Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 647 – 655

doi:10.3906/mat-1702-37

Field of values of perturbed matrices and quantum states

Madjid KHAKSHOUR[1,*], **Gholamreza AGHAMOLLAEI**[2], **Alemeh SHEIKHHOSSEINI**[3]
[1]Department of Applied Mathematics, Faculty of New Science and Technology, Graduate University of Advanced Technology of Kerman, Kerman, Iran
[2]Corresponding author, Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman, Iran
[3]Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran

Received: 09.02.2017 • **Accepted/Published Online:** 12.06.2017 • **Final Version:** 24.03.2018

Abstract: In this paper, the notion of the pseudofield of values of matrices is introduced and studied. The relationship between quantum states and the field of values of matrices is mentioned. The notion of the pseudopolynomial numerical hull, as a generalization of the pseudofield of values, of matrices is introduced and some properties of this notion are investigated.

Key words: Field of values, quantum state, pseudospectrum, polynomial numerical hull

1. Introduction and preliminaries

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices equipped with the operator norm $\|.\|$ induced by the Euclidean vector norm $\|x\| = (x^*x)^{1/2}$ on $\mathbb{C}^n$, i.e. $\|A\| = max\{\|Ax\| : x \in S^1\}$, where $A \in \mathbb{M}_n$ and $S^1 = \{x \in \mathbb{C}^n : \|x\| = 1\}$ is the unit sphere. In our discussion, we assume that $D(a, r) = \{\mu \in \mathbb{C} : |\mu - a| \leq r\}$, which is the closed disk at centered $a \in \mathbb{C}$ with radius $r > 0$. We also use the convention that if z is an eigenvalue of $A \in \mathbb{M}_n$, then $\|(A - zI)^{-1}\| := \infty$. The motivation of our study comes from two branches. The first one concerns the study of pseudospectra. The theory of pseudospectra provides an analytical and graphical alternative for investigating nonnormal matrices and operators, gives a quantitative estimate of departure from nonnormality, and also gives information about the stability of the solution of a system of linear differential equations. To see the other applications of pseudospectra of matrices in the physical sciences, we refer the reader to [12] and its references. For a given $\epsilon > 0$ and a matrix $A \in \mathbb{M}_n$, the ϵ*-pseudospectrum* (*pseudospectrum* for short) of A is defined and denoted by $\sigma_\epsilon(A) = \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq 1/\epsilon\}$, where I denotes the $n \times n$ identity matrix. It is known that

$$\sigma_\epsilon(A) = \bigcup_{E \in \mathbb{M}_n, \|E\| \leq \epsilon} \sigma(A + E), \tag{1}$$

where the matrix $A + E$ is a perturbation of A, and $\sigma(.)$ denotes the spectrum, i.e. the set of all eigenvalues. Note that in the physical sciences, the eigenvalues have many applications; for instance:

- In quantum mechanics, they help us to find atomic energy levels and the frequency of a laser;

---

*Correspondence: aghamollaei@uk.ac.ir
2010 *AMS Mathematics Subject Classification:* 15A60, 15A18, 81P68

• In electrical engineering, they determine the frequency response of an amplifier and the reliability of a power system.

The following properties of the pseudospectrum of matrices are useful in our discussion. For more details, see [6, 12].

Proposition 1.1 *Let $A \in \mathbb{M}_n$ and $\epsilon > 0$. Then the following assertions are true:*

(i) $\sigma_\epsilon(\alpha A + \beta I) = \alpha \sigma_{\epsilon/|\alpha|}(A) + \beta$, *where* $\alpha, \beta \in \mathbb{C}$ *and* $\alpha \neq 0$;

(ii) $\sigma_\epsilon(A) = D(\mu, \epsilon)$ *if and only if* $A = \mu I$, *where* $\mu \in \mathbb{C}$;

(iii) If $A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, *where* A_1 *and* A_2 *are square matrices, then* $\sigma_\epsilon(A_1) \cup \sigma_\epsilon(A_2) \subseteq \sigma_\epsilon(A)$. *The equality holds if* $B = 0$; *i.e.* $\sigma_\epsilon(A_1 \oplus A_2) = \sigma_\epsilon(A_1) \cup \sigma_\epsilon(A_2)$.

