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# Some results on Hecke and extended Hecke groups

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**Abstract:** Let $q \geq 3$ be a prime number and let $\overline{H}(\lambda_q)$ be the extended Hecke group associated with $q$. In this paper, we determine the presentation of the commutator subgroup $(H(\lambda_q)\alpha)'$ of the normal subgroup $H(\lambda_q)\alpha$, where $H(\lambda_q)\alpha$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. Next we discuss the commutator subgroup $(\overline{H}_2)'(\lambda_q)$ of the principal congruence subgroup $\overline{H}_2(\lambda_q)$ of $\overline{H}(\lambda_q)$. Then we show that some quotient groups of $\overline{H}(\lambda_q)$ are generalized $M^*-$groups. Finally, we prove some results related to some normal subgroups of $\overline{H}(\lambda_q)$, especially in the case $q = 5$.

**Key words:** Extended Hecke groups, commutator subgroups, principal congruence subgroups, generalized $M^*-$groups

## 1. Introduction

In [14], Hecke introduced the Hecke groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad S(z) = -\frac{1}{z+\lambda},$$

where $\lambda$ is a fixed positive real number.

He showed that H($\lambda$) is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, for integer $q \geq 3$, $(\lambda < 2)$, or $\lambda \geq 2$. We will focus on the discrete case with $\lambda < 2$ and we denote it by $H(\lambda_q)$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and $q$,

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q, \ [10].$$

Let $\Gamma$ be a subgroup of finite index in $H(\lambda_q)$. Then $\mathcal{U}/\Gamma$ , where $\mathcal{U}$ is the upper half plane of a Riemann surface. Let $g$ and $t$ be the genus and the number of cusps of $\mathcal{U}/\Gamma$, respectively, and let $m_1, \cdots, m_k$ be the branching numbers of the branch points on $\mathcal{U}/\Gamma$. The signature of $\Gamma$ is $(g; m_1, ..., m_k; t)$.

The Hecke group $H(\lambda_q)$ can be thought of as triangle groups having an infinity as one of the parameters. As the signature of $H(\lambda_q)$ is $(0; 2, q, \infty)$, each is an infinite triangle group. Moreover, the quotient space $\mathcal{U}/H(\lambda_q)$ is a sphere with one puncture and two elliptic fixed points of order 2 and $q$. Hence the surface $\mathcal{U}/H(\lambda_q)$ is an orbifold.

Examples of these groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2, \mathbb{Z}[\lambda_q])$, for $q \geq 4$.

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In memory of my dear son Can Şahin

The extended modular group, denoted by $\overline{H}(\lambda_3) = \Pi = PGL(2,\mathbb{Z})$, is defined by adding the reflection $R(z) = 1/\ \overline{z}$ to the generators of the modular group $H(\lambda_3)$. Then the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [35] and [39] similar to the extended modular group by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $\overline{H}(\lambda_q)$. Thus the extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = I >\cong D_2 *_{C_2} D_q. \tag{1}$$

If we take $R_1(z) = \frac{1}{\overline{z}}$, $R_2(z) = -\overline{z}$, $R_3(z) = -\overline{z} - \lambda_q$, where $T = R_2R_1 = R_1R_2$ and $S = R_1R_3$; then we get the alternative presentation

$$\overline{H}(\lambda_q) =< R_1,R_2,R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_1R_3)^q = I > .$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. Since the extended Hecke groups $\overline{H}(\lambda_q)$ contain a reflection, they are proper non-Euclidean crystallographic $(NEC)$ groups [28]. Thus the quotient space $\mathcal{U}/\overline{H}(\lambda_q)$ is a Klein surface and $\mathcal{U}/H(\lambda_q)$ is the canonical double cover of $\mathcal{U}/\overline{H}(\lambda_q)$.

The Hecke groups $H(\lambda_q)$, the extended Hecke groups $\overline{H}(\lambda_q)$ and their normal subgroups have been studied for many aspects in the literature (for instances, please see [1, 2, 6, 7, 13, 17, 24, 32, 36, and 45]).

Here the map

$$\alpha : T \to RT,\ S \to S,\ R \to R, \tag{2}$$

induces an outer automorphism of $\overline{H}(\lambda_q)$, [17, p. 12]. Thus the group

$$H(\lambda_q)\alpha =< RT, S \mid (RT)^2 = S^q = I >\cong C_2 * C_q, \tag{3}$$

is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

Throughout this paper, we identify matrix $A$ in $GL(2,\mathbb{Z}[\lambda_q])$ with $-A$, so that they each represent the same element of $\overline{H}(\lambda_q)$. Thus we can represent the generators of $\overline{H}(\lambda_q)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Next we give some information about the principal congruence subgroups of $\overline{H}(\lambda_q)$.

The *principal congruence subgroups* $\overline{H}_p(\lambda_q)$ *of level* $p$, $p$ prime, of $\overline{H}(\lambda_q)$ are defined in [38] (see also [15] and [26]) as

$$\overline{H}_p(\lambda_q) = \left\{ A = \begin{pmatrix} a & b\lambda_q \\ c\lambda_q & d \end{pmatrix} \in \overline{H}(\lambda_q) : a \equiv d \equiv \pm 1,\ b \equiv c \equiv 0 \ (\mathrm{mod}\ p),\ \det\ A = \pm 1 \right\}.$$

$\overline{H}_p(\lambda_q)$ is always a normal subgroup of finite index in $\overline{H}(\lambda_q)$. It is easily seen that $H_p(\lambda_q) = \overline{H}_p(\lambda_q) \cap H(\lambda_q)$.

