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Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals

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Abstract: In this paper, by applying the p-adic q-integrals to a family of continuous differentiable functions on the ring of p-adic integers, we construct new generating functions for generalized Apostol-type numbers and polynomials attached to the Dirichlet character of a finite abelian group. By using these generating functions with their functional equations, we derive various new identities and relations for these numbers and polynomials. These results are generalizations of known identities and relations including some well-known families of special numbers and polynomials such as the generalized Apostol-type Bernoulli, the Apostol-type Euler, the Frobenius–Euler numbers and polynomials, the Stirling numbers, and other families of numbers and polynomials. Moreover, by the help of these generating functions, we also construct other new families of numbers and polynomials with their generating functions. By using these functions, we investigate some fundamental properties of these numbers and polynomials. Finally, we also give explicit formulas for computing the Apostol–Bernoulli and Apostol–Euler numbers.

Key words: Generalized Bernoulli numbers and polynomials, generalized Euler numbers and polynomials, Apostol–Bernoulli and Apostol–Euler numbers and polynomials, Daehee numbers and polynomials, Stirling numbers, generating function, Dirichlet character, p-adic Volkenborn integral

1. Introduction

In this paper, by using the p-adic Volkenborn integral and the p-adic fermionic integral method, we construct generating functions. With the aid of these functions, we define some new families of special numbers and polynomials. As is well known, the special numbers and polynomials have many vital applications, not only in nearly all branches mathematics but also in other fields such as physics and engineering because it is fairly easy to do mathematical computation and operations by using polynomials. Polynomials and their generating functions are also used to solve real-world problems such as in physics, engineering, and biology. Therefore, by using the p-adic integral equation method and generating functions and their functional techniques, we introduce and investigate the various fundamental properties of our new families of the Apostol-type numbers and polynomials associated with the Dirichlet character with conductor d. We also show that our new numbers and polynomials are closely related to well-known classical numbers and polynomials that are the generalized Bernoulli numbers and polynomials and the generalized Euler numbers and polynomials, the Stirling numbers, and other families of numbers and polynomials such as the Frobenius–Euler polynomials, the Apostol-type Bernoulli and Euler numbers and polynomials, and the Daehee and Changhee numbers and polynomials.

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In this paper we need the following notations and definitions:

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ denote the set of integers, the set of real numbers, the set of complex numbers, and the set of p-adic integers, respectively. $0^n = 1$ if $n = 0$ and $0^n = 0$ if $n \in \mathbb{N}$. Moreover, let $v \in \mathbb{N}_0$.

$$(x)_v = x(x-1)\cdots(x-v+1),$$

$(x)_0 = 1$ and

$$\binom{x}{v} = \frac{x(x-1)\cdots(x-v+1)}{v!} = \frac{(x)_v}{v!}$$

(cf. [1–39] and the references cited therein).

There is benefit in expressing the following comments on the λ-Bernoulli numbers and polynomials and λ-Euler numbers and polynomials, which have been studied in different sets. That is, on the set of complex numbers, we assume that $\lambda \in \mathbb{C}$, and on set of p-adic numbers numbers or p-adic integrals, we assume that $\lambda \in \mathbb{Z}_p$.

In order to define our new families of special numbers and polynomials, we also need the next well-known classical numbers and polynomials with their generating functions:

The Apostol–Bernoulli polynomials, $\mathcal{B}_n(x;\lambda)$, are defined as follows:

$$F_A(t,x;\lambda) = \frac{te^{tx}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!}. \tag{1.1}$$

Observe that

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$$

denotes the Apostol–Bernoulli numbers (cf. [18, 28, 37–39] and the references cited therein). Note that $B_n = \mathcal{B}_n(0;1)$ denotes the classical Bernoulli numbers (cf. [1–39] and the references cited therein).

Let $d \in \mathbb{N}$ and $(\mathbb{Z}/d\mathbb{Z})^*$ denote the unit group of reduced residue class modulo d. Throughout this paper, χ is a Dirichlet character with modulo d, which is a group homomorphism, i.e.

$$\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{C}\setminus\{0\}$$

(cf. [2]).

The generalized Apostol–Bernoulli numbers attached to the Dirichlet character, $\mathcal{B}_{n,\chi}(\lambda)$, are defined as follows:

$$\sum_{j=0}^{d-1} \frac{\lambda^j e^{tj} t\chi(j)}{\lambda^d e^{td} - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\chi}(\lambda)\frac{t^n}{n!}, \tag{1.2}$$

(cf. [1, 12, 19, 21, 39] and the references cited therein).

By combining (1.2) with (1.1), one can easily get

$$\mathcal{B}_{n,\chi}(\lambda) = d^{n-1}\sum_{j=0}^{d-1} \lambda^j \chi(j)\mathcal{B}_n\left(\frac{j}{d};\lambda^d\right).$$

If χ is a trivial character in (1.2), then the numbers $\mathcal{B}_{n,\chi}(\lambda)$ reduce to the Apostol–Bernoulli numbers; that is,

$$\mathcal{B}_n(\lambda) = \mathcal{B}_{n,1}(\lambda)$$

(cf. [1, 12, 19, 21, 27, 29, 39] and the references cited therein).

The Apostol–Euler polynomials $\mathcal{E}_n(x,\lambda)$ are defined as follows:

$$F_{P1}(t,x;\lambda)=\frac{2e^{tx}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(x,\lambda)\frac{t^n}{n!}. \tag{1.3}$$

If we substitute $x=0$ into (1.3), then we have the Apostol–Euler numbers:

$$\mathcal{E}_n(\lambda)=\mathcal{E}_n(0,\lambda).$$

Substituting $\lambda=1$ into (1.3), one easily sees that

$$E_n=\mathcal{E}_n^{(1)}(1)$$

denotes the classical Euler numbers (cf. [4–39] and the references cited therein).