Proposition 1.2 *Let* $A \in \mathbb{M}_n$. *Then:*

(i) for every $\epsilon > 0$, $\sigma(A) + D(0, \epsilon) \subseteq \sigma_\epsilon(A)$;

(ii) A *is normal (i.e.* $A^*A = AA^*$, *where* $A^* = \overline{A}^T$*) if and only if* $\sigma_\epsilon(A) = \sigma(A) + D(0, \epsilon)$ *for every* $\epsilon > 0$.

Our second motivation concerns the study of fields of values, which is useful in studying and understanding matrices and has many applications in numerical analysis, differential equations, systems theory, etc.; e.g., see [3, 8, 9] and the references cited there. For $A \in \mathbb{M}_n$, the *field of values* of A is defined and denoted by

$$W(A) = \{x^*Ax \ : \ x \in S^1\}.$$

It is known that $\sigma(A) \subseteq W(A)$. Moreover, in the following proposition, we list some important properties of the field of values of matrices. For more information, see [8, 9].

Proposition 1.3 *Let* $A \in \mathbb{M}_n$. *Then the following assertions are true:*

(i) $W(A)$ *is a compact convex set in* $\mathbb{C}$;

(ii) $W(A) \subseteq \mathbb{R}$ *if and only if* A *is Hermitian (i.e.* $A = A^*$*)*;

(iii) $W(A) \subseteq [0, \infty)$ *if and only if* A *is positive semidefinite (i.e.* $A = A^*$ *and* $x^*Ax \geq 0$ *for all* $x \in \mathbb{C}^n$*).*

In this paper, we are going to introduce and study the notions of the pseudofield of values and pseudopolynomial numerical hulls of matrices. For this, in Section 2, we state the relationship between quantum states and the field of values of matrices. Also, for a given $\epsilon > 0$, we introduce the notion of the ϵ-pseudofield of values of matrices and then we investigate some algebraic and geometrical properties of this notion. In Section 3, we introduce and study the notion of the ϵ-pseudopolynomial numerical hull, as a generalization of the ϵ-pseudofield of values of matrices.

2. Quantum states and the field of values

In quantum physics, quantum states are represented by density matrices, i.e. positive semidefinite matrices that have trace one. If a quantum state $P \in \mathbb{M}_n$ has rank one, i.e. $P = xx^*$ for some $x \in \mathbb{C}^n$ with $x^*x = 1$, then P is called a pure quantum state; otherwise, P is said to be a mixed quantum state, which can be written as a convex combination of pure quantum states; see [11]. By these facts, we state the following result for the field of values of matrices.

Proposition 2.1 *Let $A \in \mathbb{M}_n$. Then:*

$$W(A) = \{tr(AP) \ : \ P \in \mathbb{M}_n \textit{ is a pure quantum state}\}.$$

Proof By the fact that $tr(XY) = tr(YX)$ for appropriate X, Y, and also from the details at the beginning of this section, we have

$$\begin{aligned} W(A) &= \{x^*Ax \ : \ x \in S^1\} \\ &= \{tr(x^*Ax) \ : \ x \in S^1\} \\ &= \{tr(A(xx^*)) \ : \ x \in S^1\} \\ &= \{tr(AP) \ : \ P \in \mathbb{M}_n \textit{ is a pure quantum state}\}. \end{aligned}$$

Thus, the proof is complete. □

In view of Proposition 2.1, quantum states are useful in the study of the field of values of matrices, and vice versa. We introduce the notion of the pseudofield of values of matrices, which is related to the field of values of perturbed matrices.

Definition 2.2 *Let $\epsilon > 0$ and $A \in \mathbb{M}_n$. The ϵ-pseudofield of values (pseudofield of values for short) of A is defined and denoted by*

$$W_\epsilon(A) = \{\lambda \in \mathbb{C} : \lambda \in W(A+E) \textit{ for some } E \in \mathbb{M}_n \textit{ with } \|E\| \leq \epsilon\}.$$

Note that in Definition 2.2, the matrix $A + E$, where $\|E\| \leq \epsilon$, is considered as a perturbation of A. In the following theorem, we show that the ϵ-pseudofield of values of a matrix as $A \in \mathbb{M}_n$ coincides with the augmented field of values of A, which is defined, e.g., see [8, p. 103], as $W(A) + D(0, \epsilon)$. Note, by Definition 2.2, that $W_\epsilon(A) = \bigcup_{E \in \mathbb{M}_n, \|E\| \leq \epsilon} W(A+E)$. In the following theorem, we also show that the union can be taken over all the rank-one matrices $E \in \mathbb{M}_n$ with norm at most ϵ.

