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Harmonic quadrangle in isotropic plane

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Abstract: The concept of the harmonic quadrangle and the associated Brocard points are introduced and investigated in the isotropic plane by employing suitable analytic methods.

Key words: Isotropic plane, cyclic quadrangle, harmonic quadrangle

1. Introduction

The isotropic plane is a real projective metric plane whose absolute figure is a pair consisting of an *absolute point* Ω and an *absolute line* ω incident to it. If $T = (x_0 : x_1 : x_2)$ denotes any point in the plane presented in homogeneous coordinates then usually a projective coordinate system where $\Omega = (0 : 1 : 0)$ and the line ω with the equation $x_2 = 0$ is chosen.

Isotropic points are the points incident with the absolute line ω and the *isotropic lines* are the lines passing through the absolute point Ω.

Metric quantities and all the notions related to the geometry of the isotropic plane can be found in [8] and [7]. Now we recall a few facts that will be used further on, wherein we assume that $x = \dfrac{x_0}{x_2}$ and $y = \dfrac{x_1}{x_2}$.

Two lines are *parallel* if they have the same isotropic point, and two points are *parallel* if they are incident with the same isotropic line.

For $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$, two nonparallel points, a *distance* between them is defined as $d(T_1, T_2) := x_2 - x_1$. In the case of parallel points $T_1 = (x, y_1)$ and $T_2 = (x, y_2)$, a *span* is defined by $s(T_1, T_2) := y_2 - y_1$. Both quantities are directed.

Two nonisotropic lines p_1 and p_2 in the isotropic plane can be given by $y = k_i x + l_i, \quad k_i, l_i \in \mathbb{R}, i = 1, 2$, labeled by $p_i = (k_i, l_i), \quad i = 1, 2$ in line coordinates. Therefore, the angle formed by p_1 and p_2 is defined by $\varphi = \angle(p_1, p_2) := k_2 - k_1$, being directed as well. Any two points $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$ have the midpoint $M = \left(\dfrac{1}{2}(x_1 + x_2), \dfrac{1}{2}(y_1 + y_2)\right)$ and any two lines with the equations $y = k_i x + l_i \quad (i = 1, 2)$ have the bisector with the equation $y = \dfrac{1}{2}(k_1 + k_2)x + \dfrac{1}{2}(l_1 + l_2)$.

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A triangle in the isotropic plane is called *allowable* if none of its sides are isotropic (see [4]).

The classification of conics in the isotropic plane can be found in [1] and [7]. Recall that the *circle* in the isotropic plane is the conic touching the absolute line ω at the absolute point Ω. The equation of such a circle is given by $y = ux^2 + vx + w, \quad u \neq 0, \quad u, v, w \in \mathbb{R}$.

As the principle of duality is valid in the projective plane, it is preserved in the isotropic plane as well.

2. Cyclic quadrangle in isotropic plane

The geometry of the cyclic quadrangle in the isotropic plane appeared first in [9]. The diagonal triangle and diagonal points were introduced, and several properties concerning them were discussed.

Let $ABCD$ be the cyclic quadrangle with

$$y = x^2 \tag{1}$$

as its circumscribed circle ([9], p. 267). Choosing

$$A = (a, a^2), B = (b, b^2), C = (c, c^2), D = (d, d^2), \tag{2}$$

with a, b, c, d being mutually different real numbers, where $a < b < c < d$, the next lemma is obtained.

Lemma 1 *([9], p. 267) For any cyclic quadrangle $ABCD$ there exist four distinct real numbers a, b, c, d such that, in the defined canonical affine coordinate system, the vertices have the form (2), the circumscribed circle has the equation (1), and the sides are given by*

$$\begin{aligned} &AB \ldots y = (a+b)x - ab, \qquad && DA \ldots y = (a+d)x - ad, \\ &BC \ldots y = (b+c)x - bc, && AC \ldots y = (a+c)x - ac, \\ &CD \ldots y = (c+d)x - cd, && BD \ldots y = (b+d)x - bd. \end{aligned} \tag{3}$$

Tangents $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ of the circle (1) at the points (2) are of the form

$$\begin{aligned} &\mathcal{A} \ldots y = 2ax - a^2, \qquad && \mathcal{B} \ldots y = 2bx - b^2, \\ &\mathcal{C} \ldots y = 2cx - c^2, && \mathcal{D} \ldots y = 2dx - d^2. \end{aligned} \tag{4}$$

