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On the Fekete–Szegö type functionals for starlike and convex functions

Pawel ZAPRAWA*
Department of Mathematics, Faculty of Mechanical Engineering, Lublin University of Technology, Lublin, Poland

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Abstract: In the paper we discuss two functionals of the Fekete–Szegö type: $\Phi_f(\mu) = a_2a_4 - \mu {a_3}^2$ and $\Theta_f(\mu) = a_4 - \mu a_2a_3$ for an analytic function $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, $z \in \Delta$, ($\Delta = \{z \in \mathbb{C} : |z| < 1\}$) and a real number μ. We focus our research on the estimation of $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$, while f is either in $\mathcal{S}^*$ (the class of starlike functions) or in $\mathcal{K}$ (the class of convex functions).

Key words: Starlike functions, convex functions, Hankel determinant, functional of Fekete–Szegö type

1. Introduction

For a function f analytic in $\Delta \equiv \{z \in \mathbb{C} : |z| < 1\}$ having the power series expansion

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots \tag{1}$$

we define two functionals for a fixed real μ:

$$\Phi_f(\mu) \equiv a_2a_4 - \mu {a_3}^2 \tag{2}$$

and

$$\Theta_f(\mu) \equiv a_4 - \mu a_2a_3. \tag{3}$$

The functionals Φ_f and Θ_f are generalizations of two expressions: $a_2a_4 - {a_3}^2$ and $a_4 - a_2a_3$. The first one is known as the second Hankel determinant and it was examined in many papers. The investigation of Hankel determinants for analytic functions was started by Pommerenke (see [11, 12]). Following Pommerenke, many mathematicians published their results concerning the second Hankel determinant for various classes of univalent functions (see, for example, [2, 3, 5, 8, 10]) or multivalent functions (see [9]). The bounds of $a_2a_4 - {a_3}^2$ for typically real functions were presented in [14].

In this paper, Φ_f and Θ_f are called the Fekete–Szegö type functionals because in a similar way the expression $a_3 - {a_2}^2$ was generalized to obtain the Fekete–Szegö functional $a_3 - \mu {a_2}^2$.

Let us recall the result reported by Janteng *et al.*

Theorem 1.1 (Janteng, Halim, Darus, [4]) *The following bounds are sharp*

*Correspondence: p.zaprawa@pollub.pl
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1. *If* $f \in \mathcal{S}^*$, *then* $|a_2a_4 - {a_3}^2| \le 1$.

2. *If* $f \in \mathcal{K}$, *then* $|a_2a_4 - {a_3}^2| \le 1/8$.

The functional $a_4 - a_2a_3$ has not been discussed very often. The results for functions in $\mathcal{S}^*$ and $\mathcal{K}$ are the following.

Theorem 1.2 (Babalola, [1]) *The following bounds are sharp*

1. *If* $f \in \mathcal{S}^*$, *then* $|a_4 - a_2a_3| \le 2$.

2. *If* $f \in \mathcal{K}$, *then* $|a_4 - a_2a_3| \le 4/9\sqrt{3}$.

It is worth stating that $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$ are invariant under rotations. If f is given by (1) and $\tilde{f}(z) = e^{-i\varphi}f(ze^{i\varphi})$, $\varphi \in \mathbb{R}$, then $\tilde{f}(z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\varphi} z^n$. Hence

$$|\Phi_{\tilde{f}}(\mu)| = \left| a_2 e^{i\varphi} \cdot a_4 e^{3i\varphi} - \mu \cdot \left(a_3 e^{2i\varphi}\right)^2 \right| = |\Phi_f(\mu)| \tag{4}$$

and

$$|\Theta_{\tilde{f}}(\mu)| = \left| a_4 e^{3i\varphi} - \mu \cdot a_3 e^{2i\varphi} \cdot a_2 e^{i\varphi} \right| = |\Theta_f(\mu)|. \tag{5}$$

Due to this property, in the research on $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$, one can discuss not all functions f of a given class, but only those functions for which coefficients a_2 are nonnegative real numbers.

