1,\ 3.45E-2].$$

Now, substituting the obtained expressions into equation (13) and removing $\mathbf{X}(x)$, we find the unknown coefficients as

$$\mathbf{A} = \mathbf{G}\left(\mathbf{I} + \mathbf{P}^3 + \mathbf{P}^3\mathbf{Q}_1\right)^{-1},$$

and using (14), we get approximation solution $u_4(x) = 0.025613x^4 - 0.153725x^3 + 0.495452x^2 - 0.999373x + 0.999979$. Note that the maximum error of $u_4(x)$ is $6.643E-5$.

Now we want to employ the residual correction technique to increase the accuracy of the approximate solution.

Using equation (10) we find $R_4(x) = 0.495452x^2 - exp(0.3-x) - 1.384031x - 0.153725x^3 + 0.025613x^4 + 0.495452(x-0.3)^2 - 0.153725(x-0.3)^3 + 0.025613(x-0.3)^4 + 1.377416$. In our case, equation (11) has the following form:

$$e'''(x) = -e(x) - e(x-0.3) - R_4(x)$$

with initial conditions $e''(0) = 0.009$, $e'(0) = -0.0006$, $e(0) = 0.00005$. Applying the present method for $M = 5$ and adding the obtained solution to $u_4(x)$, we find the corrected solution as $u_{45} = -0.00511x + 0.038283x - 0.164913x + 0.499595x - 0.99996x + 0.999999$, for which the maximum error is $1.346E-5$.

It can be seen from Table 1 and Figure 1 that the present method's errors are superior to those of others [6,17]. Figure 2 shows the effect of the residual correction technique. In Figure 3, we give the graphics of the error functions for N = 8, 10. It is observed from Figure 3 that the errors decrease while the value of N increases.

Table 1. Comparison of absolute errors $e_N(x) = |u(x) - u_N(x)|$ for Example 1.

x	Hermite collocation method [6]	Jacobi rational-Gauss method [17]	Present method	
	N=8	N=26	N=8	N=10
0.2	6.2E-9	3.605E-8	6.777E-11	2.313E-13
0.4	5.76E-8	9.299E-9	3.312E-10	1.168E-12
0.6	1.796E-7	3.503E-10	8.428E-10	2.775E-12
0.8	3.735E-7	8.345E-9	1.543E-9	5.125E-12
1.0	6.368E-7	1.161E-8	2.491E-9	8.123E-12

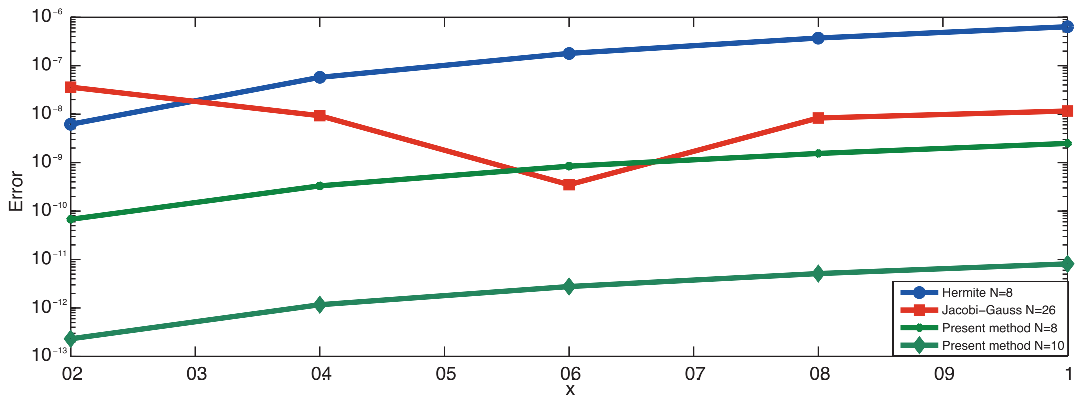

Figure 1. Comparison of $|e_N(x)|$ for Example 1.

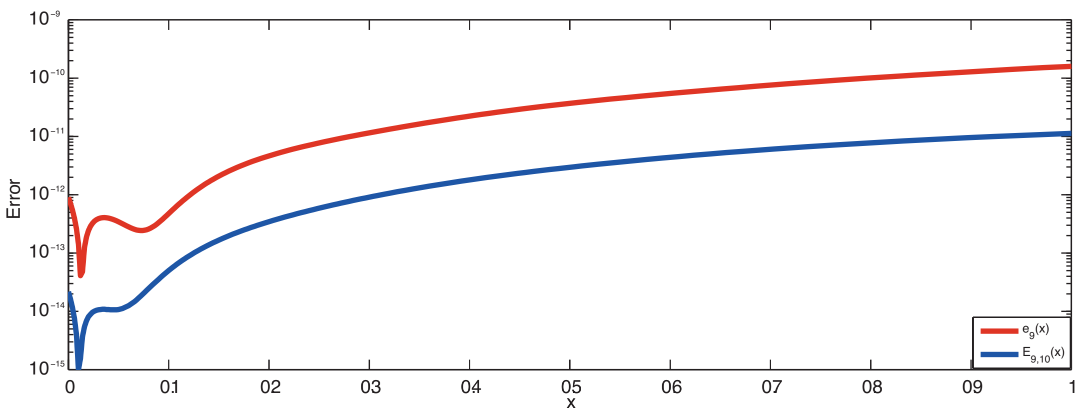

Figure 2. Graphics of $|e_9(x)|$ and $|E_{9,10}(x)|$ for Example 1.

