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A result on the maximal length of consecutive 0 digits in β-expansions

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Abstract: Let $\beta > 1$ be a real number. For any $x \in [0, 1]$, let $r_n(x, \beta)$ be the maximal length of consecutive zero digits in the first n digits of the β-expansion of x. In this note, it is proved that for any $0 < a < b < +\infty$, the set

$$E_{a,b} = \{x \in [0, 1] : \ \liminf_{n\to\infty} \frac{r_n(x, \beta)}{\log_\beta n} = a, \ \ \limsup_{n\to\infty} \frac{r_n(x, \beta)}{\log_\beta n} = b\}$$

has the full Hausdorff dimension.

Key words: β-Expansion, consecutive zero digits, Hausdorff dimension

1. Introduction

For any real number $\beta > 1$, let

$$T_\beta : \ [0, 1] \to [0, 1]$$

be the β-*transformation* defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ means the largest integer no more than ξ. Then for any $x \in [0, 1]$, T_β leads to the following series representation of x:

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots ,$$

where $\varepsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ is said to be the *n-th digit of x* with base β. The infinite digit sequence

$$\varepsilon_1(x, \beta)\varepsilon_2(x, \beta) \cdots \varepsilon_n(x, \beta) \cdots$$

is said to be the *β-expansion of x*.

For $n \geq 1$, we denote by $r_n(x, \beta)$ the maximal length of consecutive 0 digits in $\varepsilon_1(x, \beta) \cdots \varepsilon_n(x, \beta)$, i.e.

$$r_n(x, \beta) = \max\{k \geq 1 : \ \varepsilon_{i+1}(x, \beta) = \cdots = \varepsilon_{i+k}(x, \beta) = 0 \ \text{ for some } \ 0 \leq i \leq n - k\}.$$

Here we agree to define $r_n(x, \beta) = 0$ if there is no 0 digit in $\varepsilon_1(x, \beta) \cdots \varepsilon_n(x, \beta)$. Of course, $r_n(x, \beta)$ is monotonically nondecreasing with respect to n. There are many results about the growth speed of $r_n(x, \beta)$.

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For $\beta = 2$, Erdös and Rényi [3] proved that for Lebesgue almost all $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{r_n(x,2)}{\log_2 n} = 1, \tag{1.1}$$

in 1970. See also [12] for a proof of (1.1). Furthermore, Ma et al. [9] showed that the set of points that violate (1.1) is of Hausdorff dimension one. Recently, Sun and Xu [13] determined the Hausdorff dimension of the set

$$D_{a,b} = \left\{ x \in [0,1] : \quad \liminf_{n\to\infty} \frac{r_n(x,2)}{\log_2 n} = a, \quad \limsup_{n\to\infty} \frac{r_n(x,2)}{\log_2 n} = b \right\}$$

with $0 < a < b < +\infty$. They proved that the exceptional set $D_{a,b}$ has Hausdorff dimension one. In 2016, Li and Wu [8] replaced $\log_2 n$ in (1.1) by a general function $\varphi(n)$, where $\varphi : \ N \to \mathbf{R}^+$ is a monotonically increasing function with $\lim_{n\to\infty} \varphi(n) = +\infty$. They considered the following set:

$$D_\varphi = \left\{ x \in [0,1] : \quad \liminf_{n\to\infty} \frac{r_n(x,2)}{\varphi(n)} = 0, \quad \limsup_{n\to\infty} \frac{r_n(x,2)}{\varphi(n)} = +\infty \right\}.$$

Li and Wu [8] showed that the exceptional set D_φ has Hausdorff dimension 1 if $\limsup_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$; otherwise D_φ has Hausdorff dimension 0. Naturally, for any $\beta > 1$, what is the growth rate of $r_n(x,\beta)$? It is known that for $\beta = 2$, T_β is a finite expanding Markov map. However, when T_β without the Markov property, things become more difficult. Recently, Tong et al. [14] gave the answer to this question as follows.

