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An inequality on the Hodge number $h^{1,1}$ of a fibration and the Mordell–Weil rank

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Abstract: In this paper, we establish some formulas on the Mordell–Weil rank and the Hodge number $h^{1,1}$ for a fibration.

Key words: Mordell–Weil rank, Hodge number, fibration

1. Introduction

Let $f : X \to Y$ be a fibration between complex smooth projective varieties. Let X_K be the generic fiber of f defined over $K := \mathbb{C}(Y)$. When X_K is a curve or an abelian variety with a K-rational point, the Mordell–Weil group $\mathrm{MW}(f)$ of X_K/K has been defined by Shioda [12, 13] and Oguiso [10, 11]. They also established an explicit formula for the rank of $\mathrm{MW}(f)$ under some natural assumptions. We further let $\Delta \subset Y$ be the discriminant divisor of f and let $\Delta = \cup_{i=1}^{k} \Delta_i$ and $f^*(\Delta_i)_{\mathrm{red}} = \cup_{j=1}^{m_i} D_{ij}$ be the irreducible decomposition of Δ and $f^*(\Delta_i)_{\mathrm{red}}$. Then Shioda and Oguiso's formula can be given as:

$$\operatorname{rank} \mathrm{MW}(f) = \rho(X) - \rho(Y) - \operatorname{rank} \mathrm{NS}(X_K) - \sum_{i=1}^{k} (m_i - 1).$$

Here $\rho(X)$ and $\rho(Y)$ denote the Picard number of X and Y, respectively, and $\mathrm{NS}(X_K)$ the Neron–Severi group of X_K.

In [8] and [9], Mok and To also obtained some results about the finiteness of Mordell–Weil groups and the upper bounds for their ranks. Mok thinks that the problem of finding the lower bound is much more difficult (see [8, Section.3.11]). In fact, combining Shioda's formula and the inequality in [7, Theorem 1.3]:

$$h^{1,1}(X) - 2 - \sum_{i=1}^{k} (m_i - 1) \geq 2((h^1(\mathcal{O}_X) - g(Y))g(Y),$$

where $h^{1,1}(X)$ is the Hodge number of X, $h^{0,1}(X) = h^{1,0}(X) = h^1(\mathcal{O}_X)$, and $g(Y)$ is the genus of Y, we can obtain a lower bound for some fibered surfaces.

In this paper, we try to solve Mok's problems in [8]. We establish the notion of the Mordell–Weil group for a fibered variety and give its explicit formula. We also obtain some inequalities of the Hodge number $h^{1,1}$ for a fibration.

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Our paper is organized as follows:

In Section 2, we define the Mordell–Weil group of the fibration f and generalize the formula in [10]:

Theorem 1.1 *If f has a rational section and $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$, then the Mordell–Weil group* $\mathrm{MW}(f)$ *is a finitely generated abelian group, and*

$$\operatorname{rank} \mathrm{MW}(f) = \rho(X) - \rho(Y) - \operatorname{rank} \mathrm{NS}(X_K) - \sum_{i=1}^{k}(m_i - 1).$$

In Section 3, we generalize the formula in [7] to high-dimension varieties:

Theorem 1.2 *If Y is a curve, then*

$$h^{1,1}(X) - 1 - \sum_{i=1}^{k}(m_i - 1) - \operatorname{rank} \mathrm{NS}(X_K) \geq 2(h^1(\mathcal{O}_X) - g(Y))g(Y).$$

From Theorem 1.2, we get the following intersecting corollary:

Corollary 1.3 *In the situation of Theorem 1.2, if the equality holds, then $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$ or $h^1(\mathcal{O}_Y) = 0$.*

Some examples are discussed in Section 4.

2. Mordell–Weil rank for a fibered variety

The main purpose of this section is to establish the notion of the Mordell–Weil rank for a fibration and give its explicit formula.

