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On the dimension of vertex labeling of k-uniform dcsl of an even cycle

Nageswararao KARREY*, **Kizhekekunnel AUGUSTINE**
School of Physical Sciences, Department of Mathematics, Central University of Kerala, Kasaragod, India

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Abstract: In this paper, we discuss the lower bound for the dcsl index δ_k of a k-uniform dcsl of even cycle C_{2n}, $n \geq 2$, in terms of the dimension of a poset and prove that $dim(\mathscr{F}) \leq \delta_k(C_{2n})$, where $\mathscr{F}$ is the range of any k-uniform dcsl f of C_{2n}, $n \geq 2$.

Key words: k-Uniform distance compatible set labeling, dimension of the poset

1. Introduction

Acharya [1] introduced the notion of vertex *set-valuation* as a set-analogue of number valuation. For a graph $G = (V, E)$ and a nonempty set X, he defined a *set-valuation* of G as an injective *set-valued* function $f : V(G) \to 2^X$, and defined a *set-indexer* $f^\oplus : E(G) \to 2^X \setminus \{\phi\}$ as a *set-valuation* such that the induced edge labeling $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina [2] introduced a particular kind of set-valuation for which a metric, especially the cardinality of the symmetric difference, associated with each pair of vertices is k (where k is a nonnegative constant) times that of the distance between them in the graph [2]. In other words, determine those graphs $G = (V, E)$ that admit an injective set-valued function $f : V(G) \to 2^X$, where 2^X is the power set of a nonempty set X, such that, for each pair of distinct vertices u and v in G, the cardinality of the symmetric difference $f(u) \oplus f(v)$ is k times that of the usual path distance $d_G(u, v)$ between u and v in G. They [2] called such a *set-valuation* f of G a *k-uniform distance compatible set labeling* (*k-uniform dcsl*) of G, and the graph G that admits k-uniform dcsl a *k-uniform distance compatible set labeled graph* (*k-uniform dcsl graph*), and the nonempty set X corresponding to f a *k-uniform dcsl-set*. The *k-uniform dcsl index* [13] of a graph G, denoted by $\delta_k(G)$, is the minimum of the cardinalities of X, with respect to which G is a k-uniform dcsl.

A hypercube $\mathcal{H}(X)$ on a set X is a graph whose vertices are the finite subsets of X, and two vertices are joined by an edge if and only if they differ by a singleton. A partial cube is a graph that can be isometrically embedded into a hypercube [22].

A family of sets $\mathscr{F}$ is well graded if any two sets in $\mathscr{F}$ can be connected by a sequence of sets formed by single element insertion and deletion, without redundant operations, such that all intermediate sets in the sequence belong to $\mathscr{F}$. Well-graded families are of interest in several different areas of combinatorics, as various

*Correspondence: karreynageswararao@gmail.com
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families of sets or relations are well graded. Using representation theorems, well-graded families are applied to the partial cubes [7, 22, 25], and to the oriented media, which are semigroups of transformations satisfying certain axioms (see [10, 11]).

Definition 1.1 *[8] Let $\mathscr{F}$ be a family of subsets of a set X. A tight path between two distinct sets P and Q (or from P to Q) in $\mathscr{F}$ is a sequence $P_0 = P, P_1, P_2 \ldots P_n = Q$ in $\mathscr{F}$ such that $d(P,Q) =\mid P \oplus Q \mid= n$ and $d(P_i, P_{i+1}) = 1$ for $0 \le i \le n-1$.*

The family $\mathscr{F}$ is a well-graded family (or wg-family) if there is a tight path between any two of its distinct sets.

Any family $\mathscr{F}$ of subsets of X defines a graph $G_{\mathscr{F}} = (\mathscr{F}, E_{\mathscr{F}})$, where $E_{\mathscr{F}} = \{\{P,Q\} \subseteq \mathscr{F} :\mid P \oplus Q \mid= 1\}$, and we call $G_{\mathscr{F}}$ an $\mathscr{F}$-induced graph.

One may recall a *partially ordered set* (or a *poset*) $\mathbf{P}$ as a structure $(P, \preceq)$ where P is a nonempty set and '$\preceq$' is a partial order relation on P such that '$\preceq$' is *reflexive, antisymmetric*, and *transitive*. We denote $(x,y) \in \mathbf{P}$ by $x \preceq y$. Given a poset $\mathbf{P}$, the dual of $\mathbf{P}$ is a new poset $\mathbf{P}^d$ on the same set P with the new relation $x \preceq_{\mathbf{P}^d} y$, if and only if $y \preceq_{\mathbf{P}} x$. Two elements x, y of $\mathbf{P}$ are *comparable* if either $x \preceq y$ or $y \preceq x$; otherwise x, y are *incomparable*. We denote the incomparable elements x and y of $\mathbf{P}$ by $x \mid\mid y$. A poset is a *chain* if it contains no incomparable pair of elements. In this case, the partial order is a *linear order* . A poset is an *antichain* if all of its pairs are incomparable.

The size of a largest chain in a poset $\mathbf{P}$ is called the *height* of the poset, denoted by *height*$(\mathbf{P})$ (or $h(\mathbf{P})$), and the size of a largest antichain is called its *width*, denoted by *width*$(\mathbf{P})$ (or $w(\mathbf{P})$). The *greatest element* I of a poset $\mathbf{P}$ is $I \succeq x$ for all $x \in \mathbf{P}$, and the *least element* 0 is $0 \preceq x$ for all $x \in \mathbf{P}$.