The generalized Apostol–Euler numbers attached to the Dirichlet character, $\mathcal{E}_{n,\chi}(\lambda)$, are defined as follows:

$$2\sum_{j=0}^{d-1}\frac{\lambda^j e^{tj}\chi(j)}{\lambda^d e^{td}+1}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\chi}(\lambda)\frac{t^n}{n!} \tag{1.4}$$

(cf. [19, 21, 39] and the references cited therein).

By combining (1.4) with (1.3), one easily sees that

$$\mathcal{E}_{n,\chi}(\lambda)=d^n\sum_{j=0}^{d-1}\lambda^j\chi(j)\mathcal{E}_n\left(\frac{j}{d};\lambda^d\right).$$

When $\chi\equiv 1$ in (1.4), one has

$$\mathcal{E}_n(\lambda)=\mathcal{E}_{n,1}(\lambda)$$

(cf. [19, 21, 39]).

Let $u\in\mathbb{C}$ with $u\neq 1$. The Frobenius–Euler numbers are defined as follows:

$$F_f(t,u)=\frac{1-u}{e^t-u}=\sum_{n=0}^{\infty}H_n(u)\frac{t^n}{n!}, \tag{1.5}$$

(cf. [7, 25, 34, 39] and the references cited therein).

The Stirling numbers of the first kind, $S_1(n,k)$, are defined as follows:

$$F_{S1}(t,k)=\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!}. \tag{1.6}$$

By using the above generating function, we have

$$S_1(0,0)=1.$$

The other properties are given as follows:

$S_1(0,k)=0$ if $k>0$. $S_1(n,0)=0$ if $n>0$. $S_1(n,k)=0$ if $k>n$ and also

$$S_1(n+1,k)=-nS_1(n,k)+S_1(n,k-1)$$

(cf. [32, 35], and see also the references cited in each of these earlier works).

Let $k\in\mathbb{N}_0$. The Stirling numbers of the second kind, $S_2(n,k)$, are defined as follows:

$$F_S(t,k)=\frac{(e^t-1)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!}. \tag{1.7}$$

By using (1.7), an explicit formula for the numbers $S_2(n,k)$ is given by

$$S_2(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \tag{1.8}$$

From (1.7), we also have

$$S_2(0,0)=1.$$

If $k>n$, then

$$S_2(n,k)=0.$$

$S_2(n,0)=0$ if $n>0$ and also

$$S_2(n+1,k)=S_2(n,k-1)+kS_2(n,k)$$

(cf. [4, 32, 38] and the references cited therein).

The Bernoulli numbers of the second kind $b_n(0)$ are defined by means of the following generating function:

$$F_{b2}(t)=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n(0)\frac{t^n}{n!} \tag{1.9}$$

(cf. [32, p. 116]).

We also note that some authors denote the Bernoulli numbers of the second kind by C_n, which are also the so-called *Cauchy numbers*.

Integrating a falling factorial polynomial

$$(u)_n=u(u-1)\cdots(u-n+1)$$

from 0 to 1, the Bernoulli numbers of the second kind are also computed by the following integral formula:

$$b_n(0)=\int_0^1(u)_n du$$

(cf. [26, 32]; see also the references cited in each of these earlier works).

Let $\mathbb{K}$ be a field with a complete valuation. Let $C^1(\mathbb{Z}_p\to\mathbb{K})$ be a set of continuous differentiable functions.

In order to define the p-adic q-integral on $\mathbb{Z}_p$, we need the q-Haar distribution, defined by Kim [14], as follows:

$$\mu_q(x) = \mu_q(x + p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]},$$

where $q \in \mathbb{C}_p$ with $\mid 1 - q \mid_p < 1$ and

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1 \end{cases}.$$

We observe that

$$\lim_{q \to 1} [x : q] = x.$$

The p-adic q-integral of a function f is defined by Kim [14] as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x) q^x \tag{1.10}$$

where $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$.

Taking limit $q \to 1$, (1.10) reduces to the Volkenborn integral (the p-adic bosonic integral), which is used to construct the Bernoulli type numbers and polynomials and the others, as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.11}$$

where

$$\mu_1(x) = \mu_1(x + p^N\mathbb{Z}_p) = \frac{1}{p^N}$$

(cf. [33]; see also [11, 14, 21]).

Substituting

$$f(x) = \binom{x}{j}, \tag{1.12}$$

$j \in \mathbb{N}_0$, into (1.11), we have

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1} \tag{1.13}$$

(cf. [33, p. 168, Proposition 55.3]).

Kim [18] defined the fermionic p-adic integral, which is used to construct generating functions for the Euler, the Genochhi type numbers, and the others, as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{1.14}$$

where $p \neq 2$ and

$$\mu_{-1}(x) = \mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = \frac{(-1)^x}{p^N}$$

(cf. [11, 18]).

In order to give the integral of a function associated with the Dirichlet character with conductor d, we also need the following notations.

Let p be a fixed prime. Letting d be a fixed positive integer with $(p,d) = 1$, we have

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \\ \mathbb{X}_1 &= \mathbb{Z}_p, \end{aligned}$$

and

$$a + dp^N\mathbb{Z}_p = \left\{x \in \mathbb{X} \mid x \equiv a\left(mod\left(dp^N\right)\right)\right\}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

Let $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Thus, we have

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \int_{\mathbb{X}} f(x)d\mu_1(x) \tag{1.15}$$

(cf. [11, 12, 14, 16, 33]).