Theorem 2.3 *Let $A \in \mathbb{M}_n$ and $\epsilon > 0$. Then the following assertions are true:*

(i) $W_\epsilon(A) = W(A) + D(0, \epsilon)$;

(ii) If $n \geq 2$, then $\bigcup_{x,y \in \mathbb{C}^n \setminus \{0\}, \|x\|\|y\| \leq \epsilon} W(A + xy^*) = W_\epsilon(A)$.

Proof To prove (i), let $z \in W_\epsilon(A)$ be given. Then there exist a matrix $E \in \mathbb{M}_n$ and a vector $x \in S^1$ such that $\|E\| \leq \epsilon$ and $z = x^*(A+E)x = x^*Ax + x^*Ex$. Since $|x^*Ex| \leq \epsilon$, $z \in W(A) + D(0, \epsilon)$. Thus, the inclusion $\subseteq$ holds. For the converse, let $z = \lambda + \xi \in W(A) + D(0, \epsilon)$ be such that $\lambda \in W(A)$ and $\xi \in D(0, \epsilon)$. Since

$\|\xi I\| \leq \epsilon$, using Proposition 3.1(*(ii) and (iii)*), we have $z = \lambda + \xi \in W(A) + \xi = W(A + \xi I)$. By Definition 2.2, $z \in W_\epsilon(A)$; this completes the proof of (i).

To prove (ii), since for every rank-one matrix as $E := xy^* \in \mathbb{M}_n$, where $x, y \in \mathbb{C}^n \setminus \{0\}$, $\|E\| = \|x\|\|y\|$, the inclusion $\bigcup_{x,y\in\mathbb{C}^n\setminus\{0\},\|x\|\|y\|\leq\epsilon} W(A + xy^*) \subseteq W_\epsilon(A)$ follows from Definition 2.2. To prove the opposite inclusion, it is enough, by (i), to show that:

$$W(A) + D(0,\epsilon) \subseteq \bigcup_{x,y\in\mathbb{C}^n\setminus\{0\},\|x\|\|y\|\leq\epsilon} W(A + xy^*).$$

For this, let $\lambda \in W(A) + D(0, \epsilon)$ be given. Then there exist a vector $x \in S^1$ and a complex number ξ with $|\xi| \leq \epsilon$ such that $\lambda = x^*Ax + \xi$. If $\xi = 0$, then using the fact that $n \geq 2$, we can find a nonzero vector $y \in \mathbb{C}^n$ such that $\|y\| \leq \epsilon$ and $y^*x = 0 = \xi$. Thus, $\lambda = x^*(A + xy^*)x \in W(A + xy^*)$.

For the case $\xi \neq 0$, by setting $y = \bar{\xi}x$, we see that $\|y\| = |\xi| \leq \epsilon$, $y^*x = \xi$ and $\lambda = x^*(A + xy^*)x \in W(A + xy^*)$. This completes the proof of (ii). Thus, the proof is complete. □

By setting $A = 0 \in \mathbb{M}_n$, where $n \geq 2$, in Theorem 2.3, we have the following result in which we find the union of the field of values of perturbed rank-one matrices.

Corollary 2.4 *Let $n \geq 2$. Then*

$$\bigcup_{x,y\in\mathbb{C}^n\setminus\{0\},\|x\|\|y\|\leq\epsilon} W(xy^*) = D(0,\epsilon).$$

The following example shows that the result in Theorem 2.3(ii) does not hold for the case $n = 1$.

Example 2.5 *Let $A = [a] \in M_1$. Then we have*

$$\bigcup_{e\in\mathbb{C}\setminus\{0\},|e|\leq\epsilon} W(A + [e]) = D(a,\epsilon) \setminus \{a\} \neq D(a,\epsilon) = W_\epsilon(A).$$

3. Generalized field of values of perturbed matrices

In this section, we are going to introduce the notion of pseudopolynomial numerical hulls of matrices as a generalization of the pseudofield of values. The *polynomial numerical hull of order* k, where k is a positive integer, is a set of complex numbers naturally associated with a given $A \in \mathbb{M}_n$, defined and denoted by

$$V^k(A) = \{\lambda \in \mathbb{C} : \ |p(\lambda)| \leq \|p(A)\| \text{ for all } p \in \mathbb{P}_k\},$$

where $\mathbb{P}_k$ is the set of all scalar polynomials of degree k or less. This is a set designed to give more information than the spectrum alone can provide about the behavior of the matrix A under the action of polynomials and other functions; e.g., see [5, 7] and references therein. The sets $V^k(A)$, where $k \geq 1$, are generally called the *polynomial numerical hulls* of A. For the case $k = 1$, $V^k(A)$ reduces to the *field of values* of A; namely, $V^1(A) = W(A)$. This shows that the notion of polynomial numerical hulls is a generalization of the field of values of matrices.