We say that z *covers* y if and only if $y \prec z$ and $y \preceq x \preceq z$ implies either $x = y$ or $x = z$. A *Hasse diagram* of a poset $(P, \preceq)$ is a drawing in which the points of P are placed so that if y *covers* x, then y is placed at a higher level than x and joined to x by a line segment. A poset $\mathbf{P}$ is *connected* if its Hasse diagram is connected as a graph.

A *cover graph (or Hasse graph)* of a poset $(P, \preceq)$ is the graph with vertex set P such that $x, y \in P$ are adjacent if and only if one of them covers the other. All posets depicted in this paper are shown by their Hasse diagrams. A *planar drawing* of a poset $\mathbf{P}$ is a representation of the Hasse diagram of $\mathbf{P}$ such that no edges of the Hasse diagram cross each other. A *planar poset* is a poset that has a planar drawing; otherwise, it is called a *nonplanar poset*. A graph is *outer planar* if it has a crossing-free embedding in the plane such that all vertices are on the same face.

A poset $\mathbf{Q}$ is a subposet of $\mathbf{P}$ if $Q \subseteq P$, and for each pair $x, y \in \mathbf{Q}$, $x \preceq y$ in $\mathbf{Q}$ exactly if $x \preceq y$ in $\mathbf{P}$. Two posets $\mathbf{P}$ and $\mathbf{Q}$ are called *isomorphic* if there is a one-to-one correspondence $\Phi : P \to Q$ such that $x \preceq y$ in $\mathbf{P}$ if and only if $\Phi(x) \preceq \Phi(y)$ in $\mathbf{Q}$. The poset $\mathbf{Q}$ is said to be *embedded* in $\mathbf{P}$, denoted by $\mathbf{Q} \subseteq \mathbf{P}$, if $\mathbf{Q}$ is isomorphic to a subposet of $\mathbf{P}$.

A *linear extension* $\mathbf{L}$ of $\mathbf{P}$ is a linear order on the elements of $\mathbf{P}$, such that $x \preceq y$ in $\mathbf{P}$ implies $x \preceq y$ in $\mathbf{L}$ for all $x, y \in \mathbf{P}$.

Definition 1.2 *[9] A set $\mathcal{R} = \{\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_k\}$ of linear extensions of $\mathbf{P}$ is a* ***realizer*** *of $\mathbf{P}$ if $\mathbf{P} = \cap_{\mathbf{L} \in \mathcal{R}}$.*

The ***dimension*** *of $\mathbf{P}$, denoted by $dim(\mathbf{P})$, is the minimum cardinality of a realizer.*

Hiraguchi [18] proved that the dimension cannot exceed the width, and for antichains dimension can be much less than the width. He also proved that the dimension cannot exceed half the number of elements of the poset, even though there are posets of arbitrarily large dimension.

The following definition is due to Hiraguchi [18], and later Bogart [5]:

Definition 1.3 *The standard example (also called standard n-dimensional poset) $S_n(n \geq 2)$ is the poset of height two consisting of n minimal elements $a_1, \ldots, a_n$ and n maximal elements $b_1, \ldots, b_n$ such that $a_i \preceq b_j$ in S_n exactly if $i \neq j$.*

A poset $(L, \preceq)$ is a *lattice* if every pair of elements $x, y \in L$ has a *least upper bound* as join of x, y, denoted by $x \vee y$, and a *greatest lower bound* as meet of x, y, denoted by $x \wedge y$. In general, a lattice is denoted by $(L, \preceq)$. A lattice $(L, \preceq)$ is planar if its Hasse diagram drawing is planar.

Throughout this paper, by a lattice we mean a poset under set inclusion $\subseteq$. Unless otherwise mentioned, for all the terminology in graph theory and lattice theory, one may refer, respectively, to [4, 17]. Throughout this article, by a graph we mean a simple and connected graph. By dimension of vertex labeling of a k-uniform dcsl graph, we mean the dimension of the poset $\mathscr{F}$ whose elements are the vertex labeling of the k-uniform dcsl graph. Throughout this paper, by a poset we mean a planar poset.

We need the following existing results.

Theorem 1.1 *[6] Suppose that the largest antichain in the poset $\mathbf{P}$ has size r. Then $\mathbf{P}$ can be partitioned into r chains, but not fewer.*

Theorem 1.2 *[20] Suppose that the largest chain in the poset $\mathbf{P}$ has size r. Then $\mathbf{P}$ can be partitioned into r antichains, but not fewer.*

Theorem 1.3 *[23] For $n \geq 2$, S_n, the standard n-dimensional poset, $dim(S_n) = n$.*

Theorem 1.4 *[19] For every $n \geq 5$, the standard example S_n is nonplanar, but it is a subposet of a planar poset.*

The following theorem is due to Felsner, Trotter, and Wiechert.

Theorem 1.5 *[12] If the cover graph of a poset $\mathbf{P}$ is outer planar, then $dim(\mathbf{P}) \leq 4$. If $\mathbf{P}$ is a poset with an outer planar cover graph and the height of $\mathbf{P}$ is 2, then $dim(\mathbf{P}) \leq 3$.*

Proposition 1.1 *[13] For a k-uniform dcsl graph G, $\delta_k(G) \geq k.\ diam(G)$.*

Theorem 1.6 *[13] If G is k-uniform dcsl, and m is a positive integer, then G is mk-uniform dcsl.*

Theorem 1.7 *[14] The cycle C_n, $n \geq 3$, with chords is a dcsl graph if and only if n is even and the maximum number of chords is $\frac{n}{2} - 2$.*

Germina and Jinto [15] proved that the vertex labeling of any 1-uniform dcsl graph forms a wg-family, and for any wg-family $\mathscr{F}$, the $\mathscr{F}$-induced graph $G_{\mathscr{F}}$ admits a 1-uniform distance compatible set labeling. Germina and Nageswara Rao [16] proved that if $\mathscr{F}$ is a well-graded family of subsets of X whose $\mathcal{F}$-induced graph is $G_{\mathscr{F}}$ and if $C_{\mathscr{F}}$ is the cover graph of $\mathscr{F}$ with respect to set inclusion '$\subseteq$', then $C_{\mathscr{F}} \cong G_{\mathscr{F}}$.