In this section we always assume that f has a rational section O. Here we say a subvariety $S \subset X$ a rational section of f, if $f|_S : S \to Y$ is birational. Hence, X_K has a K-rational point O. We know that $\mathrm{Pic}^0(X_K)$ is a smooth projective abelian variety of dimension $h^1(\mathcal{O}_{X_K})$ over K and there is a Poincaré line bundle $\mathcal{P}$ on $X_K \times \mathrm{Pic}^0(X_K)$ such that $\mathcal{P}|_{\{O\}\times \mathrm{Pic}^0(X_K)} \cong \mathcal{O}_{\mathrm{Pic}^0(X_K)}$ (cf. [6]). If we consider $\mathcal{P}$ as a family of line bundles on $\mathrm{Pic}^0(X_K)$ parametrized by X_K, then we obtain the Albanese map of X_K:

$$\begin{aligned} a : X_K &\to \mathrm{Alb}(X_K) := \mathrm{Pic}^0(\mathrm{Pic}^0(X_K)) \\ S &\mapsto \mathcal{P}|_{\{S\}\times \mathrm{Pic}^0(X_K)}. \end{aligned}$$

Now we can define the Mordell–Weil group of f as follows:

Definition 2.1 *The Mordell–Weil group* $\mathrm{MW}(f)$ *of f is the group* $\mathrm{Alb}(X_K)/K$ *of K-rational points of* $\mathrm{Alb}(X_K)$.

Before we prove Theorem 1.1, we exhibit some easy corollaries of it.

Corollary 2.2 *If $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$, then we have*

$$h^{1,1}(X) \geq \rho(Y) + \operatorname{rank} \mathrm{NS}(X_K) + \operatorname{rank} \mathrm{MW}(f) + \sum_{i=1}^{k}(m_i - 1).$$

Corollary 2.3 *If $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$ and the general fiber F of f is regular, i.e. $h^1(\mathcal{O}_F) = 0$, then we have*

$$\rho(X) = \rho(Y) + \operatorname{rank} \mathrm{NS}(X_K) + \sum_{i=1}^{k} (m_i - 1).$$

When we apply Theorem 1.1 to a fibered surface, one can obtain an Arakelov type inequality with Mordell–Weil rank.

Corollary 2.4 *Let $\pi : S \to C$ be a nontrivial semistable genus g fibered surface with a section. Let s_1 be the number of fibers with noncompact Jacobians. If $q_\pi = q(S) - g(C) = 0$, then we have*

$$\deg \pi_* \omega_{S/C} \leq \frac{g}{2}(2g(C) - 2 + s_1) - \frac{1}{2} \operatorname{rank} \mathrm{MW}(\pi).$$

The motive of the proof of Theorem 1.1 is essentially due to Oguiso and Shioda. We just give some modifications.

Lemma 2.5 *The quotient group $\operatorname{Pic} X / f^* \operatorname{Pic} Y$ is the finitely generated abelian group of rank $\rho(X) - \rho(Y)$.*

Proof The conclusion follows from [10, Lemma 2.2]. □

Let $\operatorname{Pic}(X_K)$ be the Picard group of X_K, i.e. the group of isomorphic classes of line bundles defined over K. We have a natural surjective homomorphism:

$$\begin{array}{rcl} r_X : \operatorname{Pic}(X) & \to & \operatorname{Pic}(X_K) \\ L & \mapsto & L|_{X_K}. \end{array}$$

Since $r_X(f^* \operatorname{Pic}(Y)) = \{0\}$, the homomorphism r_X induces a surjective homomorphism:

$$r : \operatorname{Pic}(X)/f^* \operatorname{Pic}(Y) \to \operatorname{Pic}(X_K).$$

By Lemma 2.5, $\operatorname{Pic}(X_K)$ is a finitely generated abelian group. We denote the $\mathbb{Q}$-linear extension of r by

$$r_{\mathbb{Q}} : (\operatorname{Pic}(X)/f^* \operatorname{Pic}(Y)) \otimes \mathbb{Q} \to \operatorname{Pic}(X_K) \otimes \mathbb{Q}.$$

One sees that $r_{\mathbb{Q}}$ is also surjective.

Lemma 2.6 *1.* $\mathrm{MW}(f)$ *is a finitely generated abelian group and satisfies*

$$\operatorname{rank} \mathrm{MW}(f) = \operatorname{rank} \operatorname{Pic}(X_K) - \operatorname{rank} \mathrm{NS}(X_K).$$

2. $\dim \ker r_{\mathbb{Q}} = \sum_{i=1}^{k} (m_i - 1)$.

