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Boundary sentinels for the resolution of a geometrical problem

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Abstract: The aim of this paper is to estimate the shape of an unknown part of the boundary of a geometrical domain. The identification technique used to estimate this part is the observation of the solution of a diffusion problem on the known part of this boundary. This technique is based on the sentinels theory.

Key words: Boundary sentinels, observation, approximate control, fixed point, convex duality

1. Introduction

In many distributed systems governed by partial differential equations, some terms may be completely or partially unknown, as well as the initial or the boundary conditions, the operator coefficients, or the geometrical domain. Such systems are called problems with incomplete data and can be classified as inverse problems. Many models can be found in the mathematical literature, including modeling the environmental, climatic and ecological problems, etc. For more details, we refer the reader to [4, 5].

In order to identify the parameters in pollution problems governed by semilinear parabolic equations and incomplete data, Lions introduced the so-called sentinels method given in [5]. The concept of the sentinels is related to the following three objects: a state equation, an observation function, and a control function to be determined. Many papers used the famous definition of Lions in the theoretical aspect. See, for example, Massengo and Nacoulima [8] and [2, 9–11, 13, 14] for the general setting. This definition is also used in numerical studies such as [1, 2, 4].

In 1997, Bodart [2] applied the sentinels method to estimate the shape of an unknown part of the boundary Γ of a sufficiently regular domain Ω in $\mathbb{R}^2$. The shape of one side of Ω is known and set to a given temperature, and the shape of the other side has been estimated by the observation of the temperature in the middle of the domain.

However, for some physical problems (for example, in the management of oil wells, in the determination of an inaccessible part of a lake, etc.), observation in the inside is difficult or impossible to be explored, which leads to putting the observatory on the boundary. Such a problem will be the objective of this paper, where we suppose that the known part of the domain is shared in two parts. On the first we put the source of the temperature and on the other part we observe the distribution of the temperature in the domain. The problem is modeled by a partial differential equation and can be solved with the method that will be presented in sequel.

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Let $\Omega_0 \subset \mathbb{R}^2$ be an open subset with sufficiently smooth boundary, $\partial\Omega_0 = \Gamma^* \cup \Gamma_0$, with $\Gamma^* \cap \Gamma_0 = \emptyset$. We define a deformation Ω_α of the domain Ω_0 by

$$\Omega_\alpha = \{x + \alpha(x) U(x), \ x \in \Omega_0\}, \tag{1}$$

where U is a known transverse vector field of class C^∞, and $\alpha(x)$ is a C^2 function such that Γ^* remains invariant by the deformation αU. Thus, the boundary of the open set Ω_α, i.e. $\partial\Omega_\alpha = \Gamma^* \cup \Gamma_\alpha$, is of class C^2 and $\Gamma^* \cap \Gamma_\alpha = \emptyset$. Hence, Γ_α is a deformation of Γ_0 :

$$\Gamma_\alpha = \{x + \alpha(x) U(x), \ x \in \Gamma_0\}. \tag{2}$$

We assume that $\Gamma^* = \Gamma_1^* \cup \Gamma_2^*$, $\Gamma_1^* \cap \Gamma_2^* = \Phi$ with $meas\Gamma_1^* \neq 0$ and $meas\Gamma_2^* \neq 0$.

Let $y = y(x,t;\alpha)$ be the solution of

$$\begin{cases} \dfrac{\partial y}{\partial t} - \triangle y & = 0 \quad \text{in } Q_\alpha = \Omega_\alpha \times]0,T[\,, \\ \quad y & = f \quad \text{on } \Sigma_1^* = \Gamma_1^* \times]0,T[\,, \\ \quad \dfrac{\partial y}{\partial n} & = 0 \quad \text{on } \Sigma_2^* = \Gamma_2^* \times]0,T[\,, \\ \quad y & = 0 \quad \text{on } \Sigma_\alpha = \Gamma_\alpha \times]0,T[\,, \\ y(.,0) & = 0 \quad \text{in } \Omega_\alpha, \end{cases} \tag{3}$$