In the following proposition, we list some useful properties of the polynomial numerical hulls of matrices, which will be useful in our discussion.

Proposition 3.1 *([1, 2]); Let $A \in \mathbb{M}_n$ and $1 \leq k \leq n$ be a positive integer. Then the following assertions are true:*

(i) $V^k(A)$ *is a compact set in* $\mathbb{C}$*;*

(ii) $\sigma(A) = V^m(A) \subseteq \cdots \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq \cdots \subseteq V^1(A) = W(A)$, *where* $m \geq n$*;*

(iii) $V^k(\alpha A + \beta I) = \alpha V^k(A) + \beta$, *where* $\alpha, \beta \in \mathbb{C}$*;*

(iv) $V^k(U^*AU) = V^k(A)$, *where* $U \in \mathbb{M}_n$ *is unitary;*

(v) $V^k(A^T) = V^k(A)$ *and* $V^k(A^*) = \overline{V^k(A)} := \{\overline{\lambda} \ : \ \lambda \in V^k(A)\}$*;*

(vi) $V^k(A) = \{z \in \mathbb{C} : (z, z^2, ..., z^k) \in conv(W(A, A^2, ..., A^k))\}$, *where* $conv(.)$ *denotes the convex hull and* $W(A_1, A_2, ..., A_m) := \{(x^*A_1x, x^*A_2x, ..., x^*A_mx) : x \in S^1\}$ *is the joint field of values of* $(A_1, A_2, ..., A_m) \in \mathbb{M}_n^m$.

Now we introduce the notion of pseudopolynomial numerical hulls of square complex matrices, which is related to the polynomial numerical hulls of perturbed matrices.

Definition 3.2 *Let $\epsilon > 0$ and $A \in \mathbb{M}_n$. The ϵ-pseudopolynomial numerical hull (pseudopolynomial numerical hull for short) of order k of A is defined and denoted by*

$$V_\epsilon^k(A) = \{\lambda \in \mathbb{C} : \lambda \in V^k(A+E) \text{ for some } E \in \mathbb{M}_n \text{ with } \|E\| \leq \epsilon\}.$$

The sets $V_\epsilon^k(A)$, where $k \in \mathbb{N}$, are generally called the ϵ-pseudopolynomial numerical hulls of A.

Using Definition 3.2, for every matrix $A \in \mathbb{M}_n$, we have the following useful observation:

$$V_\epsilon^k(A) = \bigcup_{E \in \mathbb{M}_n, \|E\| \leq \epsilon} V^k(A+E). \tag{2}$$

Moreover, we have the following result.

Proposition 3.3 *Let $A \in \mathbb{M}_n$. Then*

$$V^k(A) = \bigcap_{\epsilon > 0} V_\epsilon^k(A).$$

Proof The inclusion $\subseteq$ follows from (2). To prove the converse, let $\lambda \in \bigcap_{\epsilon>0} V_\epsilon^k(A)$ be given. We will show that $\lambda \in V^k(A)$. By Definition 3.2, we have that for every $m \in \mathbb{N}$, there exists a matrix $E_m \in \mathbb{M}_n$ such that $\|E_m\| \leq \dfrac{1}{m}$ and $\lambda \in V^k(A + E_m)$. Now, let $p \in \mathbb{P}_k$ be given. Then $|p(\lambda)| \leq \|p(A + E_m)\|$. Since $E_m \longrightarrow 0$ as $m \longrightarrow \infty$, and $p(.)$ and $\|.\|$ are continuous functions, $|p(\lambda)| \leq \|p(A)\|$. Therefore, $\lambda \in V^k(A)$, and so the proof is complete. □

In the following theorem, we state some basic and essential properties of the pseudopolynomial numerical hulls of matrices. Part (ii) of this theorem shows that the notion of pseudopolynomial numerical hulls is a generalization of the pseudofield of values of matrices.