Proof See [10, Lemma 2.3 and Lemma 2.4]. Note that (2) is just proved for an abelian fibered variety in [10, Lemma 2.4], but the proof still works for the general case. □

One sees that the formula in Theorem 1.1 follows from Lemma 2.6.

Remark 2.7 *The condition $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$ in Theorem 1.1 is natural. For example, the Stein factorization of an irregular variety with positive Albanese fiber dimension satisfies this condition (cf. [5, 14]).*

3. A new inequality on the Hodge number $h^{1,1}$ for a fibered variety

It is interesting to establish an inequality on the Hodge number $h^{1,1}$ for a fibered variety. The following inequality is well known (see [10]):

$$h^{1,1}(X) - \rho(Y) - \sum_{i=1}^{k}(m_i - 1) - \operatorname{rank}\operatorname{NS}(X_K) \geq 0.$$

In this section, we give a new inequality on the Hodge number $h^{1,1}$ for a fibered variety. The inequality is a generalization of the inequality in [7].

We always assume that Y is a curve in this section. Denote by F the general fiber of f, and set $q_f = h^0(X, \Omega_X) - h^0(Y, \Omega_Y)$, $\dim X = n$.

First, we recall the below two lemmas:

Lemma 3.1 *Let Γ be an irreducible component of a fiber of f. We have:*

1. $\Gamma F = 0$;
2. *if* $\sum_{i=1}^{k}\sum_{j=1}^{m_i-1} x_{ij} D_{ij}\Gamma = 0$ *for any* Γ, *then* $x_{ij} = 0$.

Proof Let $S \subset X$ be a normal projective surface that is a complete intersection of $n-2$ sufficiently general very ample divisors. It induces a fibration of algebraic surface $S \to C$. Hence, we obtain the lemma by Zarski's lemma. □

Lemma 3.2 *Let $\alpha \in f^*H^0(Y, \Omega_Y)$, $\omega \in H^0(X, \Omega_X)$. If*

$$H(\alpha \wedge \omega, \alpha \wedge \omega) := \int_X \alpha \wedge \bar{\alpha} \wedge \omega \wedge \bar{\omega} \wedge \Omega^{n-2} = 0,$$

here $\Omega = c_1(A)$ for a very ample line bundle A, and then $\omega = f^*\beta$ for some $\beta \in f^*H^0(Y, \Omega_Y)$.

Proof Since $H(\alpha \wedge \omega, \alpha \wedge \omega) = 0$, one sees that $\alpha \wedge \omega \in H^{2,0}$. By Lefschetz decomposition, we know that $H(\alpha \wedge \omega, \alpha \wedge \omega)$ is positive-defined, $\alpha \wedge \omega = 0$ (see [4, Section0.7]). It follows that $\omega = h\alpha$, where h is a meromorphic function on X. This implies

$$\omega \in H^0(X, f^*\Omega_Y) = f^*H^0(Y, f_*f^*\Omega_Y) = f^*H^0(Y, \Omega_Y).$$

That is, $\omega = f^*\beta$ for some $\beta \in H^0(Y, \Omega_Y)$. □

Now we can prove Theorem 1.2. We assume that $h^1(\mathcal{O}_Y) \neq 0$. One can write $H^0(X, \Omega_X) = V_1 \oplus V_0$, where $V_0 = f^*H^0(Y, \Omega_Y)$, and $\dim V_1 = q_f$. Let

$$V_0' = \mathbb{C} < \alpha_1 >, \quad V_1 = \mathbb{C} < \theta_1, \cdots, \theta_{q_f} >,$$

where $0 \neq \alpha_1 \in V_0$, and $\{\theta_1, \cdots, \theta_{q_f}\}$ is a base of V_1. We define a linear map:

$$h: \; V_0' \otimes \bar{V}_1 \oplus \bar{V}_0' \otimes V_1 \to H^{1,1}(X)$$

such that $h(x \otimes y) = x \wedge y$ and $x \otimes y \in V_0' \otimes \bar{V}_1 \oplus \bar{V}_0' \otimes V_1$.

Let V_2 be the subgroup of $\mathrm{Pic}(X)$, generated by the classes of the components of all fibers. The Chern class induces a homomorphism $c_1 : V_2 \to H^{1,1}(X)$. We have $\mathrm{rank}\,\mathrm{Im}\, c_1 \geq \rho(Y) + \sum_{i=1}^{k}(m_i - 1)$.