where $f = \tilde{f} \mid \Gamma_1^*$ and $\tilde{f} \in L^2\left(]0,T[\,, H^{\frac{3}{2}}(\Gamma)\right)$. It is known that (see [6, 7])

$$y \in L^2\left(]0,T[\,, H^2(\Omega_\alpha)\right), y\mid_\Gamma \in L^2\left(]0,T[\,, H^{\frac{3}{2}}(\Gamma)\right)$$

and

$$\frac{\partial y}{\partial n}\mid_\Gamma \in L^2\left(]0,T[\,, H^{\frac{1}{2}}(\Gamma)\right), \text{ where } \Gamma = \partial\Omega_\alpha.$$

For all α we put $O = \Gamma_2^* \times]0,T[$. The solution y of (3) is supposed to be observed in O and we shall call this observation y_{obs}.

The vector field U is supposed to remain transverse to Γ_α in a sufficiently large neighborhood of α (that means that the initial guess Ω_0 is not too far from the exact domain Ω_α (deformation without shearing)).

We aim to build a sequence of functions $\left(\alpha^k\right)_{k=0\ldots\infty}$ locally converging in some sense, starting from an initial guess α^0. This will give an approximation of the final shape of the deformed boundary part Γ_α. The computation of α^{k+1} from α^k will be done by the method of sentinels.

The paper is organized as follows: in Section 2, we present the sentinels method; first, we give the definition of the sentinel function S from $l^2(\mathbb{R}) \times l^2(\mathbb{R})$ into $l^2(\mathbb{R})$, and second, we prove the existence and the uniqueness of such sentinels in three steps. We start by proving that the existence and the uniqueness of the sentinels is reduced to the resolution of an approximate control problem. Then we solve the control problem, finally, by a convex duality process where we characterize the solution to obtain the optimal conditions on the sentinels. In Section 3, we use the sentinels function to construct the iterative scheme and we prove the convergence of the sequences by using the fixed point method. Here, an approximation of the unknown boundary is given.

Finally, performing observations on a part of the boundary can solve many environmental and industrial problems, for which the access to the middle is difficult or impossible, and it can minimize the use of energy.

2. Application of the sentinels method

Let us consider the following parameterization of the set Γ_α :

$$\Gamma_\alpha = \{x(s) + \alpha(s) U(s),\ s \in [0,1], x(s) \in \Gamma_0\}, \tag{4}$$

where α is a C^2 function over the interval $[0,1]$, i.e. α belongs to the space $L^2(]0,1[)$. Writing α in a basis of functions $(b_j)_{j=1...\infty}$ in $C^2(0,1)$, we keep using $\alpha \in l^2(\mathbb{R})$ for the infinite coordinate vector of the function α in this basis, and hence (4) now is written as

$$\Gamma_\alpha = \left\{ x(s) + \sum_{j=1}^{\infty} \alpha_j b_j U(s),\ s \in [0,1], x(s) \in \Gamma_0 \right\}. \tag{5}$$

Then our goal here is to build a sequence $\left(\alpha^k\right)_{k=0...\infty}$ converging in $l^2(\mathbb{R})$. We define the sentinel function by:

$$\left| \begin{array}{ccc} S : l^2(\mathbb{R}) \times l^2(\mathbb{R}) & \longrightarrow & l^2(\mathbb{R}) \\ (\tilde{\alpha}, \alpha) & \longmapsto & \left(\int_O w_i(\tilde{\alpha})\, y(\alpha)\, d\Sigma \right)_{i=1...\infty} \end{array} \right., \tag{6}$$

where $y(\alpha) = y(x,t;\alpha)$ is the solution of (3), $\Sigma = \Gamma \times]0,T[$ and the functions $(w_i(\tilde{\alpha}))_{i=1...\infty}$ are expected to be found in some way. Now we have following result:

Proposition 1 *(Existence and uniqueness of the sentinel): There exists a unique family of functions* $(w_i(\tilde{\alpha}))_{i=1...\infty}$, *which ensures the existence and the uniqueness of the sentinel function* $S(\tilde{\alpha}, \alpha)$ *defined in* (6) *such that:*

$$w_i(\tilde{\alpha}) \in L^2(O)\ , i = 1...\infty \textit{ has a minimal norm,} \tag{7}$$

$$D_\alpha S(\tilde{\alpha}, \tilde{\alpha}) = ID + M,\ \forall \tilde{\alpha} \in l^2(\mathbb{R}), \tag{8}$$

where ID *is the identity operator and* $M \in \mathcal{L}\left(l^2(\mathbb{R})\right)$ *with*

$$\|(M_i)\|_{l^2(IR)} = \frac{\varepsilon}{i},\ \textit{for } i = 1...\infty. \tag{9}$$

Here, (M_i) *is the* i^{th} *line of the infinite matrix* M, *and* $D_\alpha S(\tilde{\alpha}, \tilde{\alpha})$ *denote the derivative of* S *with respect to its second parameter computed at* $(\tilde{\alpha}, \tilde{\alpha})$.

Remark 1 *Condition* (8) *makes sense, since* $y(x,t;\alpha)$ *is differentiable with respect to* α_j *(see [2, 15, 16]).*

Remark 2 *For a fixed* $\tilde{\alpha}$, $S_{\tilde{\alpha}}(\alpha) = S(\tilde{\alpha}, \alpha)$ *is a sentinel in the sense of Lions* [5].

Proof (of Proposition 1): The proof will be done in three steps:

First step: Conditions (7) and (8) will be rewritten into a control problem; the function $y(x,t;\alpha)$ is differentiable with respect to α as shown in [15] and we shall note

$$y_{\alpha_j} = \frac{\partial y(\tilde{\alpha})}{\partial \alpha_j}, _{j=1...\infty}$$

the derivative of $y(x,t;\alpha)$ with respect to α_j at $\tilde{\alpha}$, which is the solution of the following system:

$$\begin{cases} \dfrac{\partial y_{\alpha_j}}{\partial t} - \triangle y_{\alpha_j} & = 0 & \text{in } Q_{\tilde{\alpha}} = \Omega_{\tilde{\alpha}} \times]0,T[\,, \\ y_{\alpha_j} & = 0 & \text{on } \Sigma_1^* = \Gamma_1^* \times]0,T[\,, \\ \dfrac{\partial y_{\alpha_j}}{\partial n} & = 0 & \text{on } \Sigma_2^* = \Gamma_2^* \times]0,T[\,, \\ y_{\alpha_j} & = -b_j\left(\nabla y\left(\tilde{\alpha}\right).U\right) & \text{on } \Sigma_{\tilde{\alpha}} = \Gamma_{\tilde{\alpha}} \times]0,T[\,, \\ y_{\alpha_j}\left(.,0\right) & = 0 & \text{in } \Omega_{\tilde{\alpha}}, \end{cases} \tag{10}$$

where $y(\tilde{\alpha}) = y(x,t;\tilde{\alpha})$ solves (3) with data $\tilde{\alpha}$. Thus, the general element of the infinite matrix $D_\alpha S(\tilde{\alpha},\tilde{\alpha})$ is

$$\left(D_\alpha S\left(\tilde{\alpha},\tilde{\alpha}\right)\right)_{ij} = \int_O w_i\left(\tilde{\alpha}\right) y_{\alpha_j} d\Sigma. \tag{11}$$