Theorem 1.1 ([14]) *Let $\beta > 1$ be a real number.*

(i) For Lebesgue almost all $x \in [0,1]$, we have

$$\lim_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = 1.$$

(ii) Let $\alpha > 0$ and

$$E_\alpha = \left\{ x \in [0,1] : \quad \lim_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = \alpha \right\}.$$

Then the set E_α has Hausdorff dimension 1.

In this note, we consider the following kind of exceptional set for $r_n(x,\beta)$. For any $0 < a < b < +\infty$, let

$$E_{a,b} = \left\{ x \in [0,1] : \quad \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = a, \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = b \right\}.$$

Intuitively, the set $E_{a,b}$ is small because it consists of points that cannot satisfy the above law in Theorem 1.1. However, we prove the following dimensional result of the set $E_{a,b}$.

Theorem 1.2 *For any real numbers $0 < a < b < +\infty$, the Hausdorff dimension of the set $E_{a,b}$ is full.*

For other results about $r_n(x,\beta)$, see [2, 4]. Along another direction, when T is the Gauss map inducing continued fraction expansions, a result similar to (1.1) was proved by Wang and Wu [15]. Moreover, they got the Hausdorff dimension of a class of related exceptional sets in [15].

2. Preliminary

In this section, we list some basic properties of β-expansions and give some notations. We write uv for the concatenation of words u and v. In particular, u^i denotes the i times self-concatenation of u for $i \geq 1$. Denote by $|u|$ the length of the word u.

Definition 2.1 *We say that a finite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ or an infinite word $\varepsilon_1\varepsilon_2\cdots$ is β-*admissible*, if there exists $x \in [0,1)$ such that $\varepsilon_i(x,\beta) = \varepsilon_i$ for all $1 \leq i \leq n$ or $i \geq 1$, respectively.*

Let us denote by Σ_β^n the set of admissible words of length n and Σ_β the set of all admissible words of infinite length. We define the lexicographical order $<_{lex}$ between two infinite words as follows:

$$\varepsilon_1\varepsilon_2\cdots <_{lex} \varepsilon_1'\varepsilon_2'\cdots$$

if there exists some integer $k \geq 1$ satisfying $\varepsilon_j = \varepsilon_j'$ for all $1 \leq j < k$ and $\varepsilon_k < \varepsilon_k'$. In fact, we can extend the order $<_{lex}$ to finite words by identifying a finite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ with the infinite word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n0^\infty$ where ξ^∞ means the periodic sequence $\xi\xi\cdots$. Now we define an infinite word $\varepsilon_1^*\varepsilon_2^*\cdots$ from the β-expansions of 1. If there exists an integer $m \geq 1$ such that $\varepsilon_m(1,\beta) \geq 1$ but $\varepsilon_n(1,\beta) = 0$ for all $n > m$, then we write

$$\varepsilon_1^*(1,\beta)\varepsilon_2^*(1,\beta)\cdots = (\varepsilon_1(1,\beta)\cdots(\varepsilon_m(1,\beta)-1))^\infty.$$

Otherwise, we write

$$\varepsilon_1^*(1,\beta)\varepsilon_2^*(1,\beta)\cdots = \varepsilon_1(1,\beta)\varepsilon_2(1,\beta)\cdots.$$

We list some basic properties about admissible words in the following lemma.

Lemma 2.1 ([10, 11]) *(i) An infinite word $\epsilon_1\epsilon_2\cdots \in \Sigma_\beta$ if and only if*

$$\forall k \geq 1, \quad \epsilon_k\epsilon_{k+1}\cdots <_{lex} \varepsilon_1^*(1,\beta)\varepsilon_2^*(1,\beta)\cdots.$$

(ii) For any $x, y \in [0,1]$, $x < y$ if and only if $\varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots <_{lex} \varepsilon_1(y,\beta)\varepsilon_2(y,\beta)\cdots$. Moreover, if $1 < \beta < \beta'$, then

$$\Sigma_\beta \subset \Sigma_{\beta'}.$$

(iii) For any $\beta > 1$,

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \beta^{n+1}/(\beta-1),$$

where $\sharp$ means the number of elements of a finite set.