Let

$$E^d f(x) = f(x+d).$$

The following integral equation was defined by Kim [21, Theorem 3]:

$$q^d \int\limits_{\mathbb{Z}_p} E^d f(x)\, d\mu_{-q}(x) - (-1)^d \int\limits_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^{d-j-1} q^j f(j), \tag{1.16}$$

where d is a positive integer. Substituting $d = 1$ into (1.16), one has

$$\int\limits_{\mathbb{Z}_p} (qf(x+1) + f(x))\, d\mu_{-q}(x) = (q+1)\, f(0).$$

When $q \to 1$ in the above integral equation, we easily see that

$$\int\limits_{\mathbb{Z}_p} (f(x+1) + f(x))\, d\mu_{-1}(x) = 2f(0)$$

(cf. [21]). Substituting (1.12) into the above integral equation, Kim et al. [10] gave the following formula:

$$\int\limits_{\mathbb{Z}_p} \binom{x}{j} d\mu_{-1}(x) = \frac{(-1)^j}{2^j}. \tag{1.17}$$

Let us give a brief summary of our results as follows:

In Section 2, by using the fermionic p-adic integral method, we construct a generating function for generalized Apostol-type numbers and polynomials attached to the Dirichlet character χ. A new family of generalized Apostol-type numbers are defined by the Dirichlet character with even and odd conductors. In other subsections, we define generalized Apostol–Changhee numbers attached to the Dirichlet character with odd conductor. By using generating functions and their functional equations, we derive many identities and relations associated with the generalized Apostol–Daehee numbers and polynomials, Apostol–Changhee numbers and polynomials, Stirling numbers, Bernoulli numbers of the second kind, Frobenius–Euler polynomials, generalized Bernoulli numbers, and generalized Euler numbers. Finally, we give the p-adic Volkenborn integral representations of these numbers with some combinatorial sums.

2. Generating functions for generalized Apostol-type numbers

By applying the fermionic p-adic q-integral on the set of $\mathbb{X}$ to the following function

$$f(x,t;\lambda)=\lambda^{x}(1+\lambda t)^{x}\chi(x), \tag{2.1}$$

where $\lambda\in\mathbb{Z}_p$, we construct generating functions for the generalized Apostol–Changhee numbers and polynomials attached to Dirichlet character χ with conductor d. By using these functions with their functional equations, we study and investigate some fundamental properties of these numbers and polynomials. We also show that these numbers and polynomials are related to the Stirling numbers, the Frobenius–Euler polynomials, the generalized Bernoulli numbers, the generalized Euler numbers, and the Daehee numbers and polynomials. Finally, fermionic integral representation of these numbers can be given.

Substituting (2.1) into (1.16), we get

$$\int\limits_{\mathbb{X}}\lambda^{x}(1+\lambda t)^{x}\chi(x)d\mu_{-q}\left(x\right)=\frac{[2]}{\left(\lambda q\right)^{d}(1+\lambda t)^{d}-(-1)^{d}}\sum_{j=0}^{d-1}(-1)^{d-j-1}\chi(j)\left(\lambda q\right)^{j}(1+\lambda t)^{j}, \tag{2.2}$$

where $\lambda\in\mathbb{Z}_p$.

We have two cases in the above integral equation. In order to construct the generalized Apostol-type numbers and polynomials and other related numbers and polynomials, we peruse these cases, which are associated with a conductor of the Dirichlet character, in the next sections.

2.1. Generating functions for generalized Apostol–Changhee numbers and polynomials attached to the Dirichlet character with odd conductor

Here we give generating functions for generalized Apostol–Changhee numbers and polynomials associated with the Dirichlet character with odd conductor. By aid of these functions, we not only investigate many fundamental properties of these numbers and polynomials, but also derive various identities related to the generalized Apostol–Daehee and Apostol–Changhee numbers and polynomials, the Stirling numbers, the Bernoulli numbers of the second kind, the generalized Bernoulli numbers, the generalized Euler numbers, and the Frobenius–Euler polynomials.

Let d be an odd integer. If χ is the Dirichlet character with conductor d, then equation (2.2) reduces to the following equation:

$$\int\limits_{\mathbb{X}}\lambda^{x}(1+\lambda t)^{x}\chi(x)d\mu_{-q}\left(x\right)=\frac{[2]}{\left(\lambda q\right)^{d}(1+\lambda t)^{d}+1}\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}(1+\lambda t)^{j}. \tag{2.3}$$

By using the above integral equation, we define the generalized Apostol–Changhee numbers and polynomials by means of the following generating functions, respectively:

$$F_{\mathfrak{C}}(t;\lambda,q,\chi)=\frac{(1+q)\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\,(\lambda q)^{j}\,(1+\lambda t)^{j}}{(\lambda q)^{d}\,(1+\lambda t)^{d}+1}=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^{n}}{n!} \tag{2.4}$$

and

$$F_{\mathfrak{C}}(t,z;\lambda,q,\chi)=F_{\mathfrak{C}}(t;\lambda,q,\chi)(1+\lambda t)^{z}=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}. \tag{2.5}$$

From the above generating functions, we get

$$\mathfrak{Ch}_{n,\chi}(\lambda,q)=\mathfrak{Ch}_{n,\chi}(0;\lambda,q).$$

By using (2.4) and (2.5), we get

$$\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\binom{z}{n}\lambda^{n}t^{n}\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^{n}}{n!}.$$