By [38], we know that if $p \geq 3$ is a prime number, then $\overline{H}_p(\lambda_q) = H_p(\lambda_q)$ and if $p = 2$, then $\overline{H}(\lambda_q)/\overline{H}_2(\lambda_q) \cong H(\lambda_q)/H_2(\lambda_q)$. Thus, the groups $H_2(\lambda_q)$ and $\overline{H}_2(\lambda_q)$ are very important.

The principal congruence subgroups $H_2(\lambda_3) = \Gamma(2)$, $\overline{H}_2(\lambda_3) = \Pi(2)$ and $\Gamma(4) = \Pi(4)$ of $\Gamma$ and $\Pi$, respectively, have been studied extensively in the literature, for example, in relation to number theory, modular forms, modular curves, Belyi's theory, and graph theory (for instance, see [8, 11, 12, 21, 22, 33, and 40]).

Some normal subgroups (the first and the second commutator subgroups $\overline{H}'(\lambda_q)$ and $\overline{H}''(\lambda_q)$, the principal congruence subgroups $\overline{H}_p(\lambda_q)$ and the $m-$th power subgroups $\overline{H}^m(\lambda_q)$) of $\overline{H}(\lambda_q)$, $q \geq 3$ prime number, have been studied by Ikikardes, Koruoğlu, Sahin, and Bizim in [38, 42, 43, 44]. For $q \geq 3$ a prime number, they proved the following results:

a) There are exactly 3 normal subgroups of index 2 in $\overline{H}(\lambda_q)$. Namely, $H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q$, $\overline{H}_0(\lambda_q) =< R, S, TST \mid R^2 = S^q = (TST)^q = (RS)^2 = (RTST)^2 = I >\cong D_q *_{\mathbb{Z}_2} D_q$, and $H(\lambda_q)\alpha =< TR, S \mid (TR)^2 = S^q = I >\cong C_2 * C_q$.

b) There is exactly one normal subgroup of index 4 in $\overline{H}(\lambda_q)$. Namely, $\overline{H}'(\lambda_q) =< S, TST \mid S^q = (TST)^q = I >\cong C_2 * C_q$ .

c) There are exactly 2 normal subgroups of index $2q$ in $\overline{H}(\lambda_q)$. Namely, $H^q(\lambda_q) =< T > * < STS^{q-1} > * \ldots * < S^{q-1}TS >\cong \underbrace{C_2 * C_2 * \cdots * C_2}_{q \text{ times}}$, and $\overline{H}_2(\lambda_q) =< TR > * < RSTS > * \ldots * < RS^{q-1}TS^{q-1} >\cong \underbrace{C_2 * C_2 * \cdots * C_2}_{q \text{ times}}$.

d) The second commutator subgroup $\overline{H}''(\lambda_q)$ of $\overline{H}(\lambda_q)$ is a normal subgroup of index $4q^2$ in $\overline{H}(\lambda_q)$. Namely $\overline{H}''(\lambda_q)$ is a free group with basis $[S, TST]$, $[S, TS^2T], ..., [S, TS^{q-1}T]$, $[S^2, TST]$, $[S^2, TS^2T], ..., [S^2, TS^{q-1}T]$, ..., $[S^{q-1}, TST]$, $[S^{q-1}, TS^2T]$, ..., $[S^{q-1}, TS^{q-1}T]$.

e) The group $(H^2)'(\lambda_q)$ is equal to the second commutator subgroup $\overline{H}''(\lambda_q)$ and it has index $q$ in $H'(\lambda_q)$.

f) The group $(H^q)'(\lambda_q)$ is a free group of rank $1 + (q-2)2^{q-1}$, of signature $((q-3)2^{q-2} + 1;\ \infty^{(2^{q-1})})$ and of index $2^{q+1}q$ in $\overline{H}(\lambda_q)$.

Using the above results, we get the following subgroup diagram in Figure 1.

On the other hand, when Sahin et al. were studying in [42] some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$, for $q \geq 3$ prime, they came across an interesting general fact. If a bordered surface group $\Gamma$ is a normal subgroup of finite index in $\overline{H}(\lambda_q)$, then $\overline{H}(\lambda_q)/\Gamma$ is a group of automorphisms of the bordered Klein surface $X = \mathcal{U}/\overline{H}(\lambda_q)$. Moreover, the automorphism groups $G$ of order $\frac{4q}{(q-2)}(g-1)$ that act on compact bordered Klein surfaces $X$ of algebraic genus $g \geq 2$ are finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_q)$, where $q \geq 3$ is an integer. For example, the groups of orders $|G| = 12(g-1)$, $|G| = 8(g-1)$, $|G| = \frac{20}{3}(g-1)$, respectively, are the finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_3)$, $\overline{H}(\lambda_4)$, or $\overline{H}(\lambda_5)$ [41] and [3]. Here the orders of these groups are the highest three among the automorphism groups of the compact Klein surfaces of algebraic genus $g \geq 2$ (see [31, p. 221, proposition 1]). The groups of order $|G| = 12(g-1)$ are $M^*-$groups. These groups were first introduced in [30], and have been studied in several papers [4] and [5].