Lemma 3.3 *Let $H_1 \in \mathrm{Pic}(X_K)$ be a line bundle of X_K, and let $H \in \mathrm{Pic}(X)$ be a preimage of H_1 under r_X. Here r_X is the restriction map considered in Section 2. Then we have $c_1(H) \notin \mathrm{Im}\, h + (\mathrm{Im}\, c_1) \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof We suppose that $c_1(H) = \alpha + \beta \in \mathrm{Im}\, h + (\mathrm{Im}\, c_1) \otimes_{\mathbb{R}} \mathbb{C}$ for some $\alpha \in \mathrm{Im}\, h, \beta \in (\mathrm{Im}\, c_1) \otimes_{\mathbb{R}}$. Let F be a general fiber. By the definition of h, one can see easily that $\alpha|_F = 0$. On the other hand, Lemma 3.2 implies $\beta|_F = 0$. Hence, $c_1(H)|_F = 0$, i.e. $HF = 0$, a contradiction. □

The following lemmas are due to [7, Lemmas 2.2 and 2.3]. The original proof for the surface case works for the general case after some modifications. For the reader's convenience, we would like to recall the proof.

Lemma 3.4 *1.* $\dim_{\mathbb{C}} \mathrm{Im}\, h = 2q_f$;

2. $\mathrm{Im}\, h \cap (\mathrm{Im}\, c_1) \otimes_{\mathbb{R}} \mathbb{C} = 0$.

Proof (1) We only need to prove that h is injective.

Suppose that there is a nonzero element in the kernel of h such that

$$\sum_{j=1}^{q_f} a_j \alpha_1 \wedge \bar{\theta}_j + \sum_{j=1}^{q_f} b_j \bar{\alpha}_1 \wedge \theta_j = 0.$$

By wedging $\bar{\alpha}_1 \wedge \theta_l$ on both sides, one gets

$$\sum_{j=1}^{q_f} a_j \alpha_1 \wedge \bar{\theta}_j \wedge \bar{\alpha}_1 \wedge \theta_l = 0.$$

Let $\omega_1 = \sum_{j=1}^{q_f} \bar{a}_j \theta_j$. We have $\alpha_1 \wedge \bar{\alpha}_1 \wedge \bar{\omega}_1 \wedge \theta_l = 0$. It implies that

$$H(\alpha_1 \wedge \omega_1, \alpha_1 \wedge \omega_1) = \int_X \alpha_1 \wedge \bar{\alpha}_1 \wedge \bar{\omega}_1 \wedge \omega_1 \wedge \Omega^{n-2} = 0.$$

From Lemma 3.2, we have $\omega_1 = f^* \beta_1$ for some $\beta_1 \in H^0(Y, \Omega_Y)$. Thus, $\omega_1 \in V_0' \cap V_1$, i.e. $\omega_1 = 0$. It implies that all a_j are zero. Similarly, we have $b_j = 0$. It is a contradiction.

(2) Note that $\mathrm{Im}\, c_1 \subseteq H^{1,1}(X)$. Let $\omega = c_1(F)$ and let $\omega_{ij} = c_1(D_{ij})$. Assume that

$$x \cdot \omega + \sum_{i=1}^{k} \sum_{j=1}^{m_i - 1} x_{ij} \cdot \omega_{ij} = t \in \mathrm{Im}\, h,$$

for some $x, x_{ij} \in \mathbb{C}$.

One sees that for any component Γ in a fiber of f, $t|_\Gamma = 0$ and $\omega|_\Gamma = 0$. Hence,

$$\sum_{i=1}^{i} \sum_{j=1}^{m_i - 1} x_{ij} \cdot \omega_{ij}|_\Gamma = 0,$$

i.e.

$$\sum_{i=1}^{i}\sum_{j=1}^{m_i-1} x_{ij}\cdot D_{ij}\Gamma = 0 \quad \text{for any} \quad \Gamma.$$

From Lemma 3.1, it follows that $x_{ij}=0$. Thus, we deduce that $x\cdot c_1(F)=t\in \operatorname{Im} h$, for some $x\in\mathbb{C}$. If $x\neq 0$, then $c_1(F)\in \operatorname{Im} h$.