The points of intersection of the tangents in (4) are

$$\begin{aligned} &T_{AB} = \mathcal{A} \cap \mathcal{B} = (\tfrac{a+b}{2}, ab), \qquad && T_{BC} = \mathcal{B} \cap \mathcal{C} = (\tfrac{b+c}{2}, bc), \\ &T_{AC} = \mathcal{A} \cap \mathcal{C} = (\tfrac{a+c}{2}, ac), && T_{BD} = \mathcal{B} \cap \mathcal{D} = (\tfrac{b+d}{2}, bd), \\ &T_{AD} = \mathcal{A} \cap \mathcal{D} = (\tfrac{a+d}{2}, ad), && T_{CD} = \mathcal{C} \cap \mathcal{D} = (\tfrac{c+d}{2}, cd). \end{aligned} \tag{5}$$

The diagonal triangle of the cyclic quadrangle is formed by the intersection points of the opposite sides of the quadrangle: $U = AC \cap BD$, $V = AB \cap CD$, and $W = AD \cap BC$. An allowable triangle introduced in [4] concerns each triangle whose sides are nonisotropic lines. According to [9] the *allowable cyclic quadrangle* is the cyclic quadrangle having the allowable diagonal triangle. Hence:

Lemma 2 *The diagonal points U, V, W of the allowable cyclic quadrangle $ABCD$ are of the form*

$$U = \left(\frac{ac - bd}{a + c - b - d}, \frac{ac(b + d) - bd(a + c)}{a + c - b - d}\right),$$
$$V = \left(\frac{ab - cd}{a + b - c - d}, \frac{ab(c + d) - cd(a + b)}{a + b - c - d}\right), \tag{6}$$
$$W = \left(\frac{ad - bc}{a + d - b - c}, \frac{ad(b + c) - bc(a + d)}{a + d - b - c}\right),$$

and the sides of the diagonal triangle are given with

$$UV \ldots y = \frac{2(ad - bc)}{a + d - b - c}x - \frac{ad(b + c) - bc(a + d)}{a + d - b - c},$$
$$UW \ldots y = \frac{2(ab - cd)}{a + b - c - d}x - \frac{ab(c + d) - cd(a + b)}{a + b - c - d}, \tag{7}$$
$$VW \ldots y = \frac{2(ac - bd)}{a + c - b - d}x - \frac{ac(b + d) - bd(a + c)}{a + c - b - d},$$

where $a + c - b - d \neq 0, \quad a + b - c - d \neq 0, \quad a + d - b - c \neq 0$.

Note 1 *Conditions* $a + c - b - d \neq 0, \quad a + b - c - d \neq 0$, *and* $a + d - b - c \neq 0$ *are the conditions for the cyclic quadrangle* $ABCD$ *to be allowable.*

3. On the harmonic quadrangle in the isotropic plane

In this section we investigate the cyclic quadrangle with a special property.

Theorem 1 *Let $ABCD$ be an allowable cyclic quadrangle with vertices given by (2), sides by (3), and tangents of its circumscribed circle (1) at its vertices given by (4). These are the equivalent statements:*

1. *the point $T_{AC} = \mathcal{A} \cap \mathcal{C}$ is incident with the diagonal BD;*
2. *the point $T_{BD} = \mathcal{B} \cap \mathcal{D}$ is incident with the diagonal AC;*
3. *the equality*
$$d(A, B) \cdot d(C, D) = -d(B, C) \cdot d(D, A) \tag{8}$$
holds;
4. *the equality*
$$2(ac + bd) = (a + c)(b + d) \tag{9}$$
holds.

Proof Let us first prove the equivalence of statements 1 and 4.

The point $T_{AC} = (\frac{a+c}{2}, ac)$ is obviously incident with the tangents $\mathcal{A}$ and $\mathcal{C}$ from (4), and therefore $T_{AC} = \mathcal{A} \cap \mathcal{C}$. On the other hand, T_{AC} is incident with the line BD from (3) providing

$$ac = (b+d)\frac{a+c}{2} - bd,$$

being statement 4.

The equivalence of statements 2 and 4 can be proved in an analogous way. The following equality,

$$d(A,B) \cdot d(C,D) - d(B,C) \cdot d(A,D) = (b-a)(d-c) - (c-b)(d-a) = 2(ac+bd) - (a+c)(b+d),$$

proves the equivalency between 3 and 4. □

A cyclic quadrangle will be referred to as a *harmonic quadrangle* if it satisfies one and hence all of the equivalent conditions presented in Theorem 1.

Since properties 1–3 have completely geometrical sense, property 4 does not depend on the choice of the affine coordinate system.