Sahin et al. defined the generalized $M^*-$groups in [41] similar to the $M^*-$groups. A finite group $G$ is called a generalized $M^*-$group if it is generated by three distinct nontrivial elements $r_1, r_2, r_3$ that satisfy the relations

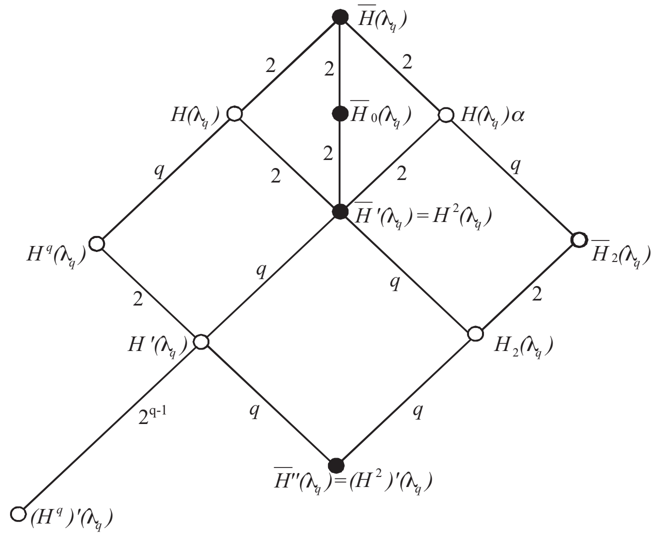


**Figure 1**.

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^2 = (r_1 r_3)^q = I,$$

$q \geq 3$ prime, and other relations that make the group finite. It is clear that the order of $G$ is $\frac{4q}{(q-2)}(g-1)$, $g \geq 2$ integer.

In [41], Sahin et al. showed that a finite group of order at least $4q$ is a generalized $M^*-$group if and only if it is the homomorphic image of the extended Hecke group $\overline{H}(\lambda_q)$, $q \geq 3$ prime. Thus, if the index of a normal subgroup $N$ in $\overline{H}(\lambda_q)$ is greater than $2q$, then the quotient group $\overline{H}(\lambda_q)/N$ is a generalized $M^*-$group.

The aim of this work is to generalize some results known for the extended modular group to the extended Hecke groups $\overline{H}(\lambda_q)$, for $q \geq 3$ prime (especially, $\overline{H}(\lambda_5)$). We obtain the group structures and the signatures of the commutator subgroups $(H(\lambda_q)\alpha)'$ and $(\overline{H}_2)'(\lambda_q)$ of $\overline{H}(\lambda_q)$. For this, we apply the Reidemeister–Schreier method, the permutation method, and the Riemann–Hurwitz formula. Next, we discuss some normal subgroups and how they are related under the map $\alpha$ introduced in (2). We also give some relations between some normal subgroups of $\overline{H}(\lambda_q)$ and a figure that explain the effect of $\alpha$. Furthermore, we show that some of the quotient groups of $\overline{H}(\lambda_q)$ are generalized $M^*-$groups. Finally, we determine the generators of $H_4(\lambda_5)$ and prove that $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$ and $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$.

## 2. Commutator subgroups of $H(\lambda_q)\alpha$ and $\overline{H}_2(\lambda_q)$

In this section, we study the first commutator subgroups $(H(\lambda_q)\alpha)'$ and $(\overline{H}_2)'(\lambda_q)$ of $\overline{H}(\lambda_q)$, for $q \geq 3$ a prime number.

**Theorem 2.1** *Let $q \geq 3$ be a prime number. Then*

*i)* $|H(\lambda_q)\alpha : (H(\lambda_q)\alpha)'| = 2q$,

*ii) the group $(H(\lambda_q)\alpha)'$ is a free group of rank $(q-1)$ with basis $TSTS$, $TS^2TS^2$, ..., $TS^{q-1}TS^{q-1}$.*

**Proof**

i) The quotient group $H(\lambda_q)\alpha/(H(\lambda_q)\alpha)'$ is the group obtained by adding the relation $TRS = STR$ to the relations of $H(\lambda_q)\alpha$ in (3). Then

$$H(\lambda_q)\alpha/(H(\lambda_q)\alpha)' \cong C_2 \times C_q.$$

Therefore, we obtain $|H(\lambda_q)\alpha : (H(\lambda_q)\alpha)'| = 2q$.

ii) We choose $\Sigma = \{I,\ S,\ S^2, ..., S^{q-1},\ TR,\ TRS,\ TRS^2, ..., TRS^{q-1}\}$ as a Schreier transversal for $(H(\lambda_q)\alpha)'$. According to the Reidemeister–Schreier method (see [29]), we get the generators of $(H(\lambda_q)\alpha)'$ as the following.

$$\begin{array}{ll}
I.TR.(TR)^{-1} = I, & I.S.(S)^{-1} = I,\\
S.TR.(TRS)^{-1} = STRS^{q-1}RT, & S.S.(S^2)^{-1} = I,\\
S^2.TR.(TRS^2)^{-1} = S^2TRS^{q-2}RT, & S^2.S.(S^3)^{-1} = I,\\
\vdots & \vdots\\
S^{q-1}.TR.(TRS^{q-1})^{-1} = S^{q-1}TRSRT, & S^{q-1}.S.(I)^{-1} = I,\\
TR.TR.(I)^{-1} = I, & TR.S.(TRS)^{-1} = I,\\
TRS.TR.(S)^{-1} = TRSTRS^{q-1}, & TRS.S.(TRS^2)^{-1} = I,\\
TRS^2.TR.(S^2)^{-1} = TRS^2TRS^{q-2}, & TRS^2.S.(TRS^3)^{-1} = I,\\
\vdots & \vdots\\
TRS^{q-1}.TR.(S^{q-1})^{-1} = TRS^{q-1}TRS, & TRS^{q-1}.S.(TR)^{-1} = I.
\end{array}$$

Here $STRS^{q-1}RT = STST$, $S^2TRS^{q-2}RT = S^2TS^2T$, $S^{q-1}TRSRT = S^{q-1}TS^{q-1}T$, $TRSTRS^{q-1} = TS^{q-1}TS^{q-1}$, $TRS^2TRS^{q-2} = TS^{q-2}TS^{q-2}$ and $TRS^{q-1}TRS = TSTS$, as $TR = RT$ and $SR = RS^{q-1}$. Also as $(STST)^{-1} = TS^{q-1}TS^{q-1}$, $(S^2TS^2T)^{-1} = TS^{q-2}TS^{q-2}$ and $(S^{q-1}TS^{q-1}T)^{-1} = TSTS$, the generators of $(H(\lambda_q)\alpha)'$ are $TSTS$, $TS^2TS^2$, ..., $TS^{q-1}TS^{q-1}$.