Making the Cauchy product of the above right-hand side of the two infinite series, we get

$$\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}\mathfrak{Ch}_{j,\chi}(\lambda,q)\right)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^{n}}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 1 *Let $n\in\mathbb{N}_0$. Then we have*

$$\mathfrak{Ch}_{n,\chi}(z;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}\mathfrak{Ch}_{j,\chi}(\lambda,q).$$

Remark 1 *If $q\to 1$ and $\lambda=1$ and $\chi\equiv 1$, then (2.5) reduces to the generating function for the Changhee polynomials:*

$$\frac{2}{t+2}(1+t)^{z}=\sum_{n=0}^{\infty}Ch_{n}(z)\frac{t^{n}}{n!}.$$

From this equation, we see that

$$Ch_{n}=Ch_{n}(0),$$

where Ch_n denotes the Changhee numbers (cf. [10, 22], and also see [6, 8–10, 15]).

By using (2.3), a fermionic p-adic q-integral representation for the generalized Changhee numbers is given by the following theorem:

Theorem 2 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{Ch}_{n,\chi}(\lambda, q) = \int_{\mathbb{X}} \lambda^{x+n}(x)_n \chi(x) d\mu_{-q}(x). \tag{2.6}$$

By using (2.3), we also get the following functional equation:

$$F_{\mathfrak{C}}(t, x; \lambda, q, \chi) = \frac{(1+q)}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j F_{P1}\left(d \log(1+\lambda t), \frac{j}{d}; (\lambda q)^d\right).$$

Combining the above equation with (1.3) and (2.4), we obtain

$$\frac{(1+q)}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j \sum_{n=0}^{\infty} d^n \mathcal{E}_n\left(\frac{j}{d}; (\lambda q)^d\right) \frac{(\log(1+\lambda t))^n}{n!} = \sum_{m=0}^{\infty} \mathfrak{C}h_{m,\chi}(\lambda, q) \frac{t^m}{m!}.$$

Combining (1.6) with the above equation, we obtain

$$\frac{[2]}{2} \sum_{j=0}^{d-1} \chi(j) (-\lambda q)^j \sum_{n=0}^{\infty} d^n \mathcal{E}_n\left(\frac{j}{d}; (\lambda q)^d\right) \sum_{m=0}^{\infty} S_1(m, n) \frac{(\lambda t)^m}{m!} = \sum_{m=0}^{\infty} \mathfrak{C}h_{m,\chi}(\lambda, q) \frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, since $S_1(m, n) = 0$, $m < n$, we arrive at the following theorem:

Theorem 3 *Let m be a nonnegative integer. Then we have*

$$\mathfrak{Ch}_{m,\chi}(\lambda, q) = \sum_{j=0}^{d-1} (-q)^j \chi(j) \sum_{n=0}^{m} \lambda^{j+m} d^n \mathcal{E}_n\left(\frac{j}{d}; (\lambda q)^d\right) S_1(m, n). \tag{2.7}$$

By combining (2.7) with the following widely known interesting formula including the generalized Euler numbers,

$$\mathfrak{Ch}_{n,\chi}(\lambda) = d^n \sum_{j=0}^{d-1} (-1)^j \lambda^j \chi(j) \mathcal{E}_n\left(\frac{j}{d}; \lambda^d\right),$$

we arrive at the following corollary:

Corollary 1 *Let $m \in \mathbb{N}_0$. Then we have*

$$\mathfrak{Ch}_{m,\chi}(\lambda, q) = \sum_{n=0}^{m} \mathcal{E}_{n,\chi}(q\lambda) S_1(m, n). \tag{2.8}$$

Remark 2 *If $q \to 1$, $\lambda = 1$ and $\chi \equiv 1$, then (2.8) reduces to the following well-known result, which was proved by Kim et al. (cf. [10]):*

$$Ch_m = \sum_{n=0}^{m} E_n S_1(m, n).$$

Substituting $\lambda t = e^u - 1$ into (2.4), we get

$$\frac{[2]}{(\lambda q)^d e^{du} + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j e^{ju} = \sum_{n=0}^{\infty} \mathfrak{Ch}_{n,\chi}(\lambda, q) \frac{(e^u - 1)^n}{n!}. \tag{2.9}$$

By substituting (1.7) into the above equation, since $S_2(m,n) = 0$ with $n > m$, we get

$$\sum_{m=0}^{\infty} \frac{[2]}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j d^m \mathcal{E}_m \left(\frac{j}{d}; (\lambda q)^d \right) \frac{u^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m,n)}{\lambda^n} \frac{u^m}{m!}.$$

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4

$$\frac{[2]}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j d^m \mathcal{E}_m \left(\frac{j}{d}; (\lambda q)^d \right) = \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m,n)}{\lambda^n}$$

or

$$\mathcal{E}_m, \chi(\lambda) = \frac{2}{[2]} \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m,n)}{\lambda^n}. \tag{2.10}$$

Remark 3 *If $\chi \equiv 1$, $q \to 1$, and $\lambda = 1$, then (2.10) reduces to the following well-known result, which was proved by Kim et al. (cf. [10]):*

$$E_m = \sum_{n=0}^{m} Ch_n S_2(m,n).$$

Replacing λ by $-\lambda$ and $1 - \lambda t = e^u$ in equation (2.4), we get

$$u \sum_{n=0}^{\infty} (-1)^{n+1} \mathfrak{Ch}_{n,\chi}(-\lambda, q) \frac{(e^u - 1)^n}{\lambda^n n!} = [2] \sum_{m=0}^{\infty} \mathcal{B}_{m,\chi}(\lambda q) \frac{u^m}{m!}.$$