Considering i as fixed, condition (8) now reads

$$\int_O w_i\left(\tilde{\alpha}\right) y_{\alpha_j} d\Sigma = \delta_{ij} + (M)_{ij}\,,\ {}_{j=1\ldots\infty}, \tag{12}$$

where the matrix M is defined as in Proposition 1. Since the elements of $L^2(O)$ and $H^{\frac{1}{2}}(O)$ are (respectively) the restrictions of the $L^2(\Sigma)$ and $H^{\frac{1}{2}}(\Sigma)$ elements and since $H^{\frac{1}{2}}(\Sigma)$ is dense in $L^2(\Sigma)$ [12], then for every element $w_i(\tilde{\alpha})$ in $L^2(O)$ let us consider a sequence $(w_{in}(\tilde{\alpha}))_{n\in IN} \subset H^{\frac{1}{2}}(O)$ converging to $w_i(\tilde{\alpha})$, and for every $n \in \mathbb{N}$, let $q_{in} \in L^2\left(]0,T[\,, H^2\left(\Omega_{\tilde{\alpha}}\right)\right)$, be the solution of the following adjoint problem:

$$\begin{cases} -\dfrac{\partial q_{in}}{\partial t} - \triangle q_{in} & = 0 & \text{in } Q_{\tilde{\alpha}}, \\ q_{in} & = 0 & \text{on } \Sigma/\Sigma_2^*, \\ \dfrac{\partial q_{in}}{\partial n} & = w_{in}\left(\tilde{\alpha}\right) & \text{on } \Sigma_2^* \\ q_{in}\left(T\right) & = 0. \end{cases} ,n \in IN. \tag{13}$$

Due to the regularity of the solutions and by a suitable integration by parts, we get

$$\int_O w_{in}\left(\tilde{\alpha}\right) y_{\alpha_j} d\Sigma = \int_{\Sigma_{\tilde{\alpha}}} b_j\left(\nabla y\left(\tilde{\alpha}\right).U\right)(\alpha) \frac{\partial q_{in}}{\partial n} d\Sigma.$$

Taking $n \longrightarrow +\infty$ to obtain

$$\int_O w_i\left(\tilde{\alpha}\right) y_{\alpha_j} d\Sigma = \int_{\Sigma_{\tilde{\alpha}}} b_j\left(\nabla y\left(\tilde{\alpha}\right).U\right)(\alpha) \frac{\partial q_i}{\partial n} d\Sigma, \tag{14}$$

where $q_i = \lim\limits_{n\longrightarrow+\infty} q_{in}$ ($q_i \in L^2\left(]0,T[\,, H^{\frac{1}{2}}\left(\Omega_{\tilde{\alpha}}\right)\right)$), we define a linear continuous operator $B \in L\left(L^2(O);l^2(\mathbb{R})\right)$ by

$$B : \left| \begin{array}{ccc} L^2(O) & \longrightarrow & l^2(\mathbb{R}) \\ w_i\left(\tilde{\alpha}\right) & \longmapsto & \left(\int_{\Sigma_{\tilde{\alpha}}} b_j\left(\nabla y\left(\tilde{\alpha}\right).U\right)(\alpha) \frac{\partial q_i}{\partial n} d\Sigma\right)_{j=1\ldots\infty} \end{array} \right. .$$

Equation (14) allows us to rewrite (11) as

$$(D_\alpha S(\tilde{\alpha},\tilde{\alpha}))_{ij} = (Bw_i(\tilde{\alpha}))_j\,. \tag{15}$$

This is a control problem, i.e. to find $w_i(\tilde{\alpha}) \in L^2(O)$ of minimal norm such that $Bw_i(\tilde{\alpha}) = z$ with $z \in l^2(\mathbb{R})$, but this is an exact controllability type problem, and what we will show is that we can accomplish approximate controllability, which is sufficient for the numerical applications.

Second step: Approximate controllability result: We are going to prove that the range of B is dense, and to do this we shall establish that its adjoint is injective.