Using the permutation method (see [46]) and the Riemann–Hurwitz formula, we get the signature of $(H(\lambda_q)\alpha)'$ as $(0; \underbrace{\infty, \infty, \cdots, \infty}_{q \text{ times}}) = (0;\ \infty^{(q)})$. □

It is clear that the group $(H(\lambda_q)\alpha)'$ is a subgroup of $H(\lambda_q)$. From [27], there are only two normal subgroups of index $2q$ in $H(\lambda_q)$, for $q \geq 3$ prime. Namely, $H'(\lambda_q)$ and $H_2(\lambda_q)$. As the signature of $H_2(\lambda_q)$ is $(0;\ \infty^{(q)})$ (see [16]), we get the following result.

**Corollary 2.2** *The subgroup $(H(\lambda_q)\alpha)'$ is equal to the principal congruence subgroup $H_2(\lambda_q)$ of $H(\lambda_q)$, i.e. $(H(\lambda_q)\alpha)' = H_2(\lambda_q)$.*

**Theorem 2.3** *Let $q \geq 3$ be a prime number.*

*i)* $\left|\overline{H}_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| = 2^q$.

*ii) The group $(\overline{H}_2)'(\lambda_q)$ is a free group of rank $1 + (q-2)2^{q-1}$.*

*iii) The group $(\overline{H}_2)'(\lambda_q)$ is of index $2^{q-1}$ in $H_2(\lambda_q)$.*

**Proof**

i) If we take $k_1 = TR,\ k_2 = RSTS,\ k_3 = RS^2TS^2,\ \cdots,\ k_q = RS^{q-1}TS^{q-1}$ as the generators of $\overline{H}_2(\lambda_q)$, then the quotient group $\overline{H}_2(\lambda_q)/(\overline{H}_2)'(\lambda_q)$ is the group obtained by adding the relation $k_ik_j = k_jk_i$ to the relations of $\overline{H}_2(\lambda_q)$, for $i \neq j$ and $i, j \in \{1, 2, \cdots, q\}$. Thus we have

$$\overline{H}_2(\lambda_q)/(\overline{H}_2)'(\lambda_q) \cong \underbrace{C_2 \times C_2 \times \cdots \times C_2}_{q \text{ times}}.$$

Therefore, we obtain $\left|\overline{H}_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| = 2^q$.

ii) Let $\Sigma = \{I,\ k_1,\ k_2, ...,\ k_q,\ k_1k_2,\ k_1k_3,\ ...,\ k_1k_q,\ k_2k_3,\ k_2k_4,\ ...,\ k_2k_q,\ ..., k_{q-1}k_q,\ k_1k_2k_3,\ k_1k_2k_4,\ ..., k_1k_2k_q,\ ...,\ k_1k_2\cdots k_q\}$ be a Schreier transversal for $(\overline{H}_2)'(\lambda_q)$. Using the Reidemeister–Schreier method, we obtain the generators of $(\overline{H}_2)'(\lambda_q)$ as the following.

There are $C(q,2) = \begin{pmatrix} q \\ 2 \end{pmatrix}$ generators of the form $k_ik_jk_ik_j$, where $i < j$ and $i,\ j \in \{1, 2, \cdots, q\}$. There are $2 \times \begin{pmatrix} q \\ 3 \end{pmatrix}$ generators of the form $k_ik_jk_tk_jk_tk_i$, or $k_ik_jk_tk_ik_tk_j$, where $i < j < t$ and $i,\ j,\ t \in \{1, 2, \cdots, q\}$. There are $3 \times \begin{pmatrix} q \\ 4 \end{pmatrix}$ generators of the form $k_ik_jk_tk_uk_ik_uk_tk_j$, or $k_ik_jk_tk_uk_jk_uk_tk_i$, or $k_ik_jk_tk_uk_tk_uk_jk_i$, where $i < j < t < u$ and $i, j, t, u \in \{1, 2, \cdots, q\}$. Similarly, there are $(q-1) \times \begin{pmatrix} q \\ q \end{pmatrix}$ generators of the form $k_1k_2\cdots k_qk_1k_qk_{q-1}\cdots k_2$, or $k_1k_2\cdots k_qk_2k_qk_{q-1}\cdots k_3k_1$, or $\cdots$, or $k_1k_2\cdots k_qk_{q-1}k_qk_{q-2}\cdots k_2k_1$. Totally, there are $1 + (q-2)2^{q-1}$ generators of $(\overline{H}_2)'(\lambda_q)$.

iii) We know that $\left|H(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| = 2^q.q$ and $|H(\lambda_q) : H_2(\lambda_q)| = 2q$. Therefore we get $\left|H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| = 2^{q-1}$.