(iv) An infinite word $\varepsilon_1\varepsilon_2\cdots$ is the β-expansion of 1 for some $\beta > 1$ if and only if for all $k \geq 2$, $\varepsilon_k\varepsilon_{k+1}\cdots <_{lex} \varepsilon_1\varepsilon_2\cdots$.

For any admissible word $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$, define

$$I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) = \{x \in [0,1]: \ \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots\varepsilon_n(x,\beta) = \varepsilon_1\varepsilon_2\cdots\varepsilon_n\},$$

which is called an n-th order *cylinder*. We define the n-th order cylinder containing x for $x \in [0,1]$, denoted by $I_n(x,\beta)$, which is the set of points $y \in [0,1]$ with the property that $\varepsilon_i(y,\beta) = \varepsilon_i(x,\beta)$ for all $1 \leq i \leq n$. We write $|I_n(x,\beta)|$ for the length of $I_n(x,\beta)$. The following basic properties of cylinders are proved in [6] and [7].

Lemma 2.2 **([6, 7])** *The cylinder* $I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)$ *is an interval whose left endpoint is* $\frac{\varepsilon_1}{\beta}+\frac{\varepsilon_2}{\beta^2}+\cdots+\frac{\varepsilon_n}{\beta^n}$ *and*

$$| I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) | \leq \frac{1}{\beta^n}.$$

Here we consider a kind of cylinders with maximal lengths, which are known as full cylinders.

Definition 2.2 *For any* $\varepsilon_1\varepsilon_2\cdots\varepsilon_n \in \Sigma_\beta^n$, *we say that the cylinder* $I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)$ *is full if its length satisfies*

$$| I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n) | = \frac{1}{\beta^n}.$$

The characterization of full cylinders was obtained by Fan and Wang [6] as follows.

Lemma 2.3 **([6])** *Let* $\varepsilon_1\varepsilon_2\cdots\varepsilon_n$ *be an admissible word. The following conditions are equivalent:*

(i) *The cylinder* $I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)$ *is full;*

(ii) $T_\beta^n(I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)) = [0,1)$*;*

(iii) *For any* $w_1w_2\cdots w_m \in \Sigma_\beta^m$, *the concatenation* $\varepsilon_1\varepsilon_2\cdots\varepsilon_n w_1 w_2\cdots w_m$ *is still* β*-admissible.*

We shall make use of the following two lemmas from Bugeaud and Wang [1] to construct new full cylinders and estimate the number of full cylinders.

Lemma 2.4 **([1])** *If* $I_n(\varepsilon_1\varepsilon_2\cdots\varepsilon_n)$ *and* $I_m(w_1w_2\cdots w_m)$ *are two full cylinders, then the concatenation* $I_{n+m}(\varepsilon_1\varepsilon_2\cdots\varepsilon_n w_1 w_2 \cdot$ *is still a full cylinder.*

Lemma 2.5 **([1])** *For any* $n+1$ *consecutive cylinders of order* n, *there is at least one full cylinder.*

3. Proof of Theorem 1.2

3.1. The construction of a Cantor subset of $E_{a,b}$

Let us recall that for any $0 < a < b < +\infty$,

$$E_{a,b} = \left\{ x \in [0,1] : \ \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = a, \ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = b \right\}.$$

The main idea of the proof is to construct a Cantor subset of $E_{a,b}$ denoted by $E^*_{a,b}$ with $\dim_H E^*_{a,b} = 1$. Here we denote by $\dim_H$ the Hausdorff dimension. For convenience, we shall make use of full cylinders repeatedly in the construction of $E^*_{a,b}$. The construction is divided into four steps.