Theorem 3.4 *Let $\epsilon > 0$ and $A \in \mathbb{M}_n$. Then the following assertions are true:*

(i) $V_\epsilon^k(U^*AU) = V_\epsilon^k(A)$, *where $U \in \mathbb{M}_n$ is unitary;*

(ii) $\sigma_\epsilon(A) = V_\epsilon^m(A) \subseteq \cdots \subseteq V_\epsilon^{k+1}(A) \subseteq V_\epsilon^k(A) \subseteq \cdots \subseteq V_\epsilon^1(A) = W_\epsilon(A)$, *where $m \geq n$;*

(iii) $V_\epsilon^k(\alpha A + \beta I) = \alpha V_{\epsilon/|\alpha|}^k(A) + \beta$, *where $\alpha \neq 0$ and β are complex scalars;*

(iv) $V_\epsilon^k(A)$ *is a nonempty and compact set in $\mathbb{C}$;*

(v) $V_\epsilon^k(A^T) = V_\epsilon^k(A)$ *and* $V_\epsilon^k(A^*) = \overline{V_\epsilon^k(A)}$, *and consequently, if A is Hermitian, then $V_\epsilon^k(A)$ is symmetric with respect to the real axis;*

(vi) $V_\epsilon^k(A) = D(\mu, \epsilon)$ *if and only if $A = \mu I$, where $\mu \in \mathbb{C}$;*

(vii) If $A = A_1 \oplus A_2$, where $A_i \in M_{n_i}$ $(n_1 + n_2 = n)$, then $V_\epsilon^k(A_1) \cup V_\epsilon^k(A_2) \subseteq V_\epsilon^k(A)$. The set equality holds if $k = n$.

Proof The assertions in (i)–(v) follow easily from Proposition 3.1, Definitions 3.2 and 2.2, and relations (1) and (2). To prove (vi), let $V_\epsilon^k(A) = D(\mu, \epsilon)$. By Proposition 1.2$(i)$ and part (ii), we have $\sigma(A) + D(0, \epsilon) \subseteq D(\mu, \epsilon)$, and hence, $\sigma(A) = \{\mu\}$. Therefore, $\sigma_\epsilon(A) = D(\mu, \epsilon)$, and so, by Proposition 1.1(ii), $A = \mu I$. The converse also easily follows from part (ii), Proposition 1.1(ii), and Theorem 2.3(i). Finally, to prove (vii), it is enough to show that $V_\epsilon^k(A_1) \subseteq V_\epsilon^k(A)$. For this, let $z \in V_\epsilon^k(A_1)$ be given. By Definition 3.2, there exists a matrix $E \in M_{n_1}$ such that $\|E\| \leq \epsilon$ and $z \in V^k(A_1 + E)$. Hence, for all $p \in \mathbb{P}_k$, we have:

$$\begin{aligned}
|p(z)| &\leq \|p(A_1 + E)\| \\
&\leq \|p((A_1 + E) \oplus A_2)\| \\
&= \|p((A_1 \oplus A_2) + (E \oplus 0))\| \\
&= \|p(A + E')\|,
\end{aligned}$$

where $E' = E \oplus o$. So, $z \in V^k(A + E')$. Since $\|E'\| = \|E\| \leq \epsilon$, by (2), we see that $z \in V_\epsilon^k(A)$, and hence the inclusion holds. For the case $k = n$, the set equality also holds by part (ii) and Proposition 1.1(iii). The proof is complete. □

The following theorem states one property of $V_\epsilon^n(.)$.

Theorem 3.5 *Let $\epsilon > 0$, and $A, B \in M_n$ be such that $AB = BA$. If A or $A + B$ is normal, then*

$$\sigma_\epsilon(A + B) \subseteq \sigma(A) + \sigma_\epsilon(B).$$

Proof At first, we assume that A is normal, so there exists a unitary matrix $U \in M_n$ such that $U^*AU = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$. The commutativity of A and B implies that $U^*BU = T_1 \oplus \cdots \oplus T_k$, where $T_i \in M_{n_i}$. By

Proposition 1.1((i) and (iii)) and Theorem 3.4((i) and (ii)), we have:

$$\begin{aligned}
\sigma_\epsilon(A+B) &= \sigma_\epsilon(U^*AU+U^*BU)\\
&= \sigma_\epsilon((\lambda_1 I_{n_1}+T_1)\oplus\cdots\oplus(\lambda_k I_{n_k}+T_k))\\
&= \bigcup_{i=1}^{k}\sigma_\epsilon(\lambda_i I+T_i)\\
&= \bigcup_{i=1}^{k}(\lambda_i+\sigma_\epsilon(T_i))\\
&\subseteq \sigma(A)+\sigma_\epsilon(B).
\end{aligned}$$

The result in the case that $A+B$ is normal follows from Proposition 1.2(ii). The proof is thus complete. □

The following example shows that the condition "A or $A+B$ is normal" in Theorem 3.5 is necessary.