In this paper we obtain the estimates of $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$, while $\mu \in \mathbb{R}$ and f is either in $\mathcal{S}^*$ or in $\mathcal{K}$. Almost all presented estimates are sharp and the extremal functions are derived. Taking into account (4) and (5), it is obvious that the rotations of the derived functions are extremal too.

2. Preliminaries

In order to prove our results, we need a few lemmas concerning functions in the class $\mathcal{P}$, i.e. analytic functions p such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for all $z \in \Delta$. Let $p \in \mathcal{P}$ have the Taylor series expansion

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots, z \in \Delta. \tag{6}$$

Lemma 2.1 ([13]) *If* $p \in \mathcal{P}$, *then the sharp estimate* $|p_n| \le 2$ *holds for* $n = 1, 2, \ldots$.

Lemma 2.2 ([7]) *If* $p \in \mathcal{P}$, *then the sharp estimate* $|p_n - p_k p_{n-k}| \le 2$ *holds for* $n, k \in \mathbb{N}$, $n > k$.

Lemma 2.3 *If* $p \in \mathcal{P}$, *then the sharp estimate* $|p_n - {p_k}^2 p_{n-2k}| \le 6$ *holds for* $n, k \in \mathbb{N}$, $n > 2k$.

Lemma 2.4 ([6]) *If* $p \in \mathcal{P}$, *then*

1. $2p_2 = {p_1}^2 + x(4 - {p_1}^2)$,

2. $4p_3 = {p_1}^3 + 2p_1(4 - {p_1}^2)x - p_1(4 - {p_1}^2)x^2 + 2(4 - {p_1}^2)(1 - |x|^2)z$,

for some x *and* z *such that* $|x| \le 1$, $|z| \le 1$.

Lemma 2.3 immediately follows from Lemma 2.1 and Lemma 2.2 if we write $p_n - {p_k}^2 p_{n-2k} = (p_n - p_k p_{n-k}) + p_k(p_{n-k} - p_k p_{n-2k})$.

Applying the correspondence between the functions in $\mathcal{S}^*$ and $\mathcal{P}$

$$\frac{zf'(z)}{f(z)} = p(z) \quad , \quad f \in \mathcal{S}^* \ , \ p \in \mathcal{P} \tag{7}$$

and the expansions (1) and (6) we get

$$(n-1)a_n = \sum_{j=1}^{n-1} a_j p_{n-j} \quad , \quad n = 2, 3, \ldots . \tag{8}$$

In particular,

$$a_2 = p_1 \quad , \quad a_3 = \tfrac{1}{2}(p_2 + {p_1}^2) \quad , \quad a_4 = \tfrac{1}{3}(p_3 + \tfrac{3}{2}p_1p_2 + \tfrac{1}{2}{p_1}^3).$$

Hence, we can express $\Phi_f(\mu)$ and $\Theta_f(\mu)$ for $f \in \mathcal{S}^*$ in terms of coefficients of the corresponding function $p \in \mathcal{P}$:

$$\Phi_f(\mu) = \left(\frac{1}{6} - \frac{1}{4}\mu\right) {p_1}^4 + \frac{1}{3}p_1p_3 + \frac{1}{2}(1-\mu){p_1}^2 p_2 - \frac{1}{4}\mu {p_2}^2 \tag{9}$$

and

$$\Theta_f(\mu) = \frac{1}{3}p_3 + \frac{1}{2}(1-\mu)p_1p_2 + \frac{1}{6}(1-3\mu){p_1}^3. \tag{10}$$

3. Bounds of $|\Phi_f(\mu)|$ for starlike functions

In the main theorem of this section we establish the sharp bounds of $|\Phi_f(\mu)|$ for the class $\mathcal{S}^*$. The proof of this theorem is divided into four lemmas.