Choosing the y-axis to be incident with the diagonal point U, because of (6), $ac = bd$ follows. Since $ac < 0$ and $bd < 0$, we can use the notation

$$ac = bd = -k^2. \tag{10}$$

Thus, the diagonal point U turns into

$$U = (0, k^2). \tag{11}$$

Statement 4 yields

$$(a+c)(b+d) = -4k^2. \tag{12}$$

Next, we will show that

$$\frac{(a-c)^2(b-d)^2}{(a+c-b-d)^2} = 4k^2, \tag{13}$$

i.e.

$$(a-c)^2(b-d)^2 - 4k^2(a+c-b-d)^2 = 0.$$

Indeed, the left side of the equality given above, owing to (10) and (12), is equal to

$$\begin{aligned}
&[(a+c)^2 - 4ac][(b+d)^2 - 4bd] - 4k^2[(a+c)^2 + (b+d)^2 - 2(a+c)(b+d)] \\
&= [(a+c)^2 + 4k^2][(b+d)^2 + 4k^2] - 4k^2[(a+c)^2 + (b+d)^2 + 8k^2] \\
&= (a+c)^2(b+d)^2 - 16k^4 = 0.
\end{aligned}$$

Hence,

$$\frac{(a-c)(b-d)}{a+c-b-d} = \pm 2k.$$

As the fraction of the left side is a negative real number, we can choose k being $k > 0$, i.e. let

$$\frac{(a-c)(b-d)}{a+c-b-d} = -2k. \tag{14}$$

Because of (10), a numerator of the fraction included in a constant term of the equation (7) of the line VW amounts to

$$k^2(b+d-a-c),$$

and hence this constant term equals $-k^2$. Therefore, the line VW is given by $y=-k^2$.

From (5) we get

$$T_{AC}=\left(\frac{a+c}{2},-k^2\right),\quad T_{BD}=\left(\frac{b+d}{2},-k^2\right).$$

There are two more valid identities:

$$\begin{aligned}2(a-b)(c-d)&=(a-c)(b-d),\\ 2(a-d)(b-c)&=(a-c)(b-d).\end{aligned}\tag{15}$$

Let us consider for example the first identity in (15). By using (10) and (12) we get

$$\begin{aligned}2(a-b)(c-d)&=2(ac-ad-bc+bd)=-4k^2-2(ad+bc)\\ &=(a+c)(b+d)-2(ad+bc)=(a-c)(b-d).\end{aligned}$$

Furthermore, by adding the identities in (15) and dividing the result by 2, the equality $(a-c)(b-d)=(a-b)(c-d)+(a-d)(b-c)$ is obtained, being Ptolemy's theorem.

Due to the above discussion, such a harmonic quadrangle is said to be in a *standard position* or it is a *standard harmonic quadrangle* (see Figure 1). Every harmonic quadrangle can be transformed into one in standard position by means of an isotropic transformation. In order to prove geometric facts for each harmonic quadrangle, it is sufficient to give a proof for the standard harmonic quadrangle.

The diagonal points from (6) turn into

$$U=(0,k^2),V=\left(\frac{ab-cd}{a+b-c-d},-k^2\right),W=\left(\frac{ad-bc}{a+d-b-c},-k^2\right).\tag{16}$$

4. Properties of the harmonic quadrangle in the isotropic plane

In this section, we will prove several theorems dealing with the properties of the harmonic quadrangle. For the Euclidean version of the next theorem, see [6].

Theorem 2 *Let $ABCD$ be a harmonic quadrangle, and lines $\mathcal{A},\mathcal{B},\mathcal{C}$, and $\mathcal{D}$ the tangents to the circle (1) at the points A,B,C, and D, respectively. Assume that M_{AC} and M_{BD} are the midpoints of the diagonals AC and BD, respectively. Then the points M_{AC},B,D, and $T_{BD}=\mathcal{B}\cap\mathcal{D}$ are incident with a circle. The same is valid for the points M_{BD},A,C, and $T_{AC}=\mathcal{A}\cap\mathcal{C}$.*

Proof Let us prove the theorem for the points M_{AC},B,D, and $T_{BD}=\mathcal{B}\cap\mathcal{D}$.