Substituting (1.7) into the above equation, since $S_2(m,n) = 0$ with $n > m$, we get

$$\sum_{m=0}^{\infty} \mathcal{B}_{m,\chi}(\lambda q) \frac{u^m}{m!} = \sum_{m=0}^{\infty} \left(\frac{m}{[2]} \sum_{n=0}^{m-1} \frac{(-1)^{n+1}}{\lambda^n} \mathfrak{Ch}_{n,\chi}(-\lambda, q) S_2(m-1, n) \right) \frac{u^m}{m!}.$$

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 5 *Let m be a positive integer. Then we have*

$$\mathcal{B}_{m,\chi}(\lambda q) = \frac{m}{[2]} \sum_{n=0}^{m-1} \frac{(-1)^{n+1}}{\lambda^n} \mathfrak{Ch}_{n,\chi}(-\lambda, q) S_2(m-1, n). \tag{2.11}$$

If $\chi \equiv 1$ and $q \to 1$, then equation (2.11) reduces to the following corollary:

Corollary 2 $\mathcal{B}_m(\lambda) = \frac{m}{2}\sum_{n=0}^{m-1}\frac{(-1)^{n+1}}{\lambda^n}Ch_n(-\lambda,1)S_2(m-1,n)$.

By combining (1.5) with (2.9), we obtain

$$\sum_{m=0}^{\infty}\frac{(1+q)}{(\lambda^d q^d+1)}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j d^m H_m\left(\frac{j}{d};-\frac{1}{(\lambda q)^d}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\mathfrak{Ch}_{n,\chi}(\lambda,q)S_2(m,n)}{\lambda^n}\frac{t^m}{m!},$$

where d is an odd integer. Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 6 *Let d be an odd integer. Then we have*

$$\frac{(1+q)}{(\lambda^d q^d+1)}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j d^m H_m\left(\frac{j}{d};-\frac{1}{(\lambda q)^d}\right) = \sum_{n=0}^{m}\frac{\mathfrak{Ch}_{n,\chi}(\lambda,q)S_2(m,n)}{\lambda^n}.$$

If $\chi\equiv 1$ and $q\to 1$, then (2.11) reduces to the following corollary:

Corollary 3

$$H_m\left(-\frac{1}{\lambda}\right) = \frac{\lambda+1}{2}\sum_{n=0}^{m}\frac{Ch_n(\lambda,1)S_2(m,n)}{\lambda^n}.$$

2.2. A new family of generalized Apostol-type numbers attached to the Dirichlet character with even conductor

Here we give generating functions for new families of generalized Apostol-type numbers and polynomials attached to the Dirichlet character with even conductor. These functions give us many facilities to derive many identities and relations. These relations and identities are related to various well-known special numbers and polynomials, such as the generalized Apostol–Daehee and Apostol–Changhee numbers and polynomials, the Stirling numbers, the Bernoulli numbers of the second kind, the Frobenius–Euler polynomials, the generalized Bernoulli numbers, the generalized Euler numbers, and the Frobenius–Euler polynomials.

Let d be an even integer. If χ is the Dirichlet character with even conductor d, then equation (2.2) reduces to the following integral equation:

$$\int_{\mathbb{X}}\lambda^x(1+\lambda t)^x\chi(x)d\mu_{-q}(x) = \frac{-(1+q)}{(\lambda q)^d(1+\lambda t)^d-1}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j(1+\lambda t)^j.$$

By using the above equation, we define a new family of special numbers including generalized Apostol-type numbers by means of the following generating functions:

$$H(t;\lambda,q,\chi) = \frac{(1+q)\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j(1+\lambda t)^j}{(\lambda q)^d(1+\lambda t)^d-1} = \sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}, \tag{2.12}$$

where d is an even positive integer and $\lambda\in\mathbb{Z}_p$ with $\lambda\neq 1$.

We modify equation (2.12) as follows:

$$H(t;\lambda,q,\chi) = (1+q)\sum_{m=0}^{\infty}(-1)^m\chi(m)\,(\lambda q)^m\,(1+\lambda t)^m = \sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}.$$

If $q\to 1$ in (2.12), then we get the following generating functions for the numbers $Y_{n,\chi}(\lambda)$:

$$\frac{2\sum_{j=0}^{d-1}(-1)^j\chi(j)\lambda^j(1+\lambda t)^j}{\lambda^d(1+\lambda t)^d-1}=\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda)\frac{t^n}{n!}.$$

Using the motivation of the above generating equation, we also derive another generating function for a new family of numbers, $Y_n(\lambda)$, as follows:

$$g(t;\lambda)=\frac{2}{\lambda^2 t+\lambda-1}=\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!}. \tag{2.13}$$

The numbers $Y_n(\lambda)$ are related to various kinds of well-known numbers such as the Apostol–Bernoulli numbers, the q-Euler numbers, the Stirling numbers, the q-Changhee numbers, and the Daehee numbers. We investigate these relations in next section.