It is known that if $\mathbf{P}$ is a poset with a least element and a greatest element, and if $\mathbf{P}$ is planar, then $dim(\mathbf{P}) \leq 2$ [3]. Also, if $\mathbf{P}$ is a poset with a least element (or a greatest element), and if $\mathbf{P}$ is planar, then $dim(\mathbf{P}) \leq 3$ [24].

Analogously, we have:

Theorem 1.8 *A 1-uniform dcsl graph whose collection of vertex labeling, $\mathscr{F}$, forms a planar lattice. Then $dim(\mathscr{F}) \leq 2$.*

Theorem 1.9 *A 1-uniform dcsl graph whose collection of vertex labeling, $\mathscr{F}$, forms a planar poset, which has a least element (or a greatest element). Then $dim(\mathscr{F}) \leq 3$.*

Invoking Theorem 1.8 and Theorem 1.9:

Theorem 1.10 *If the collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle C_{2n}, $(n \geq 2)$, is a planar lattice, then $dim(\mathscr{F}) \leq 2$.*

Theorem 1.11 *If the collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle C_{2n}, $(n \geq 2)$, is a planar poset, which has a least element (or a greatest element), then $dim(\mathscr{F}) \leq 3$.*

2. Main results

[21] Since the assignment of vertex labeling of a 1-uniform dcsl graph is not unique, the problem of determining posets obtained by embedding the vertex labeling of a 1-uniform dcsl graph is same as determining the existence of different vertex labels f of a 1-uniform dcsl graph whose corresponding range $Range(f) = \mathscr{F}$, say, forms a poset under set inclusion $\subseteq$. Thus, there is a one to one correspondence between the 1-uniform dcsl f of a graph and its corresponding poset $\mathscr{F}$. Thus, it is always possible to find a 1-uniform dcsl f of a graph G so that $\mathscr{F} = Range(f)$ forms a poset under set inclusion $\subseteq$. Hence, $\mathscr{F}$ contains the collection of vertex labeling f of a 1-uniform dcsl graph G as an embedding of itself. Hence, the problem of determining the 1-uniform dcsl labeling f of a graph G is equivalent in determining the poset $\mathscr{F}$ that embeds the 1-uniform dcsl vertex labeling f of the same graph G.

Let $\mathscr{F}$ be the collection of vertex labeling of a 1-uniform dcsl graph G that forms a lattice. Then it is noticed that all the maximal chains of the poset $\mathscr{F}$ have the same length, and hence $\mathscr{F}$ is graded.

Theorem 2.1 *If the collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl graph G forms a lattice, then it is graded.*

Proof Let G be a 1-uniform dcsl graph and f be its 1-uniform dcsl.

Suppose $\mathscr{F}$ is the collection of vertex labeling of a 1-uniform dcsl graph G that forms a lattice under $\subseteq$.

Suppose $\text{Inf}\ \mathscr{F} = P$ and $\text{Sup}\ \mathscr{F} = Q$. That is, there exist unique vertices $p, q \in G$ such that $f(p) = P$ and $f(q) = Q$.

Suppose $M_1, M_2, \ldots, M_t$ are the maximal chains of $\mathscr{F}$, where

$$M_1 : f(a_0) \subseteq f(a_1) \subseteq \cdots \subseteq f(a_{M_1}),$$
$$M_2 : f(b_0) \subseteq f(b_1) \subseteq \cdots \subseteq f(b_{M_2}),$$
$$\vdots$$
$$M_t : f(t_0) \subseteq f(t_1) \subseteq \cdots \subseteq f(t_{M_t}).$$

Claim: $l(M_1) = l(M_2) = \cdots = l(M_t)$.

Since $\mathscr{F}$ is a lattice and P and Q are the infimum and supremum of $\mathscr{F}$, the infimum and supremum of each maximal chain $M_i, 1 \leq i \leq t$, is P and Q, respectively.

That is,

$$P(= f(p)) = f(a_0) = f(b_0) = \cdots = f(t_0)$$

and

$$Q(= f(q)) = f(a_{M_1}) = f(b_{M_2}) = \cdots = f(t_{M_t}).$$

Hence, since f is injective,

$$p = a_0 = b_0 = \cdots = t_0$$

and,

$$q = a_{M_1} = b_{M_2} = \cdots = t_{M_t}.$$

Also, corresponding to each M_i $(1 \leq i \leq t)$, there exists a path, say P_i, which connects both p and q such that $d(p,q) = l(M_i)$.

Hence, all the paths P_i $(1 \leq i \leq t)$ have initial vertex p and end vertex q.

Since f is a 1-uniform dcsl that is injective, and $f(a_0), f(a_{M_1}) \in M_1$ such that $p = a_0,\ q = a_{M_1}$, and $d(p,q) = l(M_1)$,

$$\mid f(a_0) \oplus f(a_{M_1}) \mid = \mid f(p) \oplus f(q) \mid = d(p,q) = l(M_1).$$

Similarly,

$$\mid f(b_0) \oplus f(b_{M_2}) \mid = \mid f(p) \oplus f(q) \mid = d(p,q) = l(M_2),$$
$$\vdots \qquad\qquad \vdots$$
$$\mid f(t_0) \oplus f(t_{M_t}) \mid = \mid f(p) \oplus f(q) \mid = d(p,q) = l(M_t).$$

Therefore, for each $1 \leq i \leq t$, $l(M_i) = d(p,q)$, and hence

$$l(M_1) = l(M_2) = \cdots = l(M_t).$$

This completes the proof. □

Let $\mathscr{F}$ be a collection of vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ that has minimum width. By minimum width, we mean the smallest among all the widths. It can be observed that the minimum width of $\mathscr{F}$ is 2 when $\mathscr{F}$ is a lattice.