The adjoint operator $B^* \in L\left(l^2(\mathbb{R}); L^2(O)\right)$ is given by:

$$B^*: \left| \begin{array}{ccc} l^2(\mathbb{R}) & \longrightarrow & L^2(O) \\ (\sigma_j)_{j=1\ldots\infty} & \longmapsto & \Phi\,|_O, \end{array} \right.$$

where Φ is the solution of the following problem:

$$\left\{ \begin{array}{lll} \dfrac{\partial \Phi}{\partial t} - \triangle \Phi & = 0 & \text{in } Q_{\tilde{\alpha}}, \\ \Phi & = 0 & \text{on } \Sigma_1^*, \\ \dfrac{\partial \Phi}{\partial n} & = 0 & \text{on } \Sigma_2^*, \\ \Phi & = -\left(\nabla y(\tilde{\alpha}).U\right) \sum\limits_{j=1}^{\infty} b_j \sigma_j & \text{on } \Sigma_{\tilde{\alpha}}, \\ \Phi(.,0) & = 0 & \text{in } \Omega_{\tilde{\alpha}}. \end{array} \right. \tag{16}$$

Indeed, from (16) we have

$$(w,\Phi)_{L^2(O)} = \sum_{j=1}^{\infty} \sigma_j \int\limits_{\Sigma_{\tilde{\alpha}}} b_j \left(\nabla y(\tilde{\alpha}).U\right)(\alpha) \frac{\partial q_i}{\partial n} d\Sigma = (\sigma, Bw)_{l^2(\mathbb{R})}\,,$$

i.e. $\Phi\,|_O = B^*\sigma$. Suppose now that $B^*\sigma = \Phi\,|_O = 0$, i.e. Φ identically vanishes in O. By the Cauchy uniqueness theorem we conclude that $\Phi = 0$ in $Q_{\tilde{\alpha}}$. Thus, we get

$$\left(\nabla y(\tilde{\alpha}).U\right) \sum_{j=1}^{\infty} b_j \sigma_j = 0.$$

Since $(b_j)_{j=1\ldots\infty}$ is a basis of $l^2(\mathbb{R})$, then we have either $\{(\nabla y(\tilde{\alpha}).U) = 0\}$ or $\{\sigma_j = 0, j = 1 \ldots \infty\}$. Decomposing the field U on the normal and tangent vectors $\nu_{\tilde{\alpha}}$ and $\tau_{\tilde{\alpha}}$ on $\Gamma_{\tilde{\alpha}}$ we get:

$$\nabla y(\tilde{\alpha}).U(x) = \nabla y(\tilde{\alpha}).(a\nu_{\tilde{\alpha}}(x)) + \nabla y(\tilde{\alpha}).(b\tau_{\tilde{\alpha}}(x)), \forall x \in \Gamma_{\tilde{\alpha}},$$

and since $y(\tilde{\alpha})$ vanishes on $\Gamma_{\tilde{\alpha}}$ we have:

$$\nabla y(\tilde{\alpha}).U(x) = a \frac{\partial y(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}}(x), \forall x \in \Gamma_{\tilde{\alpha}}.$$

By the Cauchy uniqueness theorem we have that $\frac{\partial y(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}} \mid \Gamma_{\tilde{\alpha}}$ cannot be null. Otherwise, $y(\tilde{\alpha})$ would be equal to 0 a.e. in $Q_{\tilde{\alpha}}$. Thus, $\nabla y(\tilde{\alpha}).U \neq 0$ and B^* is injective. This proves that the range of B is dense in $l^2(\mathbb{R})$, i.e.

$$\forall \rho > 0, \forall z \in l^2(\mathbb{R}), \exists w_i(\tilde{\alpha}) \in L^2(O); \|Bw_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} \leq \rho. \tag{17}$$

Third step: By a convex duality process a control fulfilling conditions (7) and (8) is exhibited; it remains to construct $w_i(\tilde{\alpha})$ as the function of the minimal norm satisfying (17), which will be done by the Fenchel–Rockafellar duality method. Let us consider the set U_{ad} defined by:

$$U_{ad} = \left\{ w \in L^2(O) \text{ s.t. } \|Bw_i - z\|_{l^2(\mathbb{R})} \leq \rho, z \in l^2(\mathbb{R}) \right\},$$

which is nonempty (from (17)) and obviously convex and closed in $L^2(O)$. Thus, there exists a unique $w_i(\tilde{\alpha})$ satisfying (7) and satisfying the following minimization problem:

$$\min_{w \in U_{ad}} \frac{1}{2} \|w\|^2_{L^2(O)}. \tag{18}$$

Let F and G be two functions defined by:

$$F(w) = \frac{1}{2}\|w\|^2_{L^2(O)} \text{ and } G(\mu) = \begin{cases} 0 \text{ if } \|\mu - z\|_{l^2(\mathbb{R})} \leq \rho, \\ +\infty \text{ otherwise.} \end{cases}$$