Step 1. Let h be the smallest integer such that $I_{h+1}(10^h)$ is full. Take an integer $N \geq 1$ large enough with the property that

$$\lfloor b\log_\beta m \rfloor \geq h+1 \quad \text{and} \quad \frac{\lfloor m^{b/a} \rfloor - m}{b\log_\beta m} \geq 2,$$

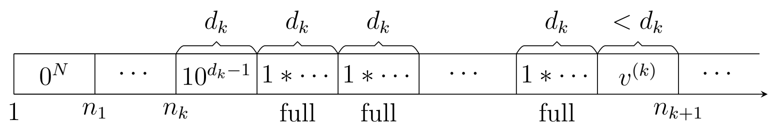

Figure. Construction of Cantor subset.

for all $m \geq N$. Let $n_1 = N$. For any $k \geq 2$, we define n_k and d_{k-1} by the following recursive formulae:

$$n_k = \lfloor n_{k-1}^{\frac{b}{a}} \rfloor, \quad d_{k-1} = \lfloor b \log_\beta n_{k-1} \rfloor.$$

Step 2. For any $k \geq 1$, we set

$$W_k = \{\varepsilon_1 \cdots \varepsilon_{d_k} \in \Sigma_\beta^{d_k} : \quad I_{d_k}(\varepsilon_1 \cdots \varepsilon_{d_k}) \text{ is full with } \varepsilon_1 = 1\}. \tag{3.1}$$

Let

$$t_k = \lfloor \frac{n_{k+1} - n_k}{d_k} \rfloor - 1. \tag{3.2}$$

Set

$$U_k = \{w^{(1)} w^{(2)} \cdots w^{(t_k)} : \quad w^{(j)} \in W_k, \text{ for all } 1 \leq j \leq t_k\}.$$

In other words, U_k consists of finite words that are possible concatenations of any t_k words from W_k.

Step 3. For any $k \geq 1$, we first define a finite word $v^{(k)}$ as follows:

$$v^{(k)} = \begin{cases} 0^{\delta_k}, & \text{if } 0 \leq \delta_k \leq h \\ 10^{\delta_k - 1}, & \text{if } h+1 \leq \delta_k < d_k, \end{cases}$$

where $\delta_k = n_{k+1} - n_k - \lfloor \frac{n_{k+1}-n_k}{d_k} \rfloor d_k$. Next, we define

$$D_k = \{10^{d_k - 1} u^{(k)} v^{(k)} : \quad u^{(k)} \in U_k\}. \tag{3.3}$$

Note that for each word in D_k, the maximal length of consecutive 0 digits is at least $d_k - 1$ and at most $d_k - 1 + h$ by our construction.

Step 4. The desired Cantor set is defined as

$$E_{a,b}^* = \{x \in [0,1] : \quad \varepsilon_1(x,\beta)\varepsilon_2(x,\beta) \cdots = 0^N \sigma^{(1)} \sigma^{(2)} \cdots, \ \sigma^{(k)} \in D_k, \ k \geq 1\},$$

which will be shown to be a subset of $E_{a,b}$ with Hausdorff dimension 1. By the properties of full cylinders, the construction is well defined (see the Figure).

The following results are obtained by direct calculations. For convenience, we list them as a lemma.

Lemma 3.1 *Let $\{n_k\}_{k \geq 1}$ and $\{d_k\}_{k \geq 1}$ be defined as above. Then:*

(1) $\lim\limits_{k\to\infty} \frac{d_k}{n_k} = 0, \quad \lim\limits_{k\to\infty} \frac{\log_\beta n_k}{\log_\beta n_{k+1}} = \frac{a}{b}$;

(2) $\lim\limits_{k\to\infty} \frac{d_k}{\log_\beta n_{k+1}} = a, \quad \lim\limits_{k\to\infty} \frac{d_k}{\log_\beta n_k} = b.$

Proposition 3.1 *$E^*_{a,b}$ is a Cantor subset of $E_{a,b}$.*

Proof We first show

$$a \le \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} \le \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} \le b. \tag{3.4}$$

By the definition of d_i, there exists some $K \ge 1$ such that $d_i > N$ for any $i \ge K$. Then for any $n > n_{K+1}$, there exists some $k \ge K+1$, such that $n_k < n \le n_{k+1}$. We distinguish two cases.