Finally, we find the signature of $(\overline{H}_2)'(\lambda_q)$ as $(q2^{q-3} - 2^{q-1} + 1; \underbrace{\infty, \infty, \cdots, \infty}_{q.2^{(q-2)} \text{ times}}) = ((q-4)2^{q-3} + 1; \infty^{(q.2^{q-2})})$. □

**Corollary 2.4** *We have* $H_2(\lambda_q) = (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)$.

**Proof** As $(H^2)'(\lambda_q)$ and $(\overline{H}_2)'(\lambda_q)$ are normal subgroups of $H'(\lambda_q)$, we obtain the chains

$$(H^2)'(\lambda_q) \subset (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \subset H_2(\lambda_q) \text{ and } (\overline{H}_2)'(\lambda_q) \subset (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \subset H_2(\lambda_q).$$

Then we get the index $\left|H_2(\lambda_q) : (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)\right|$ divides both of $q$ and $2^{q-1}$. Since $(q, 2^{q-1}) = 1$, we have $\left|H_2(\lambda_q) : (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)\right| = 1$ . Thus we get $H_2(\lambda_q) = (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)$. □

**Corollary 2.5** *We have*

*a)* $\left|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))\right| = 2^q.q^2$

*b)* $\left|H(\lambda_q) : ((H^2)'(\lambda_q)) \cap (H^q)'(\lambda_q))\right| = 2^q.q^2$

**Proof** *a)* $(H^2)'(\lambda_q)$ and $(\overline{H}_2)'(\lambda_q)$ are normal subgroups of $H(\lambda_q)$. By one of the isomorphism theorems of the groups, we have that

$$((H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q))/(H^2)'(\lambda_q) \cong (\overline{H}_2)'(\lambda_q)/((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q)).$$

As $(H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \cong H_2(\lambda_q)$, we find

$$H_2(\lambda_q)/(H^2)'(\lambda_q) \cong (\overline{H}_2)'(\lambda_q)/((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q)).$$

Then

$$\left|H_2(\lambda_q) : (H^2)'(\lambda_q)\right| = \left|(\overline{H}_2)'(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))\right|.$$

As $\left|H_2(\lambda_q) : ((H^2(\lambda_q))'\right| = q$, we get

$$\left|(\overline{H}_2)'(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))\right| = q.$$

Thus, we have

$$\left|H(\lambda_q) : ((H^2(\lambda_q))' \cap (\overline{H}_2)'(\lambda_q))\right| = \left|H(\lambda_q) : H_2(\lambda_q)\right| . \left|H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| . \left|(\overline{H}_2)'(\lambda_q) : ((H^2(\lambda_q))' \cap (\overline{H}_2)'(\lambda_q))\right|.$$

As $\left|H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)\right| = 2^{q-1}$, we obtain

$$\left|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))\right| = 2^q.q^2.$$

*b)* The proof is similar to *a)*. □

**Remark 2.6** *Under the map* $\alpha$ *in* (2), *any subgroup of* $\overline{H}(\lambda_q)$ *is mapped to a subgroup of* $\overline{H}(\lambda_q)$ *similar to the extended modular group in [19] and [20]. Indeed one finds*

$$\begin{array}{ccc} H(\lambda_q) & \overset{\alpha}{\leftrightarrow} & H(\lambda_q)\alpha \\ H^q(\lambda_q) & \leftrightarrow & \overline{H}_2(\lambda_q) \\ H'(\lambda_q) & \leftrightarrow & H_2(\lambda_q) \\ (H^q)'(\lambda_q) & \leftrightarrow & (\overline{H}_2)'(\lambda_q). \end{array}$$

*Of course, if we know the generators of any one of these subgroups, then we find the generators of its image under* $\alpha$. *The subgroups* $\overline{H}(\lambda_q)$, $\overline{H}_0(\lambda_q)$, $\overline{H}'(\lambda_q)$, *and* $\overline{H}''(\lambda_q)$ *of* $\overline{H}(\lambda_q)$ *are* $\alpha-$*invariant and hence they are characteristic subgroups. Figure 2 summarizes these results.*

*As shown in [18], if* $\mathcal{M}$ *is a regular or orientably regular hypermap corresponding to a normal subgroup* $M$ *of* $\overline{H}(\lambda_q)$, *then* $\mathcal{M}\alpha$ *is the hypermap corresponding to the normal subgroup* $M\alpha$.

**Corollary 2.7** *Let* $q \geq 3$ *be a prime number.*

*i)* *The quotient groups* $\overline{H}(\lambda_q)/H'(\lambda_q)$ *and* $\overline{H}(\lambda_q)/H_2(\lambda_q)$ *are generalized* $M^*-$*groups. These quotient groups act on surfaces of topological type* $((q-1),\ 1,\ +)$ *and* $((q-1),\ q,\ +)$ *respectively, where in the triple* $(g,k,\epsilon)$, $g$ *is the algebraic genus,* $k$ *is the number of boundary components, and* $\epsilon$ *describes the orientability of a bordered Klein surface.*

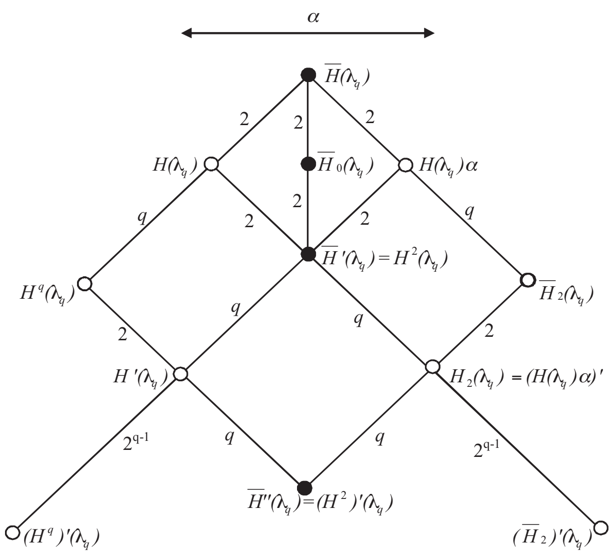


**Figure 2**. ∘ Normal subgroups of $\overline{H}(\lambda_q)$ • Characteristic subgroups of $\overline{H}(\lambda_q)$.

*ii)* *The quotient group $\overline{H}(\lambda_q)/H''(\lambda_q)$ is a generalized $M^*-$group. This quotient group acts on surfaces of topological type $(q.(q-2)+1,\ q,\ +)$.*