It is time to define a new family of the generalized Apostol-type polynomials, $Y_{n,\chi}(z;\lambda,q)$, by means of the following generating function:

$$H(t,z;\lambda,q,\chi)=(1+\lambda t)^z H(t;\lambda,q,\chi)=\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}. \tag{2.14}$$

Combining (2.14) with (2.12), we deduce that

$$Y_{n,\chi}(\lambda,q)=Y_{n,\chi}(0;\lambda,q).$$

By using (2.14) and (2.12), we obtain

$$\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(z)_n\lambda^n\frac{t^n}{n!}\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}.$$

By using the Cauchy rule of product for the above series, we get

$$\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}Y_{j,\chi}(\lambda,q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 7 *Let $n\in\mathbb{N}_0$. Let d be an even positive integer. Then we have*

$$Y_{n,\chi}(z;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}Y_{j,\chi}(\lambda,q). \tag{2.15}$$

Substituting $\lambda t = e^u - 1$ into (2.12), we get the following functional equation:

$$H(\lambda^{-1}(e^u-1);\lambda,q,\chi) = \frac{1+q}{du}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j F_A\left(du,\frac{j}{d};\lambda^d q^d\right).$$

Combining the above equation with (1.1) and (2.12), we have

$$(1+q)\sum_{n=0}^{\infty}d^{n-1}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j\mathcal{B}_n\left(\frac{j}{d};\lambda^d q^d\right)\frac{u^n}{n!} = \sum_{n=0}^{\infty}n\sum_{m=0}^{n-1}\lambda^{-m}Y_{m,\chi}(\lambda,q)S_2(n-1,m)\frac{u^n}{n!}.$$

Equating coefficients of $\frac{u^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 8 *Let $n \in \mathbb{N}$. Let d be an even positive integer. Then we have*

$$\sum_{m=0}^{n-1}\lambda^{-m}Y_{m,\chi}(\lambda,q)S_2(n-1,m) = \frac{(1+q)\,d^{n-1}}{n}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j\mathcal{B}_n\left(\frac{j}{d};\lambda^d q^d\right).$$

We give the following functional equation:

$$d\log(1+\lambda t)\mathbf{H}(t;\lambda,q,\chi) = (1+q)\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j F_A\left(d\log(1+\lambda t),\frac{j}{d};\lambda^d q^d\right).$$

Combining the above equation with (1.1) and (2.12), we obtain

$$\sum_{m=0}^{\infty}Y_{m.\chi}(\lambda,q)\frac{t^m}{m!} = (1+q)\sum_{n=0}^{\infty}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j d^{n-1}\mathcal{B}_n\left(\frac{j}{d};(\lambda q)^d\right)\frac{(\log(1+\lambda t))^{n-1}}{n!}.$$

Combining the above equation with (1.6), we have

$$\sum_{m=0}^{\infty}Y_{m.\chi}(\lambda,q)\frac{t^m}{m!} = (1+q)\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j d^{n-1}\mathcal{B}_{n+1}\left(\frac{j}{d};(\lambda q)^d\right)S_1(m,n)\frac{(\lambda t)^m}{(n+1)m!}$$

since $S_1(m,n)=0$ with $m<n$. Equating coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 9 *Let $m \in \mathbb{N}$. Let d be an even positive integer. Then we have*

$$Y_{m.\chi}(\lambda,q) = \lambda^m\sum_{n=0}^{m}\frac{(1+q)\,d^{n-1}S_1(m,n)}{n+1}\sum_{j=0}^{d-1}(-1)^j\chi(j)(\lambda q)^j\mathcal{B}_{n+1}\left(\frac{j}{d};(\lambda q)^d\right). \tag{2.16}$$

If we set $\chi \equiv 1$ and $q \to 1$ in (2.16), we get the following corollary:

Corollary 4 *Let $m \in \mathbb{N}$. Then we have*

$$Y_m(\lambda) = 2\lambda^m\sum_{n=0}^{m}\frac{\mathcal{B}_{n+1}(\lambda)\,S_1(m,n)}{n+1}.$$

2.3. Derivative and integrals of the polynomials $Y_{n,\chi}(z;\lambda,q)$

Here we give derivative formulas for the polynomials $Y_{n,\chi}(z;\lambda,q)$. By applying the p-adic Volkenborn integral, we give some summation and combinatorial sums. By differentiating equation (2.14) with respect to z, we get

$$\frac{\partial}{\partial z}H(t,z;\lambda,q,\chi)=H(t,z;\lambda,q,\chi)\log(1+\lambda t).$$

From this equation, we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda t)^{n+1}}{n+1}\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{\partial}{\partial z}Y_{n+1,\chi}(z;\lambda,q)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{\lambda^{j+1}}{(j+1)\,(n-j)!}Y_{n-j,\chi}(z;\lambda,q)t^n.$$

Comparing the coefficients of t^n on both sides of the above equation, we arrive at the following theorem:

Theorem 10 *Let n be a positive integer. Then we have*

$$\frac{\partial}{\partial z}Y_{n+1,\chi}(z;\lambda,q)=\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}j!\lambda^{j+1}Y_{n-j,\chi}(z;\lambda,q).$$

Integrating both sides of the equation (2.15) from 0 to 1, we get

$$\int_0^1 Y_{n,\chi}(z;\lambda,q)dz=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}C_{n-j}Y_{j,\chi}(\lambda,q),$$

where C_{n-j} denotes the Cauchy numbers of the first kind.