Points B,D, and T_{BD} are incident with a circle of the form

$$y=2x^2-(b+d)x+bd.\tag{17}$$

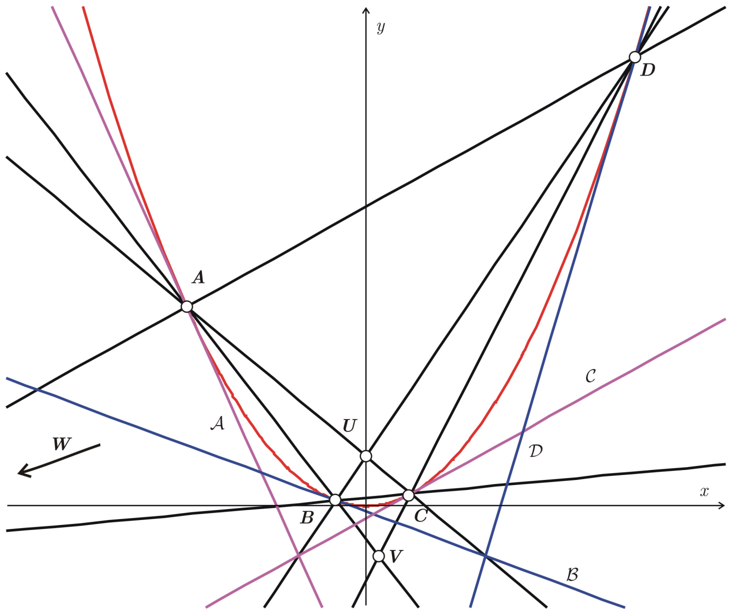

Figure 1. The harmonic quadrangle.

Applying the coordinates of $M_{AC} = (\frac{a+c}{2}, \frac{a^2+c^2}{2})$ in the equation given above, we obtain

$$\frac{a^2+c^2}{2} = \frac{(a+c)^2}{2} - \frac{1}{2}(a+c)(b+d) + bd,$$

i.e.

$$2(ac+bd) = (a+c)(b+d),$$

being the condition (9) for the cyclic quadrangle to be harmonic. Hence, M_{AC} lies on the circle (17). Because of symmetry on the real numbers a, b, c, and d, the same is valid for M_{BD}, A, C, and $T_{AC} = \mathcal{A} \cap \mathcal{C}$, whose circumscribed circle is of the form

$$y = 2x^2 - (a+c)x + ac. \tag{18}$$

□

Note 2 *Circles (17) and (18) from Theorem 2 intersect in* $(0, -k^2)$, *the point parallel to the diagonal point* U *and incident to the line* VW.

Theorem 3 *Let* $ABCD$ *be a harmonic quadrangle, and* U_{AB}, U_{BC}, U_{CD}, U_{DA} *be the intersections of the isotropic line through* U *with the sides* AB, BC, CD, DA, *respectively. Then the following equalities hold:*

$$\frac{s(U,U_{AB})}{d(A,B)} = \frac{s(U,U_{BC})}{d(B,C)} = \frac{s(U,U_{CD})}{d(C,D)} = \frac{s(U,U_{DA})}{d(D,A)}. \tag{19}$$

Proof The point U_{AB} has the coordinates $(0, -ab)$ and therefore $s(U, U_{AB}) = -k^2 - ab$. Let us prove

$$\frac{s(U,U_{AB})}{d(A,B)} = \frac{s(U,U_{BC})}{d(B,C)}.$$

It holds precisely when

$$\frac{k^2+ab}{a-b}=\frac{k^2+bc}{b-c}$$

is valid, i.e.

$$k^2(2b-a-c)+(a+c)b^2-2abc=0.$$

The latter equality holds if and only if

$$4bk^2-k^2(a+c)+b^2(a+c)=0,$$

being, after inserting $4k^2=-(a+c)(b+d)$, equivalent to

$$-b(b+d)-k^2+b^2=0,$$

i.e.

$$k^2+bd=0.$$

This completes the proof. □

Theorem 4 *Let $ABCD$ be a harmonic quadrangle and let the lines $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ be incident with the vertices A,B,C,D, respectively, and form equal angles with the sides AB,BC,CD,DA, respectively. The lines $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ form a harmonic quadrangle as well.*

Proof The lines

$$\begin{aligned}
\tilde{a} \quad &\ldots \quad y=(a+b-h)x+a(h-b)\\
\tilde{b} \quad &\ldots \quad y=(b+c-h)x+b(h-c)\\
\tilde{c} \quad &\ldots \quad y=(c+d-h)x+c(h-d)\\
\tilde{d} \quad &\ldots \quad y=(d+a-h)x+d(h-a)
\end{aligned}$$

fulfill the condition of the theorem since

$$\angle(\tilde{a},AB)=\angle(\tilde{b},BC)=\angle(\tilde{c},CD)=\angle(\tilde{d},DA)=h$$