*iii)* *The quotient groups $\overline{H}(\lambda_q)/(H^q)'(\lambda_q)$ and $\overline{H}(\lambda_q)/(\overline{H}_2)'(\lambda_q)$ are generalized $M^*-$groups. These quotient groups act on surfaces of topological type $(2^{q-1}.(q-2)+1,\ 2^{q-1},\ +)$ and $(2^{q-1}.(q-2)+1,\ q.2^{q-2},+)$ respectively.*

**Remark 2.8** *If $q=3$, then $\overline{H}(\lambda_3)/H'(\lambda_3)$ and $\overline{H}(\lambda_3)/H_2(\lambda_3)$ act on surfaces of topological type $(2,\ 1,\ +)$ and $(2,\ 3,\ +)$; $\overline{H}(\lambda_3)/H''(\lambda_3)$ acts on surfaces of topological type $(4,\ 3,\ +)$; $\overline{H}(\lambda_3)/(H^3)'(\lambda_3)$ and $\overline{H}(\lambda_3)/(\overline{H}_2)'(\lambda_3)$ act on surfaces of topological type $(5,\ 4,\ +)$ and $(5,\ 6,+)$. If $q=5$, then $\overline{H}(\lambda_5)/H'(\lambda_5)$ and $\overline{H}(\lambda_5)/H_2(\lambda_5)$ act on surfaces of topological type $(4,\ 1,\ +)$ and $(4,\ 5,\ +)$; $\overline{H}(\lambda_5)/H''(\lambda_5)$ acts on surfaces of topological type $(16,\ 5,\ +)$; $\overline{H}(\lambda_5)/(H^5)'(\lambda_5)$ and $\overline{H}(\lambda_5)/(\overline{H}_2)'(\lambda_5)$ act on surfaces of topological type $(49,\ 16,\ +)$ and $(49,\ 40,+)$. All these results coincide with some results given in [3].*

In the following, we focus on the Hecke group $H(\lambda_5)$. We know from [37] that the subgroup $(\overline{H}_2)'(\lambda_3)$ is equal to the congruence subgroup $H_4(\lambda_3)$. We want to derive a similar equation for $H(\lambda_5)$. For this we start with the special example $q=5$ in *ii*) of the proof of Theorem 2.3.

**Example 2.9** *Let $q=5$. Then $\left|\overline{H}_2(\lambda_5):(\overline{H}_2)'(\lambda_5)\right|=32$. We choose $\Sigma=\{I,\ k_1,\ k_2,\ k_3,\ k_4,\ k_5,\ k_1k_2,\ k_1k_3,\ k_1k_4,\ k_1k_5,\ k_2k_3,\ k_2k_4,\ k_2k_5,\ k_3k_4,\ k_3k_5,\ k_4k_5,\ k_1k_2k_3,\ k_1k_2k_4,\ k_1k_2k_5,\ k_1k_3k_4,\ k_1k_3k_5,\ k_1k_4k_5,\ k_2k_3k_4,\ k_2k_3k_5,\ k_2k_4k_5,\ k_3k_4k_5,\ k_1k_2k_3k_4,\ k_1k_2k_3k_5,\ k_1k_2k_4k_5,\ k_1k_3k_4k_5,\ k_2k_3k_4k_5,\ k_1k_2k_3k_4k_5\}$ as a Schreier transversal for $(\overline{H}_2)'(\lambda_5)$. Using the Reidemeister–Schreier method, we get the following generators of $(\overline{H}_2)'(\lambda_5)$ (here $\lambda=\lambda_5=\frac{1+\sqrt{5}}{2}$ and $\lambda$ is a root of the polynomial $\lambda^2-\lambda-1=0$). There are 10 generators of the form,*

$$k_1k_2k_1k_2 = \begin{pmatrix} 1 & 4\lambda \\ 0 & 1 \end{pmatrix} \qquad k_2k_4k_2k_4 = \begin{pmatrix} 48\lambda+29 & 40\lambda+24 \\ -20\lambda-12 & -16\lambda-11 \end{pmatrix}$$
$$k_1k_3k_1k_3 = \begin{pmatrix} 16\lambda+9 & 20\lambda+12 \\ 12\lambda+8 & 16\lambda+9 \end{pmatrix} \qquad k_2k_5k_2k_5 = \begin{pmatrix} 4\lambda+13 & 4\lambda+8 \\ -4\lambda-8 & -4\lambda-3 \end{pmatrix}$$
$$k_1k_4k_1k_4 = \begin{pmatrix} 16\lambda+9 & 12\lambda+8 \\ 20\lambda+12 & 16\lambda+9 \end{pmatrix} \qquad k_3k_4k_3k_4 = \begin{pmatrix} 8\lambda+5 & 8\lambda+4 \\ -8\lambda-4 & -8\lambda-3 \end{pmatrix}$$
$$k_1k_5k_1k_5 = \begin{pmatrix} 1 & 0 \\ 4\lambda & 1 \end{pmatrix} \qquad k_3k_5k_3k_5 = \begin{pmatrix} 48\lambda+29 & 20\lambda+12 \\ -40\lambda-24 & -16\lambda-11 \end{pmatrix}$$
$$k_2k_3k_2k_3 = \begin{pmatrix} 4\lambda+5 & 8\lambda+4 \\ -4\lambda & -4\lambda-3 \end{pmatrix} \qquad k_4k_5k_4k_5 = \begin{pmatrix} 4\lambda+5 & 4\lambda \\ -8\lambda-4 & -4\lambda-3 \end{pmatrix}$$