Proposition 2.1 *Let $\mathscr{F}$ be the collection of vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ that forms a lattice. Then $width(\mathscr{F}) = 2$.*

Proof Let $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$, and let f be a 1-uniform dcsl of C_{2n} $(n \geq 2)$, such that $\mathscr{F} = \{f(v) : v \in V(C_{2n})\}$ is a lattice.

Supposing $width(\mathscr{F}) < 2$, then all the members of $\mathscr{F}$ are comparable; hence, $\mathscr{F}$ is a chain, and hence, the graph associated with $\mathscr{F}$ is a path, a contradiction. Hence, $width(\mathscr{F}) \geq 2$.

By Theorem 1.8, $dim(\mathscr{F}) \geq 2$, and also the dimension of a poset is at most the width of the poset; hence, $width(\mathscr{F}) \geq 2$. Hence, we conclude that $width(\mathscr{F}) = 2$. □

From Theorem 2.1 and Proposition 2.1, one may notice that the collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle that forms a lattice is always graded and $width(\mathscr{F}) = 2$. Hence, $\mathscr{F}$ is obtained as an embedding of the collection of vertex labeling of 1-uniform dcsl of C_{2n}, $n \geq 2$, and we necessarily need a poset that has exactly two maximal chains of length n each. This lead us to define "cyclic width-2 poset".

Definition 2.1 *The cyclic width-2 poset $\mathbf{C}_n$ on $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ is defined as the poset of width 2 consisting of two chains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that, for $2 \leq i \leq n$, $a_{i-1} \preceq a_i$ and $b_{i-1} \preceq b_i$, for $1 \leq i \leq n$, $a_1 \preceq b_i$, $a_n \preceq b_n$, and for $2 \leq i \leq n$ and $1 \leq j \leq n-1$, $a_i \,||\, b_j$.*

Proposition 2.2 *The cyclic width-2 poset $\mathbf{C}_n$ on $2n$ elements is a lattice.*

Proof The proof follows from the fact that the least and greatest elements in $\mathbf{C}_n$ are a_1 and b_n, respectively. □

Proposition 2.3 *For $n \geq 2$, the cyclic width-2 poset $\mathbf{C}_n$ on $2n$ elements, $dim(\mathbf{C}_n) = 2$.*

Proof Since all the elements of $\mathbf{C}_n$ are not comparable, $dim(\mathbf{C}_n) > 1$. Also, by Proposition 2.2, $\mathbf{C}_n$ is a lattice with least element a_1 and greatest element b_n.

Hence, by Theorem 1.8, $dim(\mathbf{C}_n) = 2$. □

Proposition 2.4 *There exists a 1-uniform dcsl f of even cycle C_{2n} $(n \geq 2)$, whose range $Range(f) = \mathscr{F}$, say, can be embedded in $\mathbf{C}_n$, the cyclic width-2 poset on $2n$ elements.*

Proof Let $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$.

Let f be a 1-uniform dcsl cycle C_{2n} $(n \geq 2)$ with $X = \{1, 2, \ldots, n\}$.

Define $f : V(C_{2n}) \to 2^X$ by

$$f(v_1) = \emptyset,$$
$$f(v_j) = f(v_{j-1}) \cup \{j-1\},\ 2 \leq j \leq n,$$
$$f(v_{n+1}) = X,$$
$$f(v_{n+j}) = f(v_{n+j-1}) \setminus \{j-1\},\ 2 \leq j \leq n.$$

Then:

$$|f(v_1) \oplus f(v_i)| = i - 1 = 1.\ d(v_1, v_i),\ 2 \leq i \leq n+1,$$

$$|f(v_{n+1}) \oplus f(v_i)| = i - n - 1 = 1.\ d(v_{n+1}, v_i),\ n+2 \le i \le 2n.$$

In general, for $1 \le i < j \le 2n$,

$$|f(v_i) \oplus f(v_j)| \ = \begin{cases} 1 = d(v_i, v_j), & \text{if } v_iv_j \in E(C_{2n}) \\ l = d(v_i, v_j), & \text{otherwise;} \end{cases}$$

where $2 \le l \le n$.

Thus, f is a 1-uniform dcsl of C_{2n}.

Let $\mathscr{F} = \{f(v_i) : v_i \in V(C_{2n})\}$.

We prove that $\mathscr{F}$ is embedded in $\mathbf{C}_n$, the cyclic width-2 poset on $2n$ elements $a_1, \ldots, a_n,\ b_1, \ldots, b_n$ of width two consisting of two chains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that, for $2 \le i \le n$, $a_{i-1} \preceq a_i$ and $b_{i-1} \preceq b_i$, for $1 \le i \le n$, $a_1 \preceq b_i$, $a_n \preceq b_n$, and for $2 \le i \le n$ and $1 \le j \le n-1$, $a_i \,||\, b_j$.

Define $\Phi : \mathscr{F} \to \mathbf{C}_n$ defined by

$$\Phi(f(v_i)) \ = \begin{cases} a_i, & \text{if } 1 \le i \le n, \\ b_{2n+1-i}, & \text{if } n+1 \le i \le 2n. \end{cases}$$

Clearly,

$f(v_{l-1}) \subseteq f(v_l)$ if and only if $a_{l-1} \preceq a_l$, for $2 \le l \le n$.