Case 1: If $n_k < n \le n_k + d_k$, then $d_{k-1} - 1 \le r_n(x,\beta) \le \max\{d_k - 1,\ d_{k-1} - 1 + h\}$. Thus, for all k large enough,

$$\frac{d_{k-1}-1}{\log_\beta(n_k+d_k)} \le \frac{r_n(x,\beta)}{\log_\beta n} \le \frac{d_k - 1}{\log_\beta n_k}. \tag{3.5}$$

Case 2: If $n_k + d_k < n \le n_{k+1}$, then we have $d_k - 1 \le r_n(x,\beta) \le d_k - 1 + h$. Thus,

$$\frac{d_k - 1}{\log_\beta n_{k+1}} \le \frac{r_n(x,\beta)}{\log_\beta n} \le \frac{d_k - 1 + h}{\log_\beta n_k}. \tag{3.6}$$

By (3.5), (3.6), and Lemma 3.1, (3.4) holds.

It remains to prove that there exist subsequences $\{m_k\}_{k\ge1}$ and $\{m'_k\}_{k\ge1}$ such that $\lim_{k\to\infty} \frac{r_{m_k}(x,\beta)}{\log_\beta m_k} = a$ and $\lim_{k\to\infty} \frac{r_{m'_k}(x,\beta)}{\log_\beta m'_k} = b$, respectively. Let $m_k = n_{k+1} + d_k$. Then $d_k - 1 \le r_{m_k}(x,\beta) \le d_k - 1 + h$ for all k large enough. Thus, by Lemma 3.1, we have

$$\lim_{k\to\infty} \frac{r_{m_k}(x,\beta)}{\log_\beta m_k} = \lim_{k\to\infty} \frac{d_k}{\log_\beta(n_{k+1}+d_k)} = a.$$

Let $m'_k = n_k + d_k$. Similarly, we have $\lim_{k\to\infty} \frac{r_{m'_k}(x,\beta)}{\log_\beta m'_k} = b$. □

3.2. Hausdorff dimension of $E^*_{a,b}$

Our next goal is to get a lower bound of $\dim_H E^*_{a,b}$. For any $\beta_* < \beta$, we shall prove that $\dim_H E^*_{a,b} \ge \frac{\log \beta_*}{\log \beta}$. We first introduce the following modified mass distribution principle, which is helpful for the estimate of lower bounds of Hausdorff dimensions. The usual mass distribution principle can be found in [5].

Proposition 3.2 (**[1]**) *Let E be a Borel measurable set in $[0,1]$ and μ be a Borel measure with $\mu(E) > 0$. For some $s > 0$, there exist a constant $C > 0$ and an integer M with the property that for any $n > M$ and any n-th order cylinder I_n,*

$$\mu(I_n) \le C \mid I_n \mid^s .$$

Then $\dim_H E \ge s$.

For any $k \ge 1$, let W_k and D_k be defined by (3.1) and (3.3), respectively. We set

$$p_k := \sharp W_k$$

and

$$q_{k+1} := \sharp\{0^N \sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k)},\ \sigma^{(j)} \in D_j,\ 1 \le j \le k\}.$$

The following lemma gives lower bounds of p_k and q_k for k large enough.

Lemma 3.2 *For any $\beta_* < \beta$, there exists an integer $K(\beta_*) \ge 1$ such that for any $k > K(\beta_*)$,*

$$p_k \ge \beta_*^{d_k}$$

and

$$q_k \ge C(\beta_*)\beta_*^{\gamma_k},$$

where $\gamma_k = n_k - n_1 - 2\sum_{j=1}^{k-1} d_j$ and $C(\beta_*)$ is a constant depending only on β_*.