20 *generators of the form,*

$$k_1k_2k_3k_1k_3k_2 = \begin{pmatrix} -24\lambda-15 & -48\lambda-28 \\ -12\lambda-8 & -24\lambda-15 \end{pmatrix} \qquad k_1k_2k_3k_2k_3k_1 = \begin{pmatrix} 4\lambda+5 & -8\lambda-4 \\ 4\lambda & -4\lambda+5 \end{pmatrix}$$
$$k_1k_2k_4k_1k_4k_2 = \begin{pmatrix} -42\lambda-25 & -128\lambda-80 \\ -20\lambda-12 & -52\lambda-33) \end{pmatrix} \qquad k_1k_2k_4k_2k_4k_1 = \begin{pmatrix} 48\lambda+29 & -40\lambda-24 \\ 20\lambda+12 & -16\lambda-11 \end{pmatrix}$$
$$k_1k_2k_5k_1k_5k_2 = \begin{pmatrix} -8\lambda-3 & -20\lambda-16 \\ -4\lambda & -8\lambda-7 \end{pmatrix} \qquad k_1k_2k_5k_2k_5k_1 = \begin{pmatrix} 4\lambda+13 & -4\lambda-8 \\ 4\lambda+8 & -4\lambda-3 \end{pmatrix}$$
$$k_1k_3k_4k_1k_4k_3 = \begin{pmatrix} -36\lambda-23 & -72\lambda-44 \\ -32\lambda-20 & -64\lambda-39 \end{pmatrix} \qquad k_1k_3k_4k_3k_4k_1 = \begin{pmatrix} -4\lambda-3 & 4\lambda+2 \\ -4\lambda-2 & 4\lambda+1 \end{pmatrix}$$

$$k_1k_3k_5k_1k_5k_3 = \begin{pmatrix} -48\lambda-31 & -60\lambda-36 \\ -40\lambda-24 & -48\lambda-41) \end{pmatrix} \qquad k_1k_3k_5k_3k_5k_1 = \begin{pmatrix} 48\lambda+29 & -20\lambda-12 \\ 40\lambda+24 & -16\lambda-11 \end{pmatrix}$$
$$k_1k_4k_5k_1k_5k_4 = \begin{pmatrix} -24\lambda-15 & -20\lambda-8 \\ -32\lambda-20 & -28\lambda-15) \end{pmatrix} \qquad k_1k_4k_5k_4k_5k_1 = \begin{pmatrix} 4\lambda+5 & -4\lambda \\ 8\lambda+4 & -4\lambda-3 \end{pmatrix}$$
$$k_2k_3k_4k_2k_4k_3 = \begin{pmatrix} -100\lambda-63 & -112\lambda-68 \\ 48\lambda+28 & 52\lambda+33 \end{pmatrix} \qquad k_2k_3k_4k_3k_4k_2 = \begin{pmatrix} -16\lambda-11 & -40\lambda+44 \\ 8\lambda+4 & 16\lambda+13 \end{pmatrix}$$
$$k_2k_3k_5k_2k_5k_3 = \begin{pmatrix} -80\lambda-51 & -92\lambda-60 \\ 40\lambda+24 & 48\lambda+25 \end{pmatrix} \qquad k_2k_3k_5k_3k_5k_2 = \begin{pmatrix} -80\lambda-51 & -228\lambda-140 \\ 40\lambda+24 & 112\lambda+69 \end{pmatrix}$$
$$k_2k_4k_5k_2k_5k_4 = \begin{pmatrix} -20\lambda-11 & -12\lambda \\ 8\lambda+4 & 4\lambda+1 \end{pmatrix} \qquad k_2k_4k_5k_4k_5k_2 = \begin{pmatrix} -20\lambda-11 & -52\lambda-32 \\ -8\lambda-4 & 20\lambda+13 \end{pmatrix}$$
$$k_3k_4k_5k_3k_5k_4 = \begin{pmatrix} -206\lambda-127 & -92\lambda-56 \\ 182\lambda+108 & 80\lambda+49 \end{pmatrix} \qquad k_3k_4k_5k_4k_5k_3 = \begin{pmatrix} -44\lambda-27 & -52\lambda-32 \\ 40\lambda+20 & 44\lambda+29 \end{pmatrix}$$