Example 3.6 *Let $\epsilon>0$, and $A=B=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Clearly, A and $A+B$ are not normal. By [6, Proposition 3.1] and Proposition 1.1(ii), we have*

$$\sigma_\epsilon(A+B)=2\sigma_{\epsilon/2}(A)=D(0,\sqrt{2\epsilon+\epsilon^2})\nsubseteq D(0,\sqrt{\epsilon+\epsilon^2})=\sigma(A)+\sigma_\epsilon(B).$$

Corollary 3.7 *Let $\epsilon>0$, and let A be a normal matrix such that its spectrum is symmetric with respect to the origin. Then*

$$2\sigma_{\epsilon/2}(Re\,A\oplus iIm\,A)\subseteq\sigma(A)+\sigma_\epsilon(A^*),$$

where $Re\,A=\dfrac{1}{2}(A+A^*)$ and $Im\,A=\frac{1}{2i}(A-A^*)$.

Proof By setting $B:=A^*$ in Theorem 3.5 and using Proposition 1.1(ii), we see that:

$$2\sigma_{\epsilon/2}(Re\,A)\subseteq\sigma(A)+\sigma_\epsilon(A^*).$$

By the same manner as in the above, we also have:

$$2\sigma_{\epsilon/2}(iIm\,A)\subseteq\sigma(A)-\sigma_\epsilon(A^*).$$

Since $\sigma(A)$ is symmetric with respect to the origin and A is normal, $-\sigma_\epsilon(A^*)=\sigma_\epsilon(A^*)$. Hence, the result follows from Proposition 1.1(iii). □

Remark 3.8 *Recently, in [10], the authors defined for $\epsilon>0$ the ϵ-pseudospectral radius of $A\in M_n$ as*

$$r_\epsilon(A)=\sup\{|\lambda|:\lambda\in\sigma_\epsilon(A)\},$$

and they showed that for all commutative matrices A,B:

$$r_\epsilon(A+B)\leqslant r_\epsilon(A)+r_\epsilon(B).$$

Using Theorem 3.5, we obtain the following inequality, which is a refinement of the inequality above:

$$r_\epsilon(A+B) \leqslant r(A) + r_\epsilon(B),$$

where A *or* $A+B$ *is normal,* $AB = BA$*, and* $r(A)$ *is the spectral radius of* A*, i.e.* $r(A) = \max_{z\in\sigma(A)}|z|$.

In the following theorem, we give a connection between the augmented polynomial numerical hulls and the pseudopolynomial numerical hulls of matrices.

Theorem 3.9 *Let* $A \in \mathbb{M}_n$ *and* $\epsilon > 0$*. Then*

$$V^k(A) + D(0,\epsilon) \subseteq V^k_\epsilon(A).$$

The set equality holds if one of the following conditions is satisfied:
(i) *A is normal and $k = n$;*
(ii) *A is arbitrary and $k = 1$;*
(iii) *A is a scalar matrix and k is arbitrary.*

Proof Let $\mu = \lambda + \xi \in V^k(A) + D(0,\epsilon)$ be such that $\lambda \in V^k(A)$ and $\xi \in D(0,\epsilon)$. Since $\|\xi I\| \leq \epsilon$, Proposition 3.1(iii) and relation (2) imply that

$$\mu = \lambda + \xi \in V^k(A + \xi I) \subseteq \bigcup_{\|E\|\leq\epsilon} V^k(A+E) = V^k_\epsilon(A).$$

Thus, the inclusion holds. To investigate the second assertion, if condition (i) or (ii) holds, then the set equality follows from Theorem 3.4(ii), Proposition 1.2(ii), and Theorem 2.3(i). Finally, if condition (iii) holds, then the set equality follows from Theorem 3.4(vi). Thus, the proof is complete. □

The following example illustrates that the set equality in Theorem 3.9 does not hold in general.