Also, for $n+2 \le l \le 2n$,

$f(v_l) \subseteq f(v_{l-1})$ if and only if $b_{2n+1-l} \preceq b_{2n+2-l}$.

Further, for $n+1 \le l \le 2n$,

$f(v_1) \subseteq f(v_l)$ if and only if $a_1 \preceq b_l$.

Furthermore, for $2 \le i \le n$ and $n+2 \le j \le 2n$,

$f(v_i) \,||\, f(v_j)$ if and only if $a_i \,||\, b_j$. Hence, $\mathscr{F} \cong \mathbf{C}_n$.

Therefore, $\mathscr{F}$ is embedded in $\mathbf{C}_n$. □

Example 2.1 *Figure 1 depicts the 1-uniform dcsl vertex labeling of C_{2n} $(n \ge 2)$, which forms a lattice and is embedded in $\mathbf{C}_n$.*

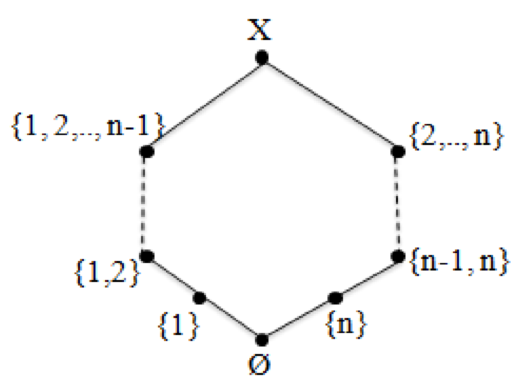

Figure 1. Vertex labeling of C_{2n} $(n \ge 2)$ that forms a lattice.

Proposition 2.5 *If the collection vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ is embedded in $\mathbf{C}_n$, then $dim(\mathscr{F}) = 2$.*

Proof Let f be the 1-uniform dcsl as in Proposition 2.4. Then $\mathscr{F} = \{f(v_i) : v_i \in V(C_{2n})\}$, which is embedded in $\mathbf{C}_n$, the cyclic width-2 poset on $2n$ elements. Then $dim(\mathscr{F}) \leq dim(\mathbf{C}_n) = 2$. If $dim(\mathscr{F}) < 2$, then every element of $\mathscr{F}$ is comparable and hence the graph associated with $\mathscr{F}$ is a path. Hence, there is no $\mathscr{F}$ whose dimension is less than that of the dimension of $\mathbf{C}_n$. Hence, we conclude that $dim(\mathscr{F}) = dim(\mathbf{C}_n) = 2$. □

Remark 2.1 *One can notice that there are posets that do not form a lattice and have width 2. Consider the following poset, $W_2 = \{a, b, x, y\}$, whose Hasse diagram is given in Figure 2. Clearly, W_2 is not a lattice. It is interesting to see that the poset W_2 is the smallest poset, which does not form a 1-uniform dcsl for an even cycle on 4 vertices.*

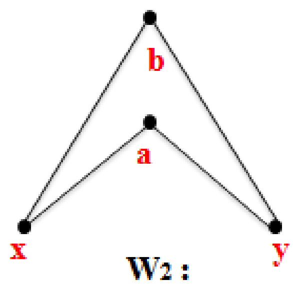

Figure 2. Hasse diagram of $W_2 = \{a, b, x, y\}$.

Remark 2.2 *It is quite interesting to see whether the converse of the Proposition 2.1 is true. That is, given a poset of width 2 forming a lattice only if the elements of the poset will be the elements of the vertex labeling of the 1-uniform dcsl of an even cycle C_{2n} $(n \geq 2)$.*

Remark 2.3 *If a poset $\mathcal{P}$ contains W_2 as an isomorphic subposet, then $\mathcal{P}$ does not form a 1-uniform dcsl even cycle since the existence of such a poset implies the noninjectivity of 1-uniform dcsl f. Hence, if $\mathscr{F}$ is a poset whose members are vertex labeling of a 1-uniform dcsl even cycle, then $\mathscr{F}$ does not contain any subposet that is isomorphic to W_2.*

Proposition 2.6 *Let $\mathscr{F}$ be a poset of width 2 whose members are vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$. Then $\mathscr{F}$ is a lattice.*

Proof Suppose, if possible, that $\mathscr{F}$ does not form a lattice, which means the poset $\mathscr{F}$, which is not lattice, of width 2, whose members are vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$. That is, $\mathscr{F}$ is isomorphic to W_2, which is a contradiction by Remark 2.3. □

From Proposition 2.1 and Proposition 2.6, we get the following result.

Theorem 2.2 *Let $\mathscr{F}$ be a collection of vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, which forms a poset. Then $width(\mathscr{F}) = 2$ if and only if $(\mathscr{F}, \subseteq)$ is a (planar) lattice.*

It is observed that there are lattices that have width 2, but to embed the vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, by Theorem 2.1, it should contain all the maximal chains of equal length. Thus, all width 2 lattices that contain equal length of maximal chains form a 1-uniform dcsl even cycle. Furthermore, the maximum height of such lattices is always $\frac{|V(C_{2n})|}{2} + 1$.

Proposition 2.7 *Let $\mathscr{F}$ be a lattice whose members are vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$. Then the maximum height of $\mathscr{F}$ is $\frac{|V(C_{2n})|}{2} + 1$.*

It has been proved that there are planar posets that are of higher dimension. For example, as proved in Theorem 1.4 and Theorem 1.3, the n-dimensional poset S_n is a planar poset for $2 \leq n \leq 4$ and $dim(S_n) = n$ for $n \geq 2$. Now we prove that the vertex labeling $\mathscr{F}$ for a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ is isomorphic to S_n if and only if $n = 3$.