Let

$$c_1(F)=\sum_{j=1}^{q_f} a_j\alpha_1\wedge\bar{\theta}_j+\sum_{j=1}^{q_f} b_j\bar{\alpha}_1\wedge\theta_j.$$

Note that $c_1(F)=c\alpha_r\wedge\bar{\alpha}_s\neq 0$, where $\alpha_r,\alpha_s\in H^0(Y,\Omega_Y)$. This implies

$$\sum_{j=1}^{q_f} a_j\alpha_1\wedge\bar{\theta}_j+\sum_{j=1}^{q_f} b_j\bar{\alpha}_1\wedge\theta_j=c\cdot\alpha_r\wedge\bar{\alpha}_s.$$

By wedging $\bar{\alpha}_1$ on both sides, one sees

$$\alpha_1\wedge\bar{\alpha}_1\wedge(\sum_{j=1}^{q_f} a_j\bar{\theta}_j)=c\cdot\alpha_r\wedge\bar{\alpha}_1\wedge\bar{\alpha}_s.$$

Since $c\cdot\alpha_r\wedge\bar{\alpha}_1\wedge\bar{\alpha}_s\in H^{1,2}(Y)=0$, one deduces that

$$\alpha_1\wedge\bar{\alpha}_1\wedge(\sum_{j=1}^{q_f} a_j\bar{\theta}_j)=0.$$

Let $\omega_2=\sum_{j=1}^{q_f} a_j\bar{\theta}_j$. By wedging ω_2 on both sides, we obtain

$$H(\alpha_1\wedge\omega_2,\alpha_1\wedge\omega_2)=\int_X \alpha_1\wedge\bar{\alpha}_1\wedge\omega_2\wedge\bar{\omega}_2\wedge\Omega^{n-2}=0.$$

By Lemma 3.2, one sees that $a_j=0$ for any j. For the same reason, $b_j=0$. Hence, $c_1(F)=0$. This is a contradiction. Thus, the lemma follows. □

Combining the above claims, we have

$$h^{1,1}(X)\geq \dim\operatorname{Im} h+\operatorname{rank}((\operatorname{Im} c_1)\otimes_{\mathbb{R}}\mathbb{C})+\operatorname{rank} c_1(\operatorname{Pic}(X_K)).$$

This proves Theorem 1.2.

4. Some examples

In this section, we give some examples. Theses examples show that the equality in Theorem 1.2 can be reached. In [1],[3], and [2], we can find more examples.

Example 4.1 *Shioda constructed an interesting example in [12, Example 5.9]. In Shoida's construction,* $f: X \to Y$ *is a elliptic fibration, where* X *is an algebraic surface and* Y *is an elliptic curve. This fibration has a unique singular fiber* (I_6^*). *One has*

$$h^{1,1}(X) = 12, \quad h^1(\mathcal{O}_X) = 1, \quad \mathrm{rankNS}(X_K) = 1,$$

$$h^1(\mathcal{O}_Y) = 1, \quad \rho(Y) = 1, \quad m_1 = 11.$$

Thus, $h^{1,1}(X) - \rho(Y) - (m_1 - 1) - \mathrm{rank\,NS}(X_K) = 2(h^1(\mathcal{O}_X) - h^1(\mathcal{O}_Y))$.

Example 4.2 *Here is an example with* $h^1(Y, \mathcal{O}_Y) = 0$.

Let $f: X \to Y = \mathbb{P}^1$ *be a family of hyperelliptic curves of genus* g *defined by the equation*

$$y^2 = t(x^2 - t)(x^2 - 2t)(x^2 - 3t) \cdots (x^2 - (g+1)t),$$

where g *is odd. One sees that* f *has two singular fibers. Each singular fiber is a multiple smooth curve of genus* $\frac{g+1}{2}$. *We have*

$$h^{1,1}(X) = 2, \quad \mathrm{rankNS}(X_K) = 1, \quad \rho(Y) = 1, \quad h^1(X, \mathcal{O}_X) = q_f = \frac{g+1}{2}.$$

Thus, $h^{1,1}(X) = \rho(Y) + \sum(m_i - 1) + \mathrm{rankNS}(X_K)$.

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