Example 3.10 *Let* $\epsilon = 1$ *and* $A = A_1 \oplus A_2 \oplus A_3$, *where* $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, *and* $A_3 = \begin{pmatrix} 2 & i \\ i & -2 \end{pmatrix}$. *By setting* $E = E_1 \oplus E_2 \oplus E_3$, *where* $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *and* $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *we see that* $\|E\| \leq 1$. *Since* A^2 *and* $(A+E)^2$ *are Hermitian, by [1, Theorem 4.2], we obtain that* $V^4(A) = \sigma(A) = \{1, 0, i, \sqrt{3}, -\sqrt{3}\}$ *and* $V^4(A+E) = \sigma(A+E) = \{1, -\frac{1}{2}, i, 2\sqrt{2}, -2\sqrt{2}\}$. *Therefore,* $2\sqrt{2} \in V^4(A+E) \subseteq V^4_1(A)$, *but* $2\sqrt{2} \notin V^4(A) + D(0,1)$. *Thus,* $V^4(A) + D(0,\ 1) \neq V^4_1(A)$.

Let $S \subseteq \mathbb{C}$ be a compact set and k be a positive integer. The polynomially convex hull of degree k of S, e.g., see [4], is defined as

$$pconv_k(S) = \{\lambda \in \mathbb{C} \ : \ |p(\lambda)| \leq \max_{z\in S} |p(z)| \ for\ all\ p \in \mathbb{P}_k\}.$$

It is clear that $S \subseteq pconv_k(S)$. If $S = pconv_k(S)$, then S is said to be a polynomially convex set. Now we state the following proposition.

Proposition 3.11 *Let $A \in \mathbb{M}_n$, $\epsilon > 0$, and S be a compact set in the complex plane such that $S \subseteq V_\epsilon^k(A)$. Then $pconv_k(S) \subseteq V_\epsilon^k(A)$. Consequently, $V_\epsilon^k(A)$ is a polynomially convex set.*

Proof Let $z \in pconv_k(S)$ and $p \in \mathbb{P}_k$ be given. Then $|p(z)| \leq \max_{t \in S} |p(t)| =: |p(z_1)|$ for some $z_1 \in S$. Since $z_1 \in S \subseteq V_\epsilon^k(A)$, Definition 3.2 implies that there exists a matrix $E \in \mathbb{M}_n$ such that $\|E\| \leq \epsilon$ and $|p(z_1)| \leq \|p(A+E)\|$. Thus, $|p(z)| \leq \|p(A+E)\|$, and hence, $z \in V_\epsilon^k(A)$. The second assertion follows from the first assertion and Theorem 3.4(iv). Therefore, the proof is complete. □

Acknowledgment

The authors wish to express their gratitude to the anonymous referees for some suggestions.

References

[1] Davis CH, Li CK, Salemi A. Polynomial numerical hulls of matrices. Linear Algebra Appl 2008; 428: 137-153.

[2] Davis CH, Salemi A. On polynomial numerical hulls of normal matrices. Linear Algebra Appl 2004; 383: 151-161.

[3] Eiermann M. Field of values and iterative methods. Linear Algebra Appl 1993; 180: 167-197.

[4] Faber V, Greenbaum A, Marshall DE. The polynomial numerical hulls of Jordan blocks and related matrices. Linear Algebra Appl 2003; 374: 231-246.

[5] Faber V, Joubert W, Knill M, Manteuffel T. Minimal residual method stronger than polynomial preconditioning. SIAM J Matrix Anal Appl 1996; 17: 707-729.

[6] Gong F, Meyerson O, Meza J, Ward A, Stoiciu M. Explicit bounds for the pseudospectra of various classes of matrices and operators. Involve 2016; 9: 517-540.

[7] Greenbaum A. Generalizations of field of values useful in the study of polynomial functions of a matrix. Linear Algebra Appl 2002; 347: 233-249.

[8] Gustafson KE, Rao DKM. Numerical Range: The Field of Values of Linear Operators and Matrices. New York, NY, USA: Springer, 1997.

[9] Horn RA, Johnson CR. Topics in Matrix Analysis. Cambridge, UK: Cambridge University Press, 1991.

[10] Kumar GK, Lui SH. On some properties of the pseudospectral radius, Electronic J Linear Algebra 2014; 27: 342-353.

[11] Nielsen MA, Chuang IL. Quantum Computing and Quantum Information. Cambridge, UK: Cambridge University Press, 2010.

[12] Trefethen LN, Embree M. Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators. Princeton, NJ, USA: Princeton University Press, 2005.