Taking into account (9) and Lemma 2.4, we can write $\Phi_f(\mu)$ as follows:

$$\begin{aligned}\Phi_f(\mu) =& \frac{1}{16}(8-9\mu){p_1}^4 + \frac{1}{24}(10-9\mu){p_1}^2(4-{p_1}^2)x \\ & - \frac{1}{48}(4-{p_1}^2)\left(4{p_1}^2 + 3(4-{p_1}^2)\mu\right)x^2 + \frac{1}{6}p_1(4-{p_1}^2)(1-|x|^2)z.\end{aligned} \tag{11}$$

Lemma 3.1 *Let $f \in \mathcal{S}^*$ and $\mu > 1$. Then $|\Phi_f(\mu)| \le 9\mu - 8$. The result is sharp.*

Proof First, assume that $\mu \ge 4/3$. As it was shown, $|\Phi_f(\mu)|$ is invariant under rotations. For this reason we can assume that p_1 is real. To shorten notation, we write p instead of p_1, $p \in [0,2]$. Hence,

$$\begin{aligned}|\Phi_f(\mu)| \le& \frac{1}{16}(9\mu-8)p^4 + \frac{1}{24}(9\mu-10)p^2(4-p^2)\varrho \\ & + \frac{1}{48}(4-p^2)\left(4p^2 + 3(4-p^2)\mu\right)\varrho^2 + \frac{1}{6}p(4-p^2)(1-\varrho^2)\ ,\end{aligned} \tag{12}$$

where $\varrho = |x| \in [0,1]$.

Denoting the right-hand side of (12) by $G(p,\varrho)$, we can write

$$G(p,\varrho) = \frac{1}{48}(4-p^2)(2-p)[3(2+p)\mu-4p]\varrho^2 + \frac{1}{24}(9\mu-10)\,p^2(4-p^2)\varrho + \frac{1}{16}(9\mu-8)\,p^4 + \frac{1}{6}p(4-p^2). \tag{13}$$

For $\mu \geq 4/3$,

$$\frac{1}{48}(4-p^2)(2-p)[3(2+p)\mu-4p] \geq \frac{1}{6}(4-p^2)(2-p) \geq 0$$

and

$$\frac{1}{24}(9\mu-10)\,p^2(4-p^2) \geq 0.$$

Consequently,

$$G(p,\varrho) \leq G(p,1) = \frac{1}{12}(3\mu-2)p^4 + \frac{1}{3}(3\mu-4)p^2 + \mu \leq G(2,1) = 9\mu-8.$$

Let now $\mu \in (1,4/3)$. Since

$$\Phi_f(\mu) = (4-3\mu)(a_2a_4 - {a_3}^2) + (3\mu-3)(a_2a_4 - \frac{4}{3}{a_3}^2)\ ,$$

by Theorem 1.1 and from the previous part of this proof,

$$|\Phi_f(\mu)| \leq (4-3\mu)\cdot 1 + (3\mu-3)\cdot 4 = 9\mu-8.$$

The extremal function is $f(z) = \frac{z}{(1-z)^2}$. □

Lemma 3.2 *If $f \in \mathcal{S}^*$, then $|\Phi_f(7/9)| \leq 1$. The result is sharp.*

Proof The formula (11) for $\mu = 7/9$ takes the form

$$\Phi_f(7/9) = a + be^{i\varphi} - ce^{2i\varphi} + dre^{i\psi}\ , \tag{14}$$

where

$$x = \varrho e^{i\varphi}\ ,\ z = re^{i\psi}\ ,\ \varphi,\psi \in [-\pi,\pi]\ ,\ \varrho, r \in [0,1]$$

and all four expressions:

$$a = \frac{1}{16}p^4\ ,\ b = \frac{1}{8}p^2(4-p^2)\varrho\ ,\ c = \frac{1}{144}(4-p^2)(5p^2+28)\varrho^2\ ,\ d = \frac{1}{6}p(4-p^2)(1-\varrho^2)$$

are nonnegative.