while $(a+b-h)a+a(h-b)=a^2$ (analogously for $\tilde{b},\tilde{c},\tilde{d}$). Denoting by $\tilde{A}=\tilde{d}\cap\tilde{a}$, $\tilde{B}=\tilde{a}\cap\tilde{b}$, $\tilde{C}=\tilde{b}\cap\tilde{c}$, and $\tilde{D}=\tilde{c}\cap\tilde{d}$, the accuracy of the following equalities is obvious:

$$\begin{aligned}
\tilde{A}&=\left(a+\frac{a-d}{d-b}h,\quad a^2+\frac{(a-d)(a+b)}{d-b}h-\frac{a-d}{d-b}h^2\right),\\
\tilde{B}&=\left(b+\frac{b-a}{a-c}h,\quad b^2+\frac{(b-a)(b+c)}{a-c}h-\frac{b-a}{a-c}h^2\right),\\
\tilde{C}&=\left(c+\frac{c-b}{b-d}h,\quad c^2+\frac{(c-b)(c+d)}{b-d}h-\frac{c-b}{b-d}h^2\right),\\
\tilde{D}&=\left(d+\frac{d-c}{c-a}h,\quad d^2+\frac{(d-c)(d+a)}{c-a}h-\frac{d-c}{c-a}h^2\right).
\end{aligned}$$

Next, some computing shows that the points $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are incident with the circle

$$(h+2k)y = 2khx^2 - h^2x - hk^2. \tag{20}$$

Indeed, for the point $\tilde{A}$:

$$\begin{aligned}
&(h+2k)y - 2khx^2 + h^2x + hk^2\\
&= (h+2k)\left[a^2 - \frac{(a+b)(a-d)}{b-d}h + \frac{a-d}{b-d}h^2\right] - 2k\left[a^2 - 2a\frac{a-d}{b-d}h + \frac{(a-d)^2}{(b-d)^2}h^2\right] + h^2\left[a - \frac{a-d}{b-d}h\right] + hk^2\\
&= a^2h - \frac{(a+b)(a-d)}{b-d}h^2 - 2\frac{(a+b)(a-d)}{b-d}hk + 2\frac{a-d}{b-d}h^2k + 4a\frac{a-d}{b-d}hk - 2\frac{(a-d)^2}{(b-d)^2}h^2k + ah^2 + hk^2\\
&= a^2h + \frac{bd-a^2}{b-d}h^2 + \frac{2hk}{b-d}\left(a^2 - ab - ad + bd\right) - 2\frac{(a-b)(a-d)}{(b-d)^2}h^2k + hk^2\\
&= a^2h + \frac{bd-a^2}{b-d}h^2 + \frac{2hk}{b-d}\left(a^2 - ab - ad + ac\right) - 2\frac{a^2 - ab - ad + ac}{(b-d)^2}h^2k + hk^2\\
&= a^2h + \frac{bd-a^2}{b-d}h^2 + 2a(a-c)hk\frac{a-b+c-d}{(a-c)(b-d)} - 2ah^2k\frac{a-c}{b-d}\cdot\frac{a-b+c-d}{(a-c)(b-d)} + hk^2\\
&= a^2h + \frac{bd-a^2}{b-d}h^2 - a(a-c)h + ah^2\frac{a-c}{b-d} + hk^2 = ach + hk^2 + \frac{bd-ac}{b-d}h^2 = 0.
\end{aligned}$$

Due to $(a-b)(c-d) = ab + cd + 2k^2$, within the following calculation we get

$$\begin{aligned}
d(\tilde{A}, \tilde{B}) &= b + \frac{b-a}{a-c}h - a - \frac{a-d}{d-b}h = b - a - \frac{a^2 + b^2 - 2ab + ab + cd - ac - bd}{(a-c)(d-b)}h\\
&= b - a - \frac{(a-b)^2 + (a-b)(c-d)}{(a-c)(d-b)}h = b - a + \frac{(a-b)(a-b+c-d)}{(a-c)(b-d)}h\\
&= b - a + \frac{(a-b)}{-2k}h = (b-a)(1 + \frac{h}{2k}).
\end{aligned}$$

Therefore,

$$d(\tilde{A}, \tilde{B}) \cdot d(\tilde{C}, \tilde{D}) = (b-a)(d-c)(1 + \frac{h}{2k})^2.$$

On the other hand,

$$d(\tilde{B}, \tilde{C}) \cdot d(\tilde{D}, \tilde{A}) = (c-b)(a-d)(1 + \frac{h}{2k})^2.$$

Since $(a-b)(c-d) = (a-d)(b-c)$, the equality $d(\tilde{A}, \tilde{B}) \cdot d(\tilde{C}, \tilde{D}) = -d(\tilde{B}, \tilde{C}) \cdot d(\tilde{D}, \tilde{A})$ is fullfiled by the points $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and the claim of the theorem is proved. □

The Euclidean case of the theorem given above can be found in [2].