By applying the p-adic integral to the equation (2.15), we arrive at the bosonic q-integral representation for the polynomials $Y_{n,\chi}(z;\lambda,q)$ as follows:

$$\int_{\mathbb{X}} Y_{n,\chi}(z;\lambda,q)d\mu_1(z)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}D_{n-j}Y_{j,\chi}(\lambda,q).$$

Since

$$D_n=(-1)^n\frac{n!}{n+1}$$

(cf. [5, 8]), we also get the following combinatorial sums:

Theorem 11

$$\int_{\mathbb{X}} Y_{n,\chi}(z;\lambda,q)d\mu_1(z)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{(n-j)!\lambda^{n-j}}{n+1-j}Y_{j,\chi}(\lambda,q).$$

By applying the fermionic p-adic integral to the equation (2.15), we arrive at the fermionic q-integral representation for the polynomials $Y_{n,\chi}(z;\lambda,q)$ as follows:

$$\int_{\mathbb{X}} Y_{n,\chi}(z;\lambda,q)d\mu_{-1}(z) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} Ch_{n-j} Y_{j,\chi}(\lambda,q).$$

Since

$$Ch_n = (-1)^n \frac{n!}{2^n}$$

(cf. [10]), we also get the following combinatorial sums:

Theorem 12

$$\int_{\mathbb{X}} Y_{n,\chi}(z;\lambda,q)d\mu_{-1}(z) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{(n-j)!\lambda^{n-j}}{2^{n-j}} Y_{j,\chi}(\lambda,q).$$

2.4. Fundamental properties of the polynomials $Y_n(z;\lambda)$ and the numbers $Y_n(\lambda)$

Here, using generating functions and functional equations, we derive recurrence relations for the numbers $Y_n(\lambda)$ and the polynomials $Y_n(z;\lambda)$. We also derive some identities and relations including the generalized Bernoulli numbers, the generalized Euler numbers, the Stirling numbers, and the q-Changhee numbers. We also give some new formulas for computing the generalized Bernoulli numbers and the generalized Euler numbers.

By using the umbral calculus method in (2.13), we get the following recurrence relation for the numbers $Y_n(\lambda)$:

Theorem 13 *Let $n \in \mathbb{N}$. Starting with*

$$Y_0(\lambda) = \frac{2}{\lambda - 1},$$

we have

$$Y_n(\lambda) = \frac{n\lambda^2}{1-\lambda} Y_{n-1}(\lambda). \tag{2.17}$$

We now give an explicit formula for the number $Y_n(\lambda)$ by the following theorem:

Theorem 14 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_n(\lambda) = 2(-1)^n \frac{n!}{\lambda - 1} \left(\frac{\lambda^2}{\lambda - 1}\right)^n. \tag{2.18}$$

Proof We assume that $\left|\lambda^2 t\right| < |\lambda - 1|$. Thus, by equation (2.13), we get

$$\sum_{n=0}^{\infty} Y_n(\lambda) \frac{t^n}{n!} = \frac{2}{\lambda - 1} \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(\lambda - 1)^n} t^n.$$

Comparing the coefficients of t^n on both sides of the above equation, we arrive at the desired result. □

We compute a few values of the numbers $Y_n(\lambda)$ by (2.17), as follows:

$$\begin{aligned} Y_1(\lambda) &= -\frac{2\lambda^2}{(\lambda-1)^2}, Y_2(\lambda) = \frac{4\lambda^4}{(\lambda-1)^3}, \\ Y_3(\lambda) &= -\frac{12\lambda^6}{(\lambda-1)^4}, Y_4(\lambda) = \frac{48\lambda^8}{(\lambda-1)^5}, \ldots \end{aligned}$$

A new family of polynomials, $Y_n(z;\lambda)$, is defined by means of the following generating function:

$$G(t,z;\lambda) = g(t;\lambda)(1+\lambda t)^z = \sum_{n=0}^{\infty} Y_n(z;\lambda)\frac{t^n}{n!} \tag{2.19}$$

so that, obviously,

$$Y_n(\lambda) = Y_n(0;\lambda).$$

By using the same proof of Theorem 7, and using (2.15), we get following formula:

$$Y_n(z;\lambda) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} (z)_{n-j} Y_j(\lambda). \tag{2.20}$$

By using (2.15), we also get

$$2(1+\lambda t)^z = \sum_{n=0}^{\infty} \lambda^2 Y_n(z;\lambda)\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} (\lambda-1)\, Y_n(z;\lambda)\frac{t^n}{n!}.$$

By using the above equation, we get

$$2\sum_{n=0}^{\infty} (z)_n\, \lambda^n \frac{t^n}{n!} = \lambda^2 \sum_{n=0}^{\infty} Y_n(z;\lambda)\frac{t^{n+1}}{n!} + (\lambda-1)\sum_{n=0}^{\infty} Y_n(z;\lambda)\frac{t^n}{n!}.$$

Equating coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain a recurrence relation for the polynomials $Y_n(z;\lambda)$ by the following theorem:

Theorem 15 *Let $n \in \mathbb{N}_0$. Then we have*

$$2\,(z)_n\, \lambda^n = n\lambda^2 Y_{n-1}(z;\lambda) + (\lambda-1)\, Y_n(z;\lambda).$$

We compute a few values of the numbers $Y_n(z;\lambda)$ by (2.20), as follows:

$$\begin{aligned} Y_0(z;\lambda) &= \frac{2}{\lambda-1}, \\ Y_1(z;\lambda) &= \frac{2\lambda}{\lambda-1}z - \frac{2\lambda^2}{(\lambda-1)^2}, \\ Y_2(z;\lambda) &= \frac{2\lambda^2}{\lambda-1}z^2 - \frac{6\lambda^3-2\lambda^2}{(\lambda-1)^2}z + \frac{4\lambda^4}{(\lambda-1)^3}, \\ Y_3(z;\lambda) &= \frac{2\lambda^3}{\lambda-1}z^3 - \frac{12\lambda^4-6\lambda^3}{(\lambda-1)^2}z^2 + \frac{22\lambda^5-14\lambda^4+4\lambda^3}{(\lambda-1)^3}z - \frac{12\lambda^6}{(\lambda-1)^4}, \ldots \end{aligned}$$