Proof We first estimate p_k. For any $k \ge 1$, write

$$W'_k = \{\varepsilon_1 \cdots \varepsilon_{d_k} \in \Sigma_\beta^{d_k} :\ I_{d_k-h-1}(\varepsilon_{h+2}\cdots\varepsilon_{d_k}) \text{ is full and } \varepsilon_1\cdots\varepsilon_{h+1} = 10^h\}$$

and

$$p'_k := \sharp W'_k.$$

Then $W'_k \subset W_k$ by the properties of full cylinders. Therefore, $p_k \ge p'_k$. From Lemma 2.1 we obtain that the number of admissible words with length $d_k - h - 1$ is at least β^{d_k-h-1}. According to Lemma 2.5, we have

$$p'_k \ge \lfloor \frac{\beta^{d_k-h-1}}{d_k - h} \rfloor$$

for any $k \ge 1$. It is easy to check that there exists $K(\beta_*) > 1$ such that for any $k > K(\beta_*)$,

$$\lfloor \frac{\beta^{d_k-h-1}}{d_k - h} \rfloor \ge \beta_*^{d_k}.$$

Thus,

$$p_k \ge p'_k \ge \beta_*^{d_k}. \tag{3.7}$$

We conclude from (3.7) that

$$\sharp U_k = p_k^{t_k} \ge (\beta_*^{d_k})^{\lfloor \frac{n_{k+1}-n_k}{d_k} \rfloor - 1} \ge \beta_*^{n_{k+1}-n_k-2d_k},$$

and hence that

$$\sharp D_k = \sharp U_k \ge \beta_*^{n_{k+1}-n_k-2d_k},$$

for any $k > K(\beta_*)$. Thus, there exists a constant $C(\beta_*) > 0$ depending only on β_* such that

$$q_k = \prod_{j=1}^{k-1} \sharp D_j \ge C(\beta_*)\beta_*^{\gamma_k},$$

where $\gamma_k = n_k - n_1 - 2\sum_{j=1}^{k-1} d_j$. □

Proposition 3.3 *The set* $E^*_{a,b}$ *has Hausdorff dimension 1.*

Proof It suffices to show that $\dim_H E^*_{a,b} \geq \frac{\log\beta_*}{\log\beta}$ for any $\beta_* < \beta$. We first define a probability measure μ on $E^*_{a,b}$ by induction. Set $\mu[0,1] = 1$ and $\mu(I_i(0^i)) = 1$, for all $1 \leq i \leq N$. For any $k \geq 1$ and any $\sigma^{(j)} \in D_j$, $1 \leq j \leq k$, we define

$$\mu(I_{n_{k+1}}(0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k)}) = \frac{\mu(I_{n_k}(0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k-1)}))}{\sharp D_k}. \tag{3.8}$$

Now we define $\mu(I_n(x,\beta))$ for any $n_k \leq n < n_{k+1}$ and any $x \in E^*_{a,b}$. Let

$$\mu(I_n(x,\beta)) = \Sigma\mu(I_{n_{k+1}}(\xi)),$$

where the sum is taken over all $\xi = 0^N\sigma^{(1)}\sigma^{(2)}\cdots\sigma^{(k)}$ with $I_{n_{k+1}}(\xi) \subseteq I_n(x,\beta)$ and $\sigma^{(j)} \in D_j$ for $1 \leq j \leq k$. Then we can extend μ to a Borel probability measure uniquely on $E^*_{a,b}$ by Kolmogorov's consistency theorem. By (3.8) and Lemma 3.2, we have

$$\mu(I_{n_k}(x,\beta)) = \frac{1}{q_k} \leq C^{-1}(\beta_*)\beta_*^{-\gamma_k},$$

for all $k > K(\beta_*)$, where $K(\beta_*)$ is the integer defined as in Lemma 3.2. For any $n_k \leq n < n_{k+1}$ with $k > K(\beta_*)$, either there exists integer l such that

$$n_k + ld_k \leq n < n_k + (l+1)d_k, \quad 0 \leq l \leq t_k$$

or

$$n_k + (t_k+1)d_k \leq n < n_{k+1},$$

where t_k is defined in (3.2).

Next we distinguish three cases.