Then problem (18) now reads

$$\min_{w \in L^2(O)} F(w) + G(Bw).$$

Applying the duality theorem of Fenchel and Rockafeller (see [3],), one gets

$$w_i(\tilde{\alpha}) = B^*\sigma^*, \tag{19}$$

where σ^* solves the dual minimization problem:

$$\min_{\sigma \in l^2(\mathbb{R})} F^*(B^*\sigma) + G^*(-\sigma)$$

with F^* and G^* being the Fenchel conjugates of F and G. It is known that $F^* = F$ and G^* is straightforward:

$$\begin{aligned} G^*(\sigma) &= \sup_{\mu \in l^2(\mathbb{R})} (\mu, \sigma)_{l^2(\mathbb{R})} - G(\mu) = \sup_{\mu \in \overline{B(0,\rho)}} (z + \mu, \sigma)_{l^2(\mathbb{R})} \\ &= (z, \sigma)_{l^2(\mathbb{R})} + \rho \|\sigma\|_{l^2(\mathbb{R})}, \end{aligned}$$

where $\overline{B(0,\rho)}$ is the $l^2(\mathbb{R})$ closed ball of center 0 and radius ρ. Then relation (18) becomes

$$\min_{\sigma \in l^2(\mathbb{R})} J(\sigma) = F(\Phi) + \rho \|\sigma\|_{l^2(\mathbb{R})} - (z, \sigma)_{l^2(\mathbb{R})}, \tag{20}$$

where Φ is the solution of (16). We know that $J(\sigma)$ is not differentiable at 0, but under suitable conditions we shall show that 0 is not an optimal point.

Lemma 1 : $\sigma^* = 0$ is the solution of (20) if and only if $\|z\|_{l^2(IR)} \leq \rho$. □

Proof First, if $\sigma^* = 0$, then (19) yields

$$w_i(\tilde{\alpha}) = 0 \quad \text{and} \quad Bw_i(\tilde{\alpha}) - z = -z.$$

Since σ^* solves (20), then

$$\|Bw_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} \leq \rho,$$

i.e. $\|z\|_{l^2(\mathbb{R})} \leq \rho$. Next, if $\|z\|_{l^2(\mathbb{R})} \leq \rho$, then $w_i(\tilde{\alpha}) = 0$ belongs to the set U_{ad}, obviously being the minimal norm element of this subset. Hence, $w_i(\tilde{\alpha}) = 0$ is then the solution of (18). Since B^* is injective, equation (19) gives $\sigma^* = 0$. From now on, we will assume that $\|z\|_{l^2(\mathbb{R})} > \rho$, bringing us into the position to give the optimality condition for σ^*. For any $\delta\sigma \in l^2(\mathbb{R})$ and $\sigma \neq 0$, one has

$$\begin{aligned} \left(\frac{\partial J}{\partial \sigma}, \delta\sigma\right)_{l^2(\mathbb{R})} &= (B^*\sigma, B^*\delta\sigma)_{l^2(\mathbb{R})} + \rho\left(\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}}, \delta\sigma\right)_{l^2(\mathbb{R})} - (z, \delta\sigma)_{l^2(\mathbb{R})} \\ &= \left(BB^*\sigma + \rho\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}} - z, \delta\sigma\right)_{l^2(\mathbb{R})}. \end{aligned}$$

Thus, σ^* is such that

$$BB^*\sigma^* - z = -\rho\frac{\sigma^*}{\|\sigma^*\|_{l^2(\mathbb{R})}}. \tag{21}$$