15 *generators of the form,*

$$k_1k_2k_3k_4k_1k_4k_3k_2 = \begin{pmatrix} 68\lambda+41 & 84\lambda+52 \\ 32\lambda+20 & 40\lambda+25 \end{pmatrix} \qquad k_1k_2k_3k_4k_2k_4k_3k_1 = \begin{pmatrix} -100\lambda-63 & 112\lambda+68 \\ -48\lambda-28 & 52\lambda+33 \end{pmatrix}$$
$$k_1k_2k_3k_4k_3k_4k_2k_1 = \begin{pmatrix} -16\lambda-11 & 40\lambda+20 \\ -8\lambda-4 & 16\lambda+13 \end{pmatrix} \qquad k_1k_2k_3k_5k_1k_5k_3k_2 = \begin{pmatrix} 80\lambda+49 & 160\lambda+100 \\ 40\lambda+24 & 80\lambda+49) \end{pmatrix}$$
$$k_1k_2k_3k_5k_2k_5k_3k_1 = \begin{pmatrix} -80\lambda-51 & 92\lambda+60 \\ -40\lambda-24 & 48\lambda+25) \end{pmatrix} \qquad k_1k_2k_3k_5k_3k_5k_2k_1 = \begin{pmatrix} -80\lambda-51 & 128\lambda+80 \\ -40\lambda-24 & 64\lambda+37 \end{pmatrix}$$
$$k_1k_2k_4k_5k_1k_5k_4k_2 = \begin{pmatrix} 80\lambda+49 & 200\lambda+120 \\ 32\lambda+20 & 80\lambda+49 \end{pmatrix} \qquad k_1k_2k_4k_5k_2k_5k_4k_1 = \begin{pmatrix} -20\lambda-11 & 12\lambda \\ -8\lambda-4 & 4\lambda+1 \end{pmatrix}$$
$$k_1k_2k_4k_5k_4k_5k_2k_1 = \begin{pmatrix} -20\lambda-11 & 52\lambda+32 \\ -8\lambda-4 & 20\lambda+13 \end{pmatrix} \qquad k_1k_3k_4k_5k_1k_5k_4k_3 = \begin{pmatrix} 120\lambda+73 & 136\lambda+84 \\ 108\lambda+60 & 120\lambda+73 \end{pmatrix}$$
$$k_1k_3k_4k_5k_3k_5k_4k_1 = \begin{pmatrix} -128\lambda-79 & 92\lambda+56 \\ -112\lambda-68 & 80\lambda+49 \end{pmatrix} \qquad k_1k_3k_4k_5k_4k_5k_3k_1 = \begin{pmatrix} -44\lambda-27 & 52\lambda+32 \\ -40\lambda-20 & 44\lambda+29 \end{pmatrix}$$
$$k_2k_3k_4k_5k_2k_5k_4k_3 = \begin{pmatrix} 368\lambda+221 & 604\lambda+372 \\ -172\lambda-112 & -288\lambda-179 \end{pmatrix} \qquad k_2k_3k_4k_5k_3k_5k_4k_2 = \begin{pmatrix} 232\lambda+145 & 588\lambda+360 \\ -112\lambda-68 & -280\lambda-175 \end{pmatrix}$$
$$k_2k_3k_4k_5k_4k_5k_3k_2 = \begin{pmatrix} 56\lambda+33 & 164\lambda+96 \\ -40\lambda-20 & -76\lambda-51 \end{pmatrix}$$

*and 4 generators of the form*

$$k_1k_2k_3k_4k_5k_1k_5k_4k_3k_2 = \begin{pmatrix} -192\lambda - 119 & -372\lambda - 228 \\ -92\lambda - 56 & -176\lambda - 111 \end{pmatrix}$$
$$k_1k_2k_3k_4k_5k_2k_5k_4k_3k_1 = \begin{pmatrix} 368\lambda + 221 & -604\lambda - 372 \\ 172\lambda + 112 & -288\lambda - 179 \end{pmatrix}$$
$$k_1k_2k_3k_4k_5k_3k_5k_4k_2k_1 = \begin{pmatrix} 232\lambda + 145 & -588\lambda - 360 \\ 112\lambda + 68 & -280\lambda - 175 \end{pmatrix}$$
$$k_1k_2k_3k_4k_5k_4k_5k_3k_2k_1 = \begin{pmatrix} 76\lambda + 53 & -164\lambda - 96 \\ 40\lambda + 20 & -76\lambda - 51 \end{pmatrix}$$

*Therefore, the subgroup* $(\overline{H}_2)'(\lambda_5)$ *is a free group of rank* 49 *and of signature* $(5;\ \infty^{(40)})$.

**Corollary 2.10** *The subgroup* $(\overline{H}_2)'(\lambda_5)$ *of* $H(\lambda_5)$ *is equal to the congruence subgroup* $H_4(\lambda_5)$, *i.e.* $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$.

**Proof** From Theorem 2.3, the group $(\overline{H}_2)'(\lambda_5)$ is a normal subgroup of index 160 in $H(\lambda_5)$. Moreover, the congruence subgroup $H_4(\lambda_5)$ is a normal subgroup of index 160 in $H(\lambda_5)$ (see [16] and [23]). Indeed, there are 4 normal subgroups of index 160 in $H(\lambda_5)$ (see [9]). However, from the previous example, all generators of the group $(\overline{H}_2)'(\lambda_5)$ are congruent to the $\pm\ I \pmod 4$. Thus $(\overline{H}_2)'(\lambda_5) \subseteq H_4(\lambda_5)$ and we get $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$. □

On the other hand, in [34], Newman and Smart showed that

$$H_6(\lambda_3) = (H^2)'(\lambda_3) \cap (H^3)'(\lambda_3).$$

Now we show that this equality is not true for the Hecke group $H(\lambda_5)$.

**Corollary 2.11** $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$.

**Proof** Since $(H^2)'(\lambda_5) \subset H'(\lambda_5)$ and $(H^5)'(\lambda_5) \subset H'(\lambda_5)$, we have $(H^2)'(\lambda_5) \cap (H^5)'(\lambda_5) \subset H'(\lambda_5)$. If $H_{10}(\lambda_5) = (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$, then $H_{10}(\lambda_5) \subset H'(\lambda_5)$. However, this is impossible since the commutator subgroup $H'(\lambda_5)$ is not congruence, from [25]. Then we get $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$. □

Finally, we formulate the following conjectures. It seems to us difficult to prove them.

**Conjecture 2.12** *i) For all* $q \geq 3$ *prime,* $(\overline{H}_2(\lambda_q))' = H_4(\lambda_q)$.

*ii) For all* $q \geq 3$ *prime,* $H_{2q}(\lambda_q) \neq (H^2)'(\lambda_q) \cap (H^q)'(\lambda_q)$.

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