Theorem 2.3 *The collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ is isomorphic to n-dimensional poset S_n if and only if $n = 3$.*

Proof The collection of vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, forming a poset that is isomorphic to S_n, when $n = 3$, is given in Figure 3.

Conversely, suppose, if possible, that there exists a 1-uniform dcsl, f of C_{2n} $(n \geq 2)$, such that $\mathscr{F} = \{f(v) : v \in V(C_{2n})\}$ forms a poset that is isomorphic to S_n, when $n \neq 3$.

Case 1: When $n < 3$. By definition of S_n, the (Hasse) graph associated to poset S_n is disconnected, which is a contradiction.

Case 2: When $n > 3$. In this case, the (Hasse) graph associated to a poset S_n is isomorphic to a chordal graph. Note that the maximum number of chords in S_n is $n(n-3)$, and due to Theorem 1.7, the maximum number of chords in a 1-uniform dcsl even cycle C_{2n} is $n-2$. We arrive a contradiction as $n(n-3) > n-2$.

Hence, $\mathscr{F} \cong S_n$ if and only if $n = 3$. □

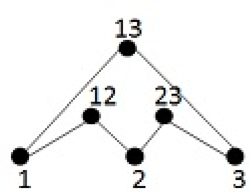

Figure 3. Vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, which is isomorphic to the 3-dimensional poset S_3.

Theorem 2.4 *If the collection of vertex labeling $\mathscr{F}$ of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ is isomorphic to n-dimensional poset S_n, then $dim(\mathscr{F}) = 3$.*

Proof Suppose $\mathscr{F}$ is the collection of vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ forming an n-dimensional poset S_n for $n = 3$. That is, $\mathscr{F} \cong S_3$. By Theorem 1.3, $dim(S_n) = n$ for $n \geq 2$ and hence $dim(\mathscr{F}) = 3$. □

Remark 2.4 *One may note that the collection of vertex labeling of a 1-uniform dcsl even cycle* C_{2n} $(n \geq 2)$ *forms a poset, but the converse need not be true. That is, there exists a poset whose elements do not form the vertex labeling of any 1-uniform dcsl even cycle* C_{2n} $(n \geq 2)$. *Also, there exists a 3-dimensional poset (but not lattice) whose Hasse graph is isomorphic to even cycle* C_{2n} *when* $n = 3$, *but their elements do not form the vertex labeling of any 1-uniform dcsl even cycle* C_{2n}.

Remark 2.5 *As remarked in Remark 2.3, the members of the 3-dimensional Chevron poset* V_6 *[3] (see Figure 4), and its dual, do not exhibit the vertex labeling of any 1-uniform dcsl even cycle* C_{2n}, *when* $n = 3$.

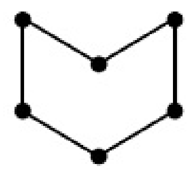

Figure 4. The Hasse diagram of the Chevron.

Remark 2.6 *Let* $G_{\mathscr{F}}$ *be an* $\mathscr{F}$*-induced graph of* $\mathscr{F}$, *and* $C_{\mathscr{F}}$ *the cover graph of* $\mathscr{F}$ *whose vertex set is* $\mathscr{F}$. *Two vertices, say* $P, Q \in \mathscr{F}$, *are adjacent if and only if either* P *covers* Q *or* Q *covers* P. *From Theorem 2.3, the poset* $\mathscr{F} \cong S_n$, *when* $n = 3$, *and the vertex labeling of any 1-uniform dcsl graph forms a wg-family, and for any wg-family* $\mathscr{F}$, *the* $\mathscr{F}$*-induced graph* $G_{\mathscr{F}}$ *admits a 1- uniform distance compatible set labeling. Hence,* $C_{\mathscr{F}} \cong G_{\mathscr{F}}$, *and hence the cover graph of* S_n *for* $n = 3$ *is a 1-uniform dcsl.*

As an immediate consequence of the Theorem 2.3 and Remark 2.6:

Proposition 2.8 *The cover graph of an* n*-dimensional poset* S_n *admits 1-uniform dcsl if and only if* $n = 3$.

Remark 2.7 *From Theorem 2.3, it is noticed that the poset* $\mathscr{F}$, *whose elements are the vertex labeling of 1-uniform dcsl even cycle* C_{2n} $(n \geq 2)$, *has height*$(\mathscr{F}) = 2$ *and width*$(\mathscr{F}) = \frac{|V(C_{2n})|}{2}$. *For, example consider the poset,* H_8, *whose Hasse diagram is given in Figure 5. The (Hasse) graph of it is isomorphic to even cycle* C_8, *and the vertex labeling constitutes a 1-uniform dcsl of height*$(\mathscr{F}) = 2$ *and width*$(\mathscr{F}) = 4 = \frac{|V(C_8)|}{2}$.

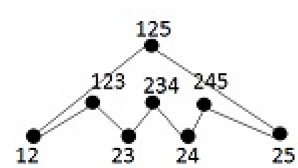

Figure 5. The Hasse diagram of H_8.

Proposition 2.9 *Let* $\mathscr{F}$ *be a collection of vertex labeling of a 1-uniform dcsl even cycle* C_{2n} $(n \geq 2)$, *which forms a poset (but not lattice). Then height*$(\mathscr{F}) = 2$ *if and only if width*$(\mathscr{F}) = \frac{|V(C_{2n})|}{2}$, *which is maximum.*

Proof Let $V(C_{2n}) = \{v_1, v_2, \ldots, v_n, \ldots, v_{2n}\}$.