Case 1: $n_k \leq n < n_k + 2d_k$. Then

$$\mu(I_n(x,\beta)) \leq \mu(I_{n_k}(x,\beta)) \leq C^{-1}(\beta_*)\beta_*^{-\gamma_k}.$$

By the definition of a full cylinder, we have

$$|\ I_n(x,\beta)\ | \geq |\ I_{n_k+2d_k}(x,\beta)\ | = \beta^{-(n_k+2d_k)}.$$

Combing the two inequalities above, we have

$$\frac{\log\mu(I_n(x,\beta))}{\log|\ I_n(x,\beta)\ |} \geq \frac{\log\beta_*^{-\gamma_k} - \log C(\beta_*)}{\log\beta^{-(n_k+2d_k)}}.$$

It follows that

$$\lim_{k\to\infty}\frac{\log\beta_*^{-\gamma_k}}{\log\beta^{-(n_k+2d_k)}} = \lim_{k\to\infty}\frac{(n_k - n_1 - 2\sum_{j=1}^{k-1} d_j)\log\beta_*}{(n_k+2d_k)\log\beta} = \frac{\log\beta_*}{\log\beta} \tag{3.9}$$

by Lemma 3.1.

Case 2: $n_k + ld_k \leq n < n_k + (l+1)d_k$ for some $2 \leq l \leq t_k$. Then

$$\begin{aligned}\mu(I_n(x,\beta)) &\leq \mu(I_{n_k+ld_k}(x,\beta)) \\ &= \mu(I_{n_k}(x,\beta)) \cdot p_k^{-(l-1)} \\ &\leq C^{-1}(\beta_*)\beta_*^{-\gamma_k} \cdot \beta_*^{-d_k(l-1)}\end{aligned}$$

by Lemma 3.2 and (3.8). By the definition of a full cylinder and Lemma 2.4, we have

$$\mid I_n(x,\beta) \mid \geq \mid I_{n_k+(l+1)d_k}(x,\beta) \mid = \beta^{-(n_k+(l+1)d_k)}.$$

Hence,

$$\frac{\log \mu(I_n(x,\beta))}{\log \mid I_n(x,\beta) \mid} \geq \frac{\log \beta_*^{-\gamma_k - d_k(l-1)} - \log C(\beta_*)}{\log \beta^{-(n_k+(l+1)d_k)}}.$$

Similarly,

$$\lim_{k\to\infty} \frac{\log \beta_*^{-\gamma_k - d_k(l-1)}}{\log \beta^{-(n_k+(l+1)d_k)}} = \frac{\log \beta_*}{\log \beta}. \tag{3.10}$$

Case 3: $n_k + (t_k+1)d_k \leq n < n_{k+1}$. Note that

$$\mu(I_n(x,\beta)) = \mu(I_{n_{k+1}}(x,\beta)) \leq C^{-1}(\beta_*)\beta_*^{-\gamma_{k+1}}$$

and

$$\mid I_n(x,\beta) \mid \geq \mid I_{n_{k+1}}(x,\beta) \mid = \beta^{-n_{k+1}}.$$

Then

$$\frac{\log \mu(I_n(x,\beta))}{\log \mid I_n(x,\beta) \mid} \geq \frac{\log \beta_*^{-\gamma_{k+1}} - \log C(\beta_*)}{\log \beta^{-n_{k+1}}} \quad \text{and} \quad \lim_{k\to\infty} \frac{\log \beta_*^{-\gamma_{k+1}}}{\log \beta^{-n_{k+1}}} = \frac{\log \beta_*}{\log \beta}. \tag{3.11}$$

By (3.9), (3.10), and (3.11), it follows that for any $\varepsilon > 0$, there exists an integer K' such that for any $n \geq n_{K'}$ and any $x \in E_{a,b}^*$

$$\mu(I_n(x,\beta)) \leq \mid I_n(x,\beta) \mid^{\log \beta_* / \log \beta - \varepsilon}.$$

Using Proposition 3.2, we conclude that

$$\dim_H E_{a,b}^* \geq \log \beta_* / \log \beta - \varepsilon.$$

Since ε is arbitrary, it follows that $\dim_H E_{a,b}^* \geq \log \beta_* / \log \beta$. □

Proof [Proof of Theorem 1.2] With the help of Proposition 3.1 and Proposition 3.3, the conclusion is obtained immediately. □

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