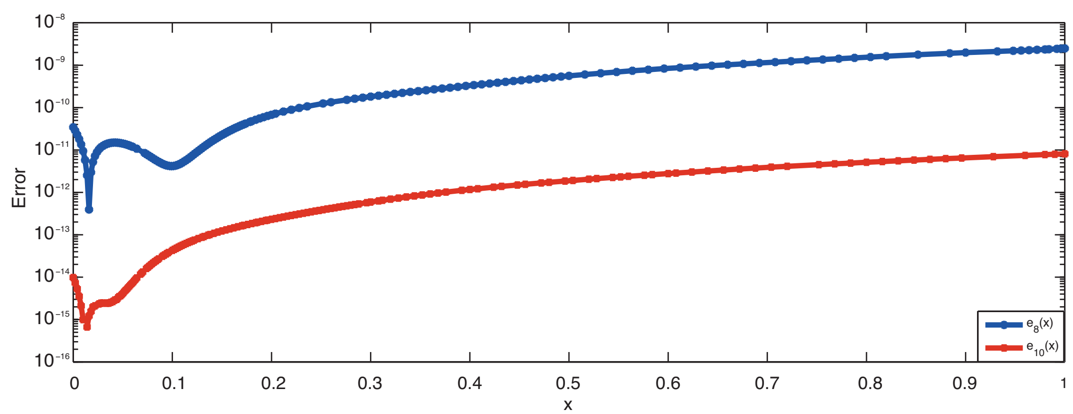

Figure 3. Comparison of $|e_N(x)|$ for N=8 and N=10, Example 1.

Example 2 *Let us consider the following differential equation with variable coefficients [5]:*

$$u'''(x) = xu''(2x) - u'(x) - u(0.5x) + x\cos 2x + \cos 0.5x$$

with initial conditions $u''(0) = -1,\ u'(0) = 0,\ u(0) = 1.$

The exact solution of problem is $u(x) = \cos x$.

In Table 2 and Figure 4, the absolute errors of the present method for $N = 8$ are compared with the absolute errors of the exponential approximation with the residual correction for $N = 8, M = 9$ and the Taylor method [6] for $N = 8$. Errors of the present method without the residual correction are better than those of the exponential approximation method [5]. Also, it is seen from Table 2 and Figure 4 that the results of our method are better than those of the Taylor method [6].

Table 2. Comparison of absolute errors $e_N(x) = |u(x) - u_N(x)|$ for Example 2.

x	Exponential approximation method [5]	Taylor method [6]	Present method
	N=8, M=9	N=8	N=8
0.2	5.167E-6	8.00E-8	1.3E-12
0.4	2.506E-5	5.12E-6	5.453E-11
0.6	5.847E-5	5.832E-5	1.607E-9
0.8	9.683E-5	3.277E-4	1.442E-8
1.0	1.119E-4	1.25E-3	6.555E-8

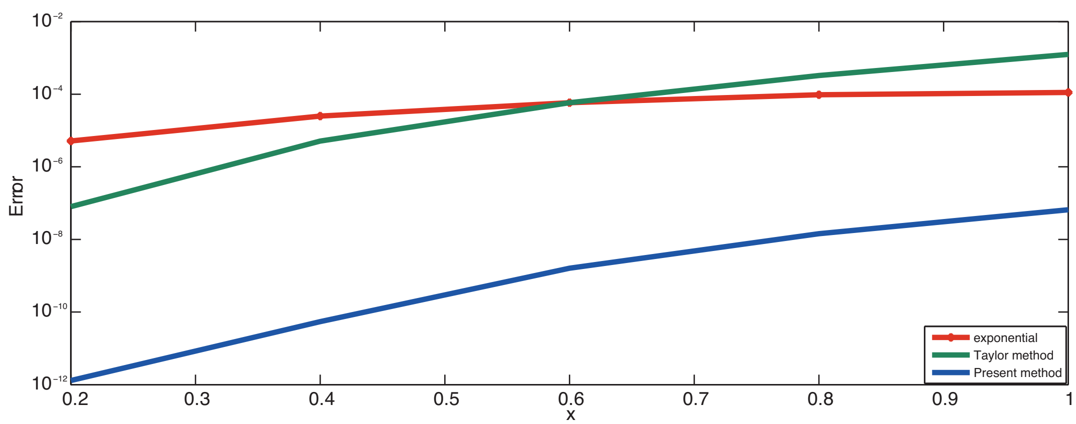

Figure 4. Comparison of $|e_N(x)|$ for Example 2.