Let $X = \{1, 2, \ldots, n, \ldots, 2n\}$, where $n = \frac{|V(C_{2n})|}{2}$, and let f be a 1-uniform dcsl of C_{2n} $(n \geq 2)$, such that $\mathscr{F} = \{f(v) : v \in V(C_{2n})\}$ forms a poset (but not lattice).

Supposing $height(\mathscr{F}) = 2$, then, by Mirsky's theorem 1.2, $\mathscr{F}$ can be partitioned into 2 antichains, but not fewer, say $\widehat{W_1}$, and $\widehat{W_2}$, and both $\widehat{W_1}$ and $\widehat{W_2}$ are of same length, say n. Thus, both $\widehat{W_1}$ and $\widehat{W_2}$ are of maximum length n. Hence, $width(\mathscr{F}) = \frac{|V(C_{2n})|}{2}(= 2)$, and hence $width(\mathscr{F})$ is maximum.

Conversely, supposing $width(\mathscr{F}) = \frac{|V(C_{2n})|}{2}(= w)$, then, by Dilworth's theorem 1.1, $\mathscr{F}$ can be partitioned into w chains, but not fewer, letting the partition be $L_1, L_2, \ldots, L_w$; hence, for $1 \leq i \leq w$, $|L_i| \leq 2$ and hence, $height(\mathscr{F}) = 2$. □

Since the cover graph of a poset $\mathscr{F}$ whose elements are the vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, which has $height(\mathscr{F}) = 2$, is an outer planar graph, by Theorem 1.5, $dim(\mathscr{F}) \leq 3$.

Thus:

Theorem 2.5 *If there exists any vertex labeling f of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, whose range $Range(f) = \mathscr{F}$, say, forms a poset (but not lattice) of $height(\mathscr{F}) = 2$, then $dim(\mathscr{F}) \leq 3$.*

Theorem 2.6 *If there exists any vertex labeling f of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ whose range $Range(f) = \mathscr{F}$, say, forms a poset, then $dim(\mathscr{F}) \leq 4$.*

Proof Since the cover graph of a poset $\mathscr{F}$ of vertex labeling of a dcsl even cycle C_{2n} $(n \geq 2)$ is outer planar, and by Theorem 1.5, $dim(\mathscr{F}) \leq 4$. □

Next, we find the dcsl index of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$. Recall that the minimum cardinality of the underlying set X such that G admits a 1-uniform dcsl is called the 1-uniform dcsl index $\delta_d(G)$ of G.

Lemma 2.1 *The 1-uniform dcsl index of C_{2n} $(n \geq 2)$ is n.*

Proof Let $V(C_{2n}) = \{v_1, v_2, \ldots, v_n, \ldots, v_{2n}\}$ and f be a dcsl labeling of C_{2n} with the underlying set as X.

First, we prove that $| X | \geq n$.

If possible, assume that C_{2n} $(n \geq 2)$ is 1-uniform dcsl with $| X | = n - 1$.

Without loss of generality, assuming $f(v_1) = X_1 = \emptyset$ and $f(v_{n+1}) = X_n = X$, then $| X_1 \oplus X_n | \leq n - 1$, whereas $d(v_1, v_{n+1}) = n$, a contradiction. Therefore, $\delta_d(C_{2n}) \geq n$.

Since, by Proposition 2.4, there exists a vertex labeling f of a 1-uniform dcsl even cycle $C_{2n}(n \geq 2)$ whose dcsl set X is of cardinality n, $\delta_d(C_{2n}) = n$. □

By Proposition 2.5 and Lemma 2.1, we have the following theorem.

Theorem 2.7 *Let $\mathscr{F}$ be a collection of vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, which forms a lattice under set inclusion '$\subseteq$'. Then $dim(\mathscr{F}) \leq \delta_d(C_{2n})$.*

Theorem 2.8 *Let the poset $\mathscr{F}$ be a set of vertex labeling of 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, which does not form a lattice with respect to set inclusion '$\subseteq$'. Then $dim(\mathscr{F}) \leq \delta_d(C_{2n})$.*

Proof Let f be a 1-uniform dcsl of C_{2n} $(n \geq 2)$, such that $\mathscr{F} = \{f(v) : v \in V(C_{2n})\}$ is the poset that does not form a lattice with respect to set inclusion '$\subseteq$'.

We divide the proof into three parts.

Part 1: First, we prove that no such poset $\mathscr{F}$ exists when $n = 2$.

If, suppose, there exists such a poset $\mathscr{F}$, then it has $width(\mathscr{F}) = 2$ and $height(\mathscr{F}) = 2$. Since $\mathscr{F}$ is not a lattice and the $width(\mathscr{F}) = 2$, $\mathscr{F} \simeq W_2$, which do not give 1-uniform dcsl for the respective graph, we arrive at a contradiction.

Part 2: When $n = 3$, if we prove that $\mathscr{F} \cong S_n$, then $dim(\mathscr{F}) = \delta_d(C_{2n})$.

When $n = 3$, the poset $\mathscr{F}$ has either $height(\mathscr{F}) = 3$ and $width(\mathscr{F}) = 3$, or $height(\mathscr{F}) = 2$ and $width(\mathscr{F}) = 3$.

Case 1: Supposing that $\mathscr{F}$ has $height(\mathscr{F}) = 3$ and $width(\mathscr{F}) = 3$, then $\mathscr{F} \cong V_6$. Then, by Proposition 2.5, the poset $\mathscr{F}$, whose elements do not form the vertex labeling of a 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$, is a contradiction.