We give the following functional equation for the generating function $G(t,z;\lambda)$:

$$G(t,z+w;\lambda)=G(t,z;\lambda)(1+\lambda t)^{w}.$$

From this equation, we get

$$\sum_{n=0}^{\infty}Y_n(z+w;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(w)_{n-j}Y_j(z;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 16 *Let $n\in\mathbb{N}_0$. Then we have*

$$Y_n(z+w;\lambda)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(w)_{n-j}Y_j(z;\lambda). \tag{2.21}$$

Substituting $w=-1$ into (2.21), we get

$$Y_n(z-1;\lambda)=n!\sum_{j=0}^{n}\binom{n}{j}(-1)_{n-j}\lambda^{n-j}Y_j(z;\lambda).$$

Theorem 17 *Let $m\in\mathbb{N}$. Then we have*

$$\mathcal{B}_m(\lambda)=\frac{m}{\lambda-1}\sum_{n=0}^{m-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{\lambda}{\lambda-1}\right)^{n}k^{m-1}. \tag{2.22}$$

Proof Substituting $\lambda t=e^u-1$ into (2.13), we get

$$\frac{2}{\lambda e^u-1}=\sum_{n=0}^{\infty}\lambda^{-n}Y_n(\lambda)\frac{(e^u-1)^n}{n!}.$$

Combining the above equation with (1.3), (1.1), and (1.7), we get

$$\sum_{m=0}^{\infty}2\mathcal{B}_m(\lambda)\frac{u^m}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{m-1}m\lambda^{-n}Y_n(\lambda)S_2(m-1,n)\frac{u^m}{m!}$$

since $S_2(m,n)=0$ with $m<n$. Equating coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we get

$$\mathcal{B}_m(\lambda)=\frac{m}{2}\sum_{n=0}^{m-1}\lambda^{-n}Y_n(\lambda)S_2(m-1,n). \tag{2.23}$$

Substituting (2.18) and (1.8) into (2.23), we arrive at the desired result. □

Remark 4 *By using equation (2.23), the Apostol–Bernoulli numbers are easily computed by values of the numbers $Y_n(\lambda)$ and the Stirling numbers of the second kind. Explicit formulas for the Apostol–Bernoulli numbers were also proved by Apostol [1, Eq-(3.7)], and also see [3].*

Remark 5 *The formula in equation (2.22) is very useful and elegant because this formula is a combinatorial sum, which gives us direct computation of the Apostol–Bernoulli numbers.*

In [23], Kim et al. defined the q-Changhee numbers by means of the following generating function:

$$T(t;q) = \frac{q+1}{qt+q+1} = \sum_{n=0}^{\infty} Ch_n(q)\frac{t^n}{n!}. \tag{2.24}$$

By using the above equation, we get the following functional equation:

$$\sum_{n=0}^{\infty} Ch_n(q)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{q}{q+1}\right)^n t^n.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 18

$$Ch_n(q) = (-1)^n\left(\frac{q}{[2]}\right)^n n!. \tag{2.25}$$

By combining (2.24) and (1.3) for $x=0$ with (2.25), we get a explicit formula for the numbers $\mathcal{E}_n\left(\frac{q}{[2]}\right)$ by the following theorem:

Corollary 5

$$\mathcal{E}_n\left(\frac{q}{[2]}\right) = (-1)^n\left(\frac{q}{[2]}\right)^n n!.$$

By using (2.24) and (1.3), we get

$$\frac{[2]}{2q}\sum_{m=0}^{\infty}\mathcal{E}_m\left(\frac{1}{q}\right)\frac{t^m}{m!} = \sum_{n=0}^{\infty} Ch_n(q)\frac{(e^t-1)^n}{q^n n!}.$$

Combining the above equation with (1.7), we get

$$\frac{[2]}{2q}\sum_{m=0}^{\infty}\mathcal{E}_m\left(\frac{1}{q}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{q^n}Ch_n(q)S_2(m,n)\frac{t^m}{m!}$$

since $S_2(m,n)=0$ with $m<n$. Combining the above equation with (2.25), we get

$$\frac{[2]}{2q}\sum_{m=0}^{\infty}\mathcal{E}_m\left(\frac{1}{q}\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^n\frac{n!}{[2]^n}S_2(m,n)\frac{t^m}{m!}.$$

By substituting (1.8) into the above equation and after equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 19

$$\mathcal{E}_m\left(\frac{1}{q}\right) = 2q\sum_{n=0}^{m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^m}{[2]^{n+1}}. \tag{2.26}$$

Remark 6 *The formula in equation (2.26) is also very useful for direct computing of the Apostol–Bernoulli numbers.*

Remark 7 *By using the Changhee numbers of the second kind, Kim et al. [23, Theorem 4] gave the following computation formula for the q-Euler numbers as follows:*

$$E_{n,q} = \sum_{k=0}^{n}\frac{1+[2]}{(q+1)^k}S_2(n,k), \tag{2.27}$$

where the numbers $E_{n,q}$ are defined by means of the following generating function:

$$\frac{1+q}{qe^t+1} = \sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}$$

(cf. [23, 30]). By combining (1.7) with (2.27), we have the following explicit formula for the numbers $E_{n,q}$ as follows:

$$E_{n,q} = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{1+[2]}{k!\,(q+1)^k}j^n.$$

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