Corollary 1 *For the harmonic quadrangles $ABCD$ and $\tilde{A}\tilde{B}\tilde{C}\tilde{D}$, the following equalities are applicable:*

$$\frac{d(\tilde{A}, \tilde{B})}{d(A, B)} = \frac{d(\tilde{B}, \tilde{C})}{d(B, C)} = \frac{d(\tilde{C}, \tilde{D})}{d(C, D)} = \frac{d(\tilde{D}, \tilde{A})}{d(D, A)} = 1 + \frac{h}{2k}.$$

Note that for $h = -2k$ all four points $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ coincide with one point, say P_1, having the coordinates

$$P_1 = \left(k, 3k^2\right). \tag{21}$$

The point P_1, called the first *Brocard point* of the quadrangle $ABCD$, is the point whose connection lines with the vertices A, B, C, D form the equal angles with the sides AB, BC, CD, and DA, respectively. Similarly, the second Brocard point P_2 is defined as the point such that its connection lines with the vertices A, B, C, D form the equal angles with sides AD, DC, CB, and BA, respectively. Observations similar to those in the proof of Theorem 4 and Corollary 1 result in

$$P_2 = \left(-k, 3k^2\right). \tag{22}$$

Some nice geometric properties of the Brocard points are described in the following two theorems and depicted in Figure 2.

Theorem 5 *Let $ABCD$ be a harmonic quadrangle and M_{AC}, M_{BD} be midpoints of the line segments $\overline{AC}$, $\overline{BD}$, respectively. Two Brocard points P_1 and P_2, diagonal point U, and two midpoints M_{AC} and M_{BD} lie on a circle. Furthermore, $d(P_1, U) = d(U, P_2)$ holds and the line P_1P_2 is parallel to the line VW.*

Proof The points P_1, P_2, and U with the coordinates given by (21), (22), and (16) apparently lie on a circle with the equation

$$y = 2x^2 + k^2.$$

It remains to prove that M_{AC} and M_{BD} are incident with the same circle. For example, that is true for the point $M_{AC} = (\frac{a+c}{2}, \frac{a^2+c^2}{2})$ because of

$$2 \cdot (\frac{a+c}{2})^2 + k^2 = 2\frac{a^2 - 2k^2 + c^2}{4} + k^2 = \frac{a^2+c^2}{2}.$$

The second statement from Theorem 5 holds since $d(P_1, U) = -k = d(U, P_2)$.

Lines P_1P_2 and VW have the equations

$$y = 3k^2, \quad y = -k^2,$$

respectively, and therefore are parallel. □

Let us now prove that the line $M_{AC}M_{BD}$ has the equation $y = (a+b+c+d)x + 3k^2$. Indeed,

$$\begin{aligned} 2y - 2(a+b+c+d)x - 6k^2 &= a^2 + c^2 - (a+c)(a+b+c+d) - 6k^2 \\ &= -2ac - (a+c)(b+d) - 6k^2 = 2k^2 + 4k^2 - 6k^2 = 0. \end{aligned}$$

The line $M_{AC}M_{BD}$ passes through the point $U' = (0, 3k^2)$, which also lies on the line P_1P_2.

Theorem 6 *Let $ABCD$ be a harmonic quadrangle, M_{AC} and M_{BD} midpoints of the line segments $\overline{AC}$ and $\overline{BD}$ respectively, and P_1 and P_2 two Brocard points of $ABCD$. Then $P_1 = WM_{AC} \cap VM_{BD}$ and $P_2 = VM_{AC} \cap WM_{BD}$.*

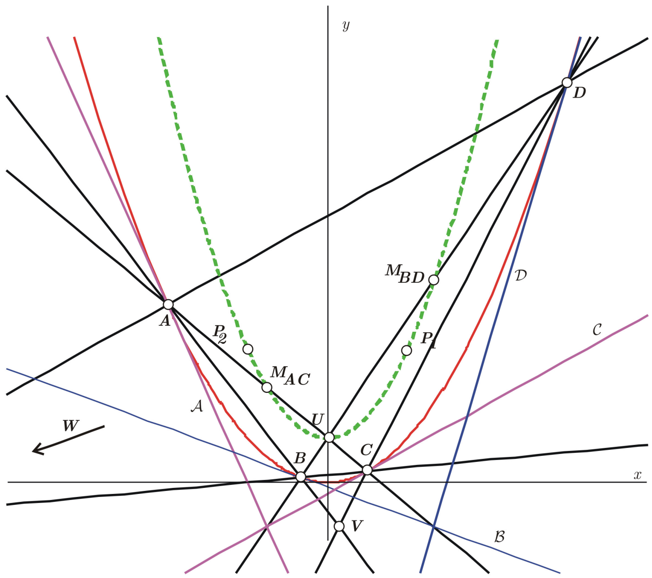

Figure 2. The visualization of Theorem 5.

Proof We prove the collinearity of the points P_1, W, and M_{AC}. Referring to (16) and (21) the slopes of lines WM_{AC} and P_1M_{AC} are obtained to be

$$\frac{(a^2+c^2+2k^2)(a+d-b-c)}{(a+c)(a+d-b-c)-2(ad-bc)} = \frac{(a^2+c^2-2ac)(a+d-b-c)}{a^2-c^2-(a-c)(b+d)} = \frac{(a-c)(a-b-c+d)}{a+c-b-d},$$
$$\frac{a^2+c^2-6k^2}{a+c-2k} = \frac{(a+c)^2-4k^2}{a+c-2k} = a+c+2k$$

and they are equal precisely when

$$(a-c)(a-b-c+d) = (a+c+2k)(a-b+c-d),$$

i.e.

$$2ad+2bc-2ac-2bd = 2k(a-b+c-d).$$

Because of (14), this is equivalent to

$$2ad+2bc-2ac-2bd = -(a-c)(b-d)$$

and this is the first equality (15). The other three collinearities can be proved in a similar manner. □

For the Euclidean version of the following theorems, see [5] (for Theorems 7 and 9) and [6] (for Theorem 8).

Theorem 7 *Let $ABCD$ be a harmonic quadrangle and M_{AC} be the midpoint of the line segment $\overline{AC}$. Then the equality*

$$d(M_{AC},A)^2 = d(M_{AC},B)\cdot d(M_{AC},D) \tag{23}$$

holds. The line AC is the bisector of the lines $M_{AC}B$ and $M_{AC}D$.

Proof The point M_{AC} is of the form $(\dfrac{a+c}{2}, \dfrac{a^2+c^2}{2})$. According to (9),

$$\begin{aligned} d(M_{AC},B)\cdot d(M_{AC},D)-d(M_{AC},A)^2 &= (\tfrac{a+c}{2}-b)(\tfrac{a+c}{2}-d)-(\tfrac{a+c}{2}-a)^2 \\ = bd-a^2+\tfrac{a+c}{2}(2a-b-d) &= bd+ac-\tfrac{1}{2}(a+c)(b+d)=0 \end{aligned}$$

is valid.

Further on, the equations of the lines $M_{AC}B, M_{AC}D$ are given by

$$\begin{aligned} M_{AC}B \ldots y &= (a+b+c-d)x+b(d-a-c), \\ M_{AC}D \ldots y &= (a-b+c+d)x+d(b-a-c). \end{aligned} \tag{24}$$

In that case, the equalities

$$\begin{aligned} \angle(M_{AC}B, M_{AC}C) &= (a+c)-(a+b+c-d)=d-b, \\ \angle(M_{AC}C, M_{AC}D) &= (a-b+c+d)-(a+c)=d-b \end{aligned}$$

prove the second part of the theorem. □

Theorem 8 *Let $ABCD$ be a harmonic quadrangle and M_{AC} be the midpoint of the line segment $\overline{AC}$. The triangles $M_{AC}DA$, $M_{AC}AB$, and CDB have equal corresponding angles.*

Proof By using (3) and (24) it is easy to prove that
$\angle(DA, AM_{AC})=\angle(AB, BM_{AC})=\angle(DB, BC)=c-d$,
$\angle(M_{AC}D, DA)=\angle(M_{AC}A, AB)=\angle(CD, DB)=b-c$, and
$\angle(AM_{AC}, M_{AC}D)=\angle(BM_{AC}, M_{AC}A)=\angle(BC, CD)=d-b$. □

Corollary 2 *The diagonal line BD is a symmedian for the triangles ABC and CDB, while the diagonal line AC is a symmedian for the triangles ABD and CDB.*