Thus, $\mathscr{F}$ does not possess $height(\mathscr{F}) = 3$ and $width(\mathscr{F}) = 3$.

Case 2: Now, supposing $\mathscr{F}$ has $height(\mathscr{F}) = 2$ and $width(\mathscr{F}) = 3$, then $\mathscr{F} \cong S_n$, and hence by Theorem 2.4, $dim(\mathscr{F}) = 3$, and by Lemma 2.1, $\delta_d(C_6) = 3$. Hence, $dim(\mathscr{F}) = \delta_d(C_6)$.

Part 3: When $n > 3$, if we prove that $dim(\mathscr{F}) \leq 4$, then $dim(\mathscr{F}) \leq \delta_d(C_{2n})$.

When $n > 3$, by Theorem 2.6, $dim(\mathscr{F}) \leq 4$, and by Lemma 2.1, $\delta_d(C_{2n}) = n$.

Hence, $dim(\mathscr{F}) \leq \delta_d(C_{2n})$. □

Theorem 2.9 *Let the poset $\mathscr{F}$ be the collection of vertex labeling for 1-uniform dcsl even cycle C_{2n} $(n \geq 2)$ whether or not it forms a lattice with respect to set inclusion '$\subseteq$'. Then $dim(\mathscr{F}) \leq \delta_d(C_{2n})$.*

Since, by Theorem 1.6, 1-uniform dcsl implies k-uniform dcsl and even cycles always admit 1-uniform dcsl, thus:

Theorem 2.10 *Even cycle C_{2n} $(n \geq 2)$ is k-uniform dcsl.*

In view of Theorem 2.10, it is interesting to find the dcsl index of a k-uniform even cycle C_{2n} $(n \geq 2)$.

Lemma 2.2 *For $n \geq 2$, $\delta_k(C_{2n}) = kn$.*

Proof By proposition 1.1, for any k-uniform dcsl-graph G, $\delta_k(G) \geq k.\ diam(G)$.

Hence, $\delta_k(C_{2n}) \geq k.\ diam(C_{2n}) = kn$; that is, $\delta_k(C_{2n}) \geq kn$.

We claim that there exists k-uniform dcsl of even cycle C_{2n}, $n \geq 2$, with underlying set X whose cardinality is kn.

Let $X = \{1, 2, \ldots, kn\}$.

Define the dcsl labeling $f : V(C_{2n}) \to 2^X$, defined by

$$\begin{aligned} f(v_1) &= \emptyset, \\ f(v_2) &= f(v_1) \cup \{1,2,\ldots,k\} = \{1,2,\ldots,k\}, \\ f(v_3) &= f(v_2) \cup \{k+1,k+2,\ldots,k+k=2k\} \\ &= \{1,2,\ldots,k,k+1,\ldots,2k\}. \end{aligned}$$

For $2 \leq j \leq n$,

$$f(v_j) = f(v_{j-1}) \cup \{(j-2)k+1,(j-2)k+2,\ldots,(j-1)k\},$$

$$\begin{aligned} \text{and } f(v_{n+1}) &= X = \{1,2,\ldots,(n-1)k,nk\}, \\ f(v_{n+2}) &= f(v_{n+1}) \setminus \{1,2,\ldots,k\} \\ &= \{k+1,k+2,\ldots,2k,\ldots,nk\}, \\ f(v_{n+3}) &= f(v_{n+2}) \setminus \{k+1,k+2,\ldots,2k\} \\ &= \{2k+1,2k+2,\ldots,3k,\ldots,nk\}. \end{aligned}$$

For $2 \leq j \leq n$,

$$f(v_{n+j}) = f(v_{n+j-1}) \setminus \{(j-2)k+1,(j-2)k+2,\ldots,(j-1)k\}.$$

Thus,

$$|f(v_1) \oplus f(v_2)| = |\{1,2,\ldots,k\}| = k = k.\ d(v_1,v_2),$$

$$|f(v_2) \oplus f(v_3)| = |\{k+1,k+2,\ldots,k+k=2k\}| = k = k.\ d(v_2,v_3),$$

$$|f(v_1) \oplus f(v_3)| = |\{1,2,\ldots,2k\}| = 2k = k.\ d(v_1,v_3).$$

Hence, in general, for $1 \leq i < j \leq 2n$,

$$|f(v_i) \oplus f(v_j)| = \begin{cases} k, & \text{if } v_iv_j \in E(G) \\ lk, & \text{if } v_iv_j \notin E(G), \end{cases}$$

where $l = d(v_i,v_j)$ and $2 \leq l \leq n$.

Hence, there exists k-uniform dcsl for C_{2n} $(n \geq 2)$, with $|\ X\ |= kn$.

Therefore, $\delta_k(C_{2n}) = kn$. □

By Theorem 1.6, note that every 1-uniform dcsl of C_{2n} $(n \geq 2)$ is a k-uniform dcsl. However, every vertex labeling of a k-uniform dcsl even cycle C_{2n} $(n \geq 2)$ need not form a connected poset, but there always exists a k-uniform dcsl of C_{2n} $(n \geq 2)$, which forms a connected poset. Hence, the Hasse diagram (poset) that embeds the vertex labeling of the 1-uniform dcsl even cycle could also embed the vertex labeling of the k-uniform dcsl even cycle when that poset is connected.

The following theorem is a consequence of Theorem 1.6, Lemma 2.2, and Theorem 2.9.

Theorem 2.11 *If the poset $\mathscr{F}$ of the collection of vertex labeling of a k-uniform dcsl even cycle C_{2n} $(n \geq 2)$ under set inclusion '$\subseteq$' is connected, then $dim(\mathscr{F}) \leq \delta_k(C_{2n})$.*

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