The estimate

$$|\Phi_f(7/9)| \leq |a + be^{i\varphi} - ce^{2i\varphi}| + d \tag{15}$$

is sharp, because it only requires properly taken ψ and $r=1$. With the notation

$$h(x) = -4acx^2 + 2b(a-c)x + (a+c)^2 + b^2 \tag{16}$$

we can write

$$|a + be^{i\varphi} - ce^{2i\varphi}|^2 = h(\cos\varphi). \tag{17}$$

A simple calculation leads to

$$\max\{h(x) : x \in [-1,1]\} = \begin{cases} h(-1) = (c-a+b)^2 & \text{for } b(a-c) \le -4ac \\ h(x_0) & \text{for } |b(a-c)| < 4ac \\ h(1) = (a-c+b)^2 & \text{for } b(a-c) \ge 4ac\ , \end{cases} \tag{18}$$

where $x_0 = \frac{b(a-c)}{4ac}$.

Therefore, the set $\Omega = [0,2] \times [0,1]$ is divided into three parts by the curves obtained from the equations: $b(a-c) = -4ac$ and $b(a-c) = 4ac$. In terms of p and ϱ they have the form:

$$(4-p^2)(5p^2+28)\varrho^2 - 2p^2(5p^2+28)\varrho - 9p^4 = 0 \tag{19}$$

and

$$(4-p^2)(5p^2+28)\varrho^2 + 2p^2(5p^2+28)\varrho - 9p^4 = 0\ , \tag{20}$$

or equivalently, in the explicit form,

$$\varrho = \varrho_k(p) \quad , \quad k = 1, 2\ ,$$

$$\varrho_k(p) = \frac{p^2}{4-p^2}\left((-1)^{k+1} + 2\sqrt{\frac{16-p^2}{5p^2+28}}\right). \tag{21}$$

Hence, we can define the sets:

$$\begin{aligned} \Omega_1 &= \{(p,\varrho) \in \Omega : p \in [0,p_0], \varrho \in [\varrho_1, 1]\} \\ \Omega_3 &= \{(p,\varrho) \in \Omega : \varrho \in [0,\varrho_2]\} \\ \Omega_2 &= \Omega \setminus \{\Omega_1 \cup \Omega_3\}\ , \end{aligned}$$

where $p_0 = \sqrt{(\sqrt{58}-4)/3} = 1.09\ldots$. The calculation of the derivatives of $\varrho_k(p)$ shows that these two functions are increasing in $[0,2]$. From (21) it follows that the curve $\varrho_1(p)$ meets the boundary of Ω in points $(0,0)$ and $(p_0, 1)$ and the curve $\varrho_2(p)$ meets the boundary of Ω in points $(0,0)$ and $(2, 3/8)$.

For $(p,\varrho) \in \Omega_1$, from (15) and (18), we have

$$\begin{aligned} |\Phi_f(7/9)| &\le c - a + b + d \\ &= \frac{1}{144}(4-p^2)(2-p)(14-5p)\varrho^2 + \frac{1}{8}p^2(4-p^2)\varrho \\ &\quad + \frac{1}{6}p(4-p^2) - \frac{1}{16}p^4. \end{aligned}$$

Since the coefficients of ϱ^2 and ϱ are positive, we can take $\varrho = 1$; so

$$|\Phi_f(7/9)| \le \frac{1}{9}(7 + 4p^2 - 2p^4)\ , \tag{22}$$

which is less than or equal to 1, even for all $p \in [0,2]$. Observe that for $p=1$ there is $|\Phi_f(7/9)|=1$.

If $(p,\varrho) \in \Omega_3$, then

$$\begin{aligned} |\Phi_f(7/9)| &\le a-c+b+d \\ &= -\frac{1}{144}(4-p^2)(2+p)(14+5p)\varrho^2+\frac{1}{8}p^2(4-p^2)\varrho \\ &\quad +\frac{1}{6}p(4-p^2)+\frac{1}{16}p^4. \end{aligned}$$

We are going to prove that the expression on the right-hand side of the inequality is less than or equal to 1. It is equivalent to showing that

$$(2+p)(14+5p)\varrho^2-18p^2\varrho+3(3p^2-8p+12) \ge 0. \tag{23}$$

However,

$$\begin{aligned} (2+p)(14+5p)\varrho^2-18p^2\varrho+3(3p^2-8p+12) =&(3p-4)^2+18p(2-p)\varrho+4(1-p\varrho)(5-4p\varrho) \\ &+(28+24p-11p^2)\varrho^2. \end{aligned}$$

Since $p\varrho \le 3/4$ for $(p,\varrho) \in \Omega_3$, all components in the above formula are nonnegative. Therefore, (23) is true in Ω_3.

For $(p,\varrho) \in \Omega_2$,

$$h(x_0)=(a+c)^2\left(1+\frac{b^2}{4ac}\right)$$

and

$$\begin{aligned} |\Phi_f(7/9)| \le (a+c)\sqrt{1+\frac{b^2}{4ac}}+d &= \frac{1}{72}\left[9p^4+(4-p^2)(5p^2+28)\varrho^2\right]\sqrt{\frac{16-p^2}{5p^2+28}}+\frac{1}{6}p(4-p^2)(1-\varrho^2) \\ &= \frac{1}{8}p^4\sqrt{\frac{16-p^2}{5p^2+28}}+\frac{1}{6}p(4-p^2)+\frac{1}{72}(4-p^2)\left[\sqrt{(5p^2+28)(16-p^2)}-12p\right]\varrho^2. \end{aligned}$$

The expression in the square brackets is positive, and thus we can estimate the whole expression by taking the greatest possible ϱ. Thus

$$|\Phi_f(7/9)| \le \begin{cases} g_1(p) & p \in [0,p_0] \\ g_2(p) & p \in [p_0,2] \end{cases},$$

where

$$g_1(p)=\frac{p^4(p+8)}{9(p+2)}\sqrt{\frac{16-p^2}{5p^2+28}}+\frac{p(5p^6-2p^5-56p^4+200p^3-48p^2+336p+672)}{18(p+2)(5p^2+28)}, \tag{24}$$

$$g_2(p)=\frac{1}{18}(p^4-2p^2+28)\sqrt{\frac{16-p^2}{5p^2+28}}. \tag{25}$$

The first bound is achieved if $\varrho=\varrho_1$, the second one if $\varrho=1$.

For $p\in[0,p_0]$, each of four functions: $\frac{p^2(p+8)}{p+2}$, $p^2\sqrt{\frac{16-p^2}{5p^2+28}}$, $\frac{p}{p+2}$, and $\frac{5p^6-2p^5-56p^4+200p^3-48p^2+336p+672}{5p^2+28}$ is nonnegative and increasing. Consequently, $g_1(p)$ is increasing and

$$\max\{g_1(p) : p\in[0,p_0]\} = g(p_0).$$

The function $g_2(p)$ for $p\in[p_0,2]$ is decreasing at the beginning; after that, it starts to increase. For this reason,

$$\max\{g_2(p) : p\in[p_0,2]\} = \max\{g(p_0), g(2)\} = g(2) = 1.$$

Combining all the discussed cases we have

$$|\Phi_f(7/9)|\le 1 \quad \text{for} \quad (p,\varrho)\in[0,2]\times[0,1].$$

This inequality is sharp. Taking $p=2$, we immediately have $|\Phi_f(7/9)|=1$. The extremal function is again $f(z)=\frac{z}{(1-z)^2}$. However, there exists another extremal function. It has been proved (see (22)) that $|\Phi_f(7/9)|=1$ also for $p=1$ and $x=-1$. If $p_1=1$, then we can deduce from Lemma 2.4 that $p_2=-1$; thus $p_2={p_1}^2-2$. It means that p_1, p_2 are the coefficients of a function $p_t(z)=\frac{1-z^2}{1-2zt+z^2}=1+2tz+(4t^2-2)z^2+\ldots$ with a suitably taken t. Comparing the coefficient of the function p_t at z with p_1 we obtain $t=1/2$. The corresponding starlike function is of the form

$$f(z)=\frac{z}{1-z+z^2}=z+z^2-z^4+\ldots.$$

Summing up, the equality $|\Phi_f(7/9)|=1$ is fulfilled for $f(z)=\frac{z}{(1-z)^2}$ or $f(z)=\frac{z}{1-z+z^2}$. □

Finally, we find the estimate of $|\Phi_f(\mu)|$, while $\mu<7/9$ and $\mu\in(7/9,1)$.

Lemma 3.3 *Let $f\in\mathcal{S}^*$ and $\mu\le 7/9$. Then $|\Phi_f(\mu)|\le 8-9\mu$. The result is sharp.*

Proof For $\mu\le 0$,

$$|\Phi_f(\mu)| = |a_2a_4-\mu {a_3}^2| \le |a_2|\cdot|a_4| + |\mu|\cdot|a_3|^2 \le 8+9|\mu| = 8-9\mu.$$

If $\mu\in(0,7/9)$, then

$$\Phi_f(\mu)=\frac{1}{7}\left[a_2a_4\,(7-9\mu)+\left(a_2a_4-\frac{7}{9}{a_3}^2\right)9\mu\right].$$

Lemma 3.2 and the previous part of this proof yield

$$|\Phi_f(\mu)|\le\frac{1}{7}\left[8\cdot(7-9\mu)+1\cdot 9\mu\right]=8-9\mu\ ,$$

with equality for $f(z)=\frac{z}{(1-z)^2}$. □

Lemma 3.4 *Let $f\in\mathcal{S}^*$ and $\mu\in(7/9,1)$. Then $|\Phi_f(\mu)|\le 1$. The result is sharp.*

Proof For $\mu \in (7/9, 1)$ we can write

$$\Phi_f(\mu) = \frac{1}{2}\left[\left(a_2a_4 - \frac{7}{9}{a_3}^2\right)(9-9\mu) + (a_2a_4 - {a_3}^2)(9\mu - 7)\right].$$

From Lemma 3.2 and Lemma 3.1,

$$|\Phi_f(\mu)| \leq \frac{1}{2}\left[1 \cdot (9-9\mu) + 1 \cdot (9\mu - 7)\right] = 1.$$

□

The results established in Lemmas 3.1–3.4 can be aggregated in the following theorem.

Theorem 3.1 *If* $f \in \mathcal{S}^*$*, then* $|\Phi_f(\mu)| \leq \max\{|9\mu - 8|, 1\}$.

4. Bounds of $|\Theta_f(\mu)|$ for starlike functions

At the beginning, observe that $\Theta_f(1) = \frac{1}{3}(p_3 - {p_1}^3)$. From Lemma 2.3 we immediately obtain the result of Theorem 1.2, point 1.

Lemma 4.1 *Let* $f \in \mathcal{S}^*$ *and* $\mu > 1$*. Then* $|\Theta_f(\mu)| \leq 6\mu - 4$*. The result is sharp.*

Proof Assume that $\mu \geq 5/3$. The formula (10) can be rewritten in the form

$$\Theta_f(\mu) = \frac{1}{3}\left[(p_3 - p_1p_2) + \frac{1}{2}(5-3\mu)p_1p_2 + \frac{1}{2}(1-3\mu){p_1}^3\right].$$

Lemma 2.1 and Lemma 2.2 result in

$$|\Theta_f(\mu)| \leq \frac{1}{3}\left[2 - 2(5-3\mu) - 4(1-3\mu)\right] = 6\mu - 4.$$

Now suppose that $\mu \in (1, 5/3)$. Since

$$\Theta_f(\mu) = \frac{1}{2}\left[(5-3\mu)(a_4 - a_2a_3) + (3\mu - 3)(a_4 - \frac{5}{3}a_2a_3)\right],$$

from the previous part of this proof and from Theorem 1.2, point 1, we obtain

$$|\Theta_f(\mu)| \leq \frac{1}{2}\left[(5-3\mu)\cdot 2 + (3\mu - 3)\cdot 6\right] = 6\mu - 4.$$

It is clear that $\Theta_f(\mu) = 6\mu - 4$ only when $p_1 = p_2 = p_3 = 2$, which means that the extremal function is $f(z) = \frac{z}{(1-z)^2}$. □

For $\mu \leq 1/3$, an application of Lemma 2.1 in (10) leads directly to

Lemma 4.2 *Let* $f \in \mathcal{S}^*$ *and* $\mu \leq 1/3$*. Then* $|\Theta_f(\mu)| \leq 4 - 6\mu$*. The result is sharp.*

Our next step is finding the bound of $|\Theta_f(2/3)|$.

Lemma 4.3 *Let* $f \in \mathcal{S}^*$. *Then* $|\Theta_f(2/3)| \le 16/9\sqrt{3} = 1.026\ldots$. *The result is sharp.*

Proof Applying (10) and Lemma 2.4 for $f \in \mathcal{S}^*$, we obtain

$$\Theta_f(2/3) = \frac{1}{12}(4 - {p_1}^2)\left[3p_1 x - p_1 x^2 + 2(1-|x|^2)z\right] \ , \tag{26}$$

where $|x| \le 1$, $|z| \le 1$.

Since $|\Theta_f(2/3)|$ is invariant under rotations, we can assume that p_1 is real, and so we write $p = p_1$, $p \in [0,2]$. Then a sharp estimate

$$|\Theta_f(2/3)| \le \frac{1}{12}(4-p^2)h(\varrho) \tag{27}$$

holds, where

$$h(\varrho) = (p-2)\varrho^2 + 3p\varrho + 2$$

and $\varrho = |x| \in [0,1]$.

It is easy to verify that if $p \in [4/5, 2]$ is fixed, then $h(\varrho)$ is strictly increasing for $\varrho \in [0,1]$, and so

$$h(\varrho) \le h(1) = 4p. \tag{28}$$

On the other hand, if p is fixed and $p \in [0, 4/5)$, then the maximal value of $h(\varrho)$ is achieved for $\varrho = \frac{3p}{2(2-p)}$; for a stated range of p, there is $\frac{3p}{2(2-p)} \in [0,1)$. In this case

$$h(\varrho) \le h\left(\frac{3p}{2(2-p)}\right) = \frac{9p^2 - 8p + 16}{4(2-p)}. \tag{29}$$

Combining (28) and (29), we can see that

$$|\Theta_f(2/3)| \le \frac{1}{12}g(p) \ ,$$

where

$$g(p) = \begin{cases} \frac{1}{4}(2+p)(9p^2 - 8p + 16) & p \in [0, 4/5) \\ 4p(4-p^2) & p \in [4/5, 2]. \end{cases} \tag{30}$$

If $p \in [0, 4/5)$, then $g(p) = \frac{1}{4}(9p^3 + 10p^2 + 32)$ is strictly increasing in $[0, 4/5)$, and so $g(p) < g(4/5)$. For $p \in [4/5, 2]$ we have $g(p) \le g(2/\sqrt{3}) = 64/3\sqrt{3}$. Since $g(4/5) < g(2/\sqrt{3})$, we obtain $|\Theta_f(2/3)| \le 16/9\sqrt{3}$.

The equality in the above estimate holds for $p_1 = 2/\sqrt{3}$, $x = -1$, $z = -1$. Consequently, $p_2 = -2/3$ and so $p_2 = {p_1}^2 - 2$. Hence, p_1, p_2 are the coefficients of $p_t(z) = \frac{1-z^2}{1-2zt+z^2}$ with a suitably taken t. However, $p_t(z) = 1 + 2tz + (4t^2 - 2)z^2 + \ldots$, and so $t = 1/\sqrt{3}$. The corresponding starlike function is

$$f(z) = \frac{z}{1 - \frac{2}{\sqrt{3}}z + z^2} = z + \frac{2\sqrt{3}}{3}z^2 + \frac{1}{3}z^3 - \frac{4\sqrt{3}}{9}z^4 + \ldots.$$

□

Now we can establish two final estimates.

Lemma 4.4 *Let* $f \in \mathcal{S}^*$.

1. *If* $\mu \in (1/3, 2/3)$, *then* $|\Theta_f(\mu)| \le \frac{16-18\sqrt{3}}{3\sqrt{3}}\mu + \frac{36\sqrt{3}-16}{9\sqrt{3}}$.

2. *If* $\mu \in (2/3, 1)$, *then* $|\Theta_f(\mu)| \le \frac{18\sqrt{3}-16}{3\sqrt{3}}\mu + \frac{16-12\sqrt{3}}{3\sqrt{3}}$.

Proof The first part of Lemma 4.4 follows from

$$\Theta_f(\mu) = (3\mu-1)(a_4 - \frac{2}{3}a_2a_3) + (2-3\mu)(a_4 - \frac{1}{3}a_2a_3) \quad , \quad \mu \in (1/3, 2/3)$$

and Lemma 4.2 and Lemma 4.3.

The second part is a consequence of Theorem 3.1, point 1, Lemma 4.3, and a formula

$$\Theta_f(\mu) = (3\mu-2)(a_4 - a_2a_3) + (3-3\mu)(a_4 - \frac{2}{3}a_2a_3) \quad , \quad \mu \in (2/3, 1).$$

□

The results presented in Lemmas 4.1–4.4 can be collected as follows.

Theorem 4.1 *If* $f \in \mathcal{S}^*$, *then*

$$|\Theta_f(\mu)| \le \max\left\{|6\mu-4|, \left(2 - \frac{16}{9\sqrt{3}}\right)|3\mu-2| + \frac{16}{9\sqrt{3}}\right\}.$$

Remark 1 The estimates in Lemma 4.4 are not sharp. They can be slightly improved, but the proof of this result does not look good. For this reason, we have decided to omit it from this discussion.

5. Bounds of $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$ for convex functions

Let $f \in \mathcal{S}^*$, $g \in \mathcal{K}$, f and g have the series expansions (1) and

$$g(z) = z + b_2z^2 + b_3z^3 + \dots\ , \tag{31}$$

respectively.

According to the Alexander relation: $f \in \mathcal{S}^*$ if and only if $g \in \mathcal{K}$, where $f(z) = zg'(z)$, there is $a_n = nb_n$. From Theorem 3.1 and Theorem 4.1 we immediately obtain the bounds of $|\Phi_f(\mu)|$ and $|\Theta_f(\mu)|$ for convex functions.

Corollary 1 *If* $f \in \mathcal{K}$, *then* $|\Phi_f(\mu)| \le \max\{|\mu-1|, 1/8\}$.

Corollary 2 *If* $f \in \mathcal{K}$, *then* $|\Theta_f(\mu)| \le \max\left\{|\mu-1|, \left(1 - 8/9\sqrt{3}\right)|\mu-1| + 4/9\sqrt{3}\right\}$.

The result in Corollary 1 is sharp. The extremal functions g can be found from the formula $zg'(z) = f(z)$, where the functions f are corresponding extremal functions in the class $\mathcal{S}^*$.

In particular, for $\mu = 1$, Corollaries 1 and 2 reduce to

Corollary 3 *If* $f \in \mathcal{K}$, *then* $|a_2a_4 - {a_3}^2| \le 1/8$.

Corollary 4 *If* $f \in \mathcal{K}$*, then* $|a_4 - a_2 a_3| \le 4/9\sqrt{3}$.

These results are given in Theorem 1.1 and Theorem 1.2. Although the estimate in Theorem 1.2, point 1 is correct, the proof given in [1] is false. The proof of Theorem 4.1, point 2, rectifies these errors.

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