Theorem 9 *Let $ABCD$ be a harmonic quadrangle. The lines l_{AB}, l_{BC}, l_{CD}, l_{DA} incident with the diagonal point U and parallel to AB, BC, CD, DA, respectively, intersect the sides of the quadrangle $ABCD$ in eight points, $l_{CD}\cap AD$, $l_{CD}\cap BC$, $l_{AB}\cap BC$, $l_{AB}\cap AD$, $l_{BC}\cap AB$, $l_{AD}\cap AB$, $l_{AD}\cap CD$, and $l_{BC}\cap CD$, which lie on a circle.*

Proof It is easy to prove that the line l_{CD} has the equation of the form

$$y=(c+d)x+k^2. \tag{25}$$

Then the coordinates of $l_{CD}\cap AD$ are given with

$$\left(\frac{ad+k^2}{a-c}, \frac{a(d^2+k^2)}{a-c}\right).$$

Because of

$$\begin{aligned}
2\left(\frac{ad+k^2}{a-c}\right)^2 &= \frac{2a^2d^2+4adk^2+2k^4}{(a-c)^2} = \frac{ad(2ad+4k^2+2bc)}{(a-c)^2} \\
&= \frac{ad(2ad-ab-ad-bc-cd+2bc)}{(a-c)^2} = \frac{ad(ad-ab+bc-cd)}{(a-c)^2} \\
&= \frac{ad(a-c)(d-b)}{(a-c)^2} = \frac{ad(d-b)}{a-c} = \frac{a(d^2+k^2)}{a-c},
\end{aligned}$$

the point given above is incident with the circle

$$y = 2x^2.$$

We can prove the same for the other seven points. □

The structure of the harmonic quadrangle allows us to obtain a rich structure of collinear points and concurrent lines. The Euclidean version of the following theorem can be found in [3].

Theorem 10 *Let $ABCD$ be a harmonic quadrangle. If $C_{AB} = AB \cap \mathcal{C}$, $D_{AB} = AB \cap \mathcal{D}$, $A_{BC} = BC \cap \mathcal{A}$, $D_{BC} = BC \cap \mathcal{D}$, $A_{CD} = CD \cap \mathcal{A}$, $B_{CD} = CD \cap \mathcal{B}$, $B_{AD} = AD \cap \mathcal{B}$, $C_{AD} = AD \cap \mathcal{C}$ are considered, the four triples of points $\{C_{AB}, A_{BC}, T_{BD}\}$, $\{A_{CD}, C_{AD}, T_{BD}\}$, $\{D_{AB}, B_{AD}, T_{AC}\}$, $\{D_{BC}, B_{CD}, T_{AC}\}$ are collinear.*

Proof The points

$$A_{CD} = \left(\frac{cd-a^2}{c+d-2a}, \frac{2acd-(c+d)a^2}{c+d-2a}\right), C_{AD} = \left(\frac{ad-c^2}{a+d-2c}, \frac{2acd-(a+d)c^2}{a+d-2c}\right)$$

are lying on the line given by

$$y = \frac{a^2(c+d)+c^2(a+d)+d^2(a+c)-6acd}{(a-c)^2+(a-d)(c-d)}x + \frac{acd(a+c+d)-a^2c^2-a^2d^2-c^2d^2}{(a-c)^2+(a-d)(c-d)}.$$

The coordinates of the point $T_{BD} = (\frac{b+d}{2}, bd)$ satisfy the equation above if and only if $-(a-d)(c-d)(ab-2ac+bc+ad-2bd+cd) = 0$, which happens exactly in the case of $ab+bc+ad+cd = 2ac+2bd$. □

References

[1] Beban-Brkić J, Šimić M, Volenec V. On foci and asymptotes of conics in isotropic plane. Sarajevo J Math 2007; 3: 257-266.

[2] Bernhart A. Polygons of pursuit. Scripta Math 1959; 24: 23-50.

[3] Bradley CJ. A Singular Cyclic Quadrangle. Available online at: http://people.bath.ac.uk/masgcs/Article88.pdf.

[4] Kolar-Šuper R, Kolar-Begović Z, Volenec V, Beban-Brkić J. Metrical relationships in standard triangle in an isotropic plane. Math Commun 2005; 10: 159-167.

[5] Langley EM. A note on Tucker's harmonic quadrilateral. Math Gaz 1923; 11: 306-309.

[6] Pamfilos P. The Associated Harmonic Quadrilateral. Forum Geom 2014; 14: 15-29.

[7] Sachs H. Ebene isotrope Geometrie. Braunschweig, Germany: Vieweg-Verlag, 1987 (in German).

[8] Strubecker K. Geometrie in einer isotropen Ebene. Math Naturwiss Unterr