Example 3 *Consider the following multi-pantograph equation [14]:*

$$u'(x) = -u(x) - u(0.8x)$$

with initial conditions $u(0) = 1$.

The exact solution of the problem is $u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \left(-1 - 0.8^{i-1}\right) x^k$ (see [3]).

The problems were considered in [13, 16] without error estimation. In this case, we can see the main advantage of the residual correction technique, which is that we can estimate the error of the approximate solution when the exact solution is not of closed form. Table 3 shows that $u_5(x)$ is a good approximation for the solution of Example 3.

Example 4 *Our last problem is*

$$u''(x) = 0.75u(x) + u(0.5x) - \frac{0.75}{x^2 + x + 1} - \frac{4}{x^2 + 2x + 4} + \frac{2(2x+1)^2}{(x^2 + x + 1)^3} - \frac{2}{(x^2 + x + 1)^2}$$

with initial conditions $u(0) = 1,\ u'(0) = -1$.

Table 3. Numerical results of estimated error $e_{5,5}(x)$ and values of $u_5(x)$ for Example 3.

x	Estimated error $\|e_{5,5}(x)\|$	Values of $u_5(x)$
0.2	7.392E-16	0.664688
0.4	5.229E-16	0.4335642
0.6	3.654E-16	0.27648307
0.8	9.907E-16	0.1714817
1.0	1.586E-15	0.1026574

The exact solution of the problem is $u(x) = \frac{1}{x^2+x+1}$.

Figure 5 shows the graphic of the error function for $N = 14$. In Table 4, a comparison is made between values of the approximate solutions and the exact solution. The maximum error is $1.898E - 10$.

Table 4. Exact and approximate values of Example 4.

x	Exact solution	Approximate solution $u_{14}(x)$	Absolute error
0.2	0.080645161290	0.80645161309	1.898E-10
0.4	0.641025641	0.6410256409	4.012E-11
0.6	0.5102040816	0.51020408145	1.821E-10
0.8	0.4098360655	0.40983606541	1.62E-10
1.0	0.90090090092	0.90090090091	6.324E-11

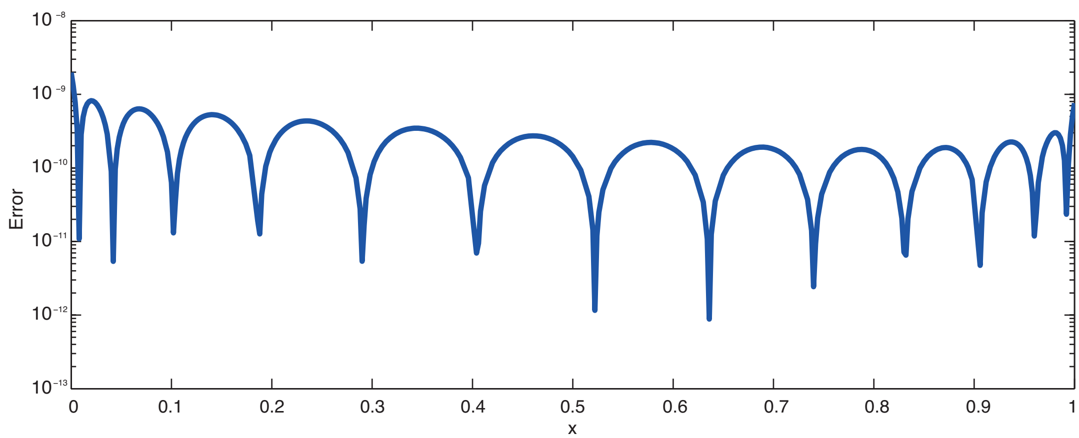

Figure 5. Graphic of $|e_{14}(x)|$ for Example 4.

6. Conclusions

In this study, a new Taylor operation method based on Taylor polynomials is presented. The advantage of the present method over other operational matrix methods is the chosen simple basis set $\mathbf{X}$ and its elements can be easily shown by the identity matrix. Also note that the integration matrix $\mathbf{P}$ mainly consist of zeros, which involve less computational work. Note that our product matrix $\mathbf{C}$ is defined more easily than in [10, 17]. In addition, the transforming of the differential equation to a system of algebraic equations is done without collocation points. Excellent agreement and high accuracy are achieved even with a small number of basis polynomials. Also, the residual correction technique is employed to increase the accuracy of the approximate solution and the estimate error of the approximate solution. It is seen from Example 3 that error estimation of the approximate solution is good even when the exact solution of the problem is unknown. Numerical results obtained by this method are compared with various numerical methods and existing exact solutions. It is observed from the tables and figures that our results are better than the results of the other compared methods.

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