Since $w_i(\tilde{\alpha}) = B^*\sigma^*$, we have $\|Bw_i(\tilde{\alpha}) - z\|_{l^2(\mathbb{R})} = \rho$. Let us choose $(z)_j$ as follows:

$$(z)_j = \delta_{ij}, j = 1...\infty,$$

where $(z)_j$ is the generic coordinate of z on the canonical basis of $l^2(\mathbb{R})$ and

$$\rho = \frac{\varepsilon}{i} \text{ with } \varepsilon > 0 \text{ sufficiently small,}$$

to get $\|z\|_{l^2(\mathbb{R})} > \rho$. Eventually (21) gives

$$(Bw_i(\tilde{\alpha}))_j = \delta_{ij} - \frac{\varepsilon}{i}\frac{\sigma_j^*}{\|\sigma^*\|_{l^2(\mathbb{R})}}, \tag{22}$$

and by combining this with (15) we obtain (8). Thus, we have proved the existence and uniqueness of a family of functions $w_i(\tilde{\alpha})$ for $i = 1...\infty$ solving (7) and (8). □

Remark 3 *From Lemma 1,the computation of σ^* can be done by either nonsmooth or smooth methods.*

Remark 4 *We remark that the differentiation of $S(\tilde{\alpha}, \alpha)$ with respect to its first variable amounts to differentiating $(w_i(\tilde{\alpha}))_{i=1...\infty}$, i.e. to differentiating twice the system (3). (See [2]).*

3. The iterative scheme construction

Now we shall construct an iterative scheme for the resolution of the problem presented in Section 2. Differentiating $S(\tilde{\alpha},\alpha)$ with respect to α at the point $(\tilde{\alpha},\tilde{\alpha})$, one gets

$$S(\tilde{\alpha},\alpha) = S(\tilde{\alpha},\tilde{\alpha}) + D_{\alpha}S(\tilde{\alpha},\tilde{\alpha}).(\alpha-\tilde{\alpha}) + o(|\alpha-\tilde{\alpha}|).$$

Taking (8) into account, we have

$$S(\tilde{\alpha},\alpha) = S(\tilde{\alpha},\tilde{\alpha}) + \alpha - \tilde{\alpha} + M(\alpha-\tilde{\alpha}) + o(|\alpha-\tilde{\alpha}|).$$

For $\tilde{\alpha}=\alpha^{k}$, $\alpha=\alpha^{k+1}$, and

$$S(\tilde{\alpha},\alpha) = S_{obs}(\tilde{\alpha}) = \left(\int_{O} w_i(\tilde{\alpha})\, y_{obs} d\Sigma\right)_{i=1\ldots\infty}, \tag{23}$$

that suggests the following iterations:

$$\alpha^{k+1} = \alpha^{k} + S_{obs}\left(\alpha^{k}\right) - S\left(\alpha^{k},\alpha^{k}\right), \tag{24}$$

such that

$$S\left(\alpha^{k},\alpha^{k}\right) = \left(\int_{O} w_i\left(\alpha^{k}\right) y\left(\alpha^{k}\right) d\Sigma\right)_{i=1\ldots\infty},$$

where $y\left(\alpha^{k}\right) = y\left(x,t;\alpha^{k}\right)$ solves (3) with data α^{k}. We can now study the convergence of our scheme.

Theorem 1 *The sequence* $\left(\alpha^{k}\right)_{k=0\ldots\infty}$ *defined by*

$$\begin{cases} \alpha^{0}\in l^{2}(\mathbb{R}) \text{ given as an initial guess,} \\ \alpha^{k+1} = \alpha^{k} + S_{obs}\left(\alpha^{k}\right) - S\left(\alpha^{k},\alpha^{k}\right), \end{cases} \tag{25}$$

converges in $l^{2}(\mathbb{R})$.

Proof Looking at (25) as a fixed point problem,

$$\alpha^{k+1} = g\left(\alpha^{k}\right),$$

where g is a map from $l^{2}(\mathbb{R})$ to itself obviously defined from (25), (6), and (23). Let us compute $g'(\mu)$ for $\mu\in l^{2}(\mathbb{R})$:

$$g'(\mu) = Id + D_{\tilde{\alpha}}S_{obs}(\mu) - D_{\tilde{\alpha}}S(\mu,\mu) - D_{\alpha}S(\mu,\mu),$$

with all the derivatives being justified. Then, in particular for

$$D_{\tilde{\alpha}}S(\mu,\mu) = D_{\tilde{\alpha}}S_{obs}(\mu)$$

$$g'(\mu) = Id - D_{\alpha}S(\mu,\mu),$$

and from (8) (in Proposition 1), we get

$$g'(\mu) = -M,$$

where $M \in L\left(l^2(\mathbb{R})\right)$ such that $\|(M_i)\|_{l^2(\mathbb{R})} = \frac{\varepsilon}{i},\ i = 1...\infty.$

Now let us compute the Hilbert–Schmidt norm of $g'(\mu)$:

$$\begin{aligned} \|g'(\mu)\|_{HS}^2 &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(g'(\mu))_{ij}^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(g'(\mu))_{ij}^2 \\ &= \sum_{i=1}^{\infty}\|(M_i)\|_{l^2(\mathbb{R})}^2 = \varepsilon^2\sum_{i=1}^{\infty}\frac{1}{i^2}. \end{aligned}$$

The value of the series in (26) can be set by choosing an appropriate value for ε. Namely, we can take ε such that

$$\|g'(\mu)\|_{HS}^2 < 1.$$

Then the iteration process (25) is locally convergent in the space $l^2(\mathbb{R})$. □

References

[1] Ainseba BE, Kernevez JP, Luce R. Application des sentinelles à l'identification des pollutions dans une rivière. Esaim-Math Model Num 1994; 28: 297-312 (in French).

[2] Bodart O, Demeestere P. Sentinels for the identification of an unknown boundary. Math Mod Meth Appl S 1997; 7: 871-885.

[3] Ekeland I, Temam R. Convex Analysis and Variational Problems. Amsterdam, the Netherlands: North-Holland, 1976.

[4] Kernevez JP. The Sentinel Method and its Application to Environmental Pollution Problems. Boca Raton, FL, USA: CRC Press, 1997.

[5] Lions JL. Sentinelles pour les Systèmes Distribués à Données Incomplètes. Paris, France: Masson, 1992 (in French).

[6] Lions JL, Magenes E. Problèmes aux Limites non Homogènes et Applications Vol 1. Paris, France: Dunod, 1968 (in French).

[7] Lions JL, Magenes E. Problèmes aux Limites non Homogènes et Applications Vol 2. Paris, France: Dunod, 1968 (in French).

[8] Massengo G, Nakoulima O. Sentinels with given sensitivity. Eur J Appl Math 2008; 19: 21-40.

[9] Massengo G, Puel JP. Boundary sentinels with given sensitivity. Rev Mat Complut 2009; 22: 165-185.

[10] Miloudi Y, Nakoulima O, Omrane A. On the instantaneous sentinels in pollution problems of incomplete data. Inverse Probl Eng 2009; 17: 451-459.

[11] Mophou GM, Velin J. Null controllability problem with constraint on the control deriving from boundary discriminating sentinels. Nonlinear Anal 2009; 71: e910-e924.

[12] Naicase S. Analyse Numérique et Equations aux Dérivées Partielles. Paris, France: Dunod, 2000 (in French).

[13] Nakoulima O. A revision of J. L. Lions' notion of sentinels. Portugal Math 2008; 65: 1-22.

[14] Nakoulima O. Optimal control for distributed systems subject to null-controllability. Application to discriminating sentinels. Esaim Contr Op Ca Va 2007; 13: 623-638.

[15] Pironneau O. Optimal Shape Design for Elliptic Systems. New York, NY, USA: Springer, 1984.

[16] Simon J. Domain variation for drag in stokes flow. Lect Notes Contr Inf 1991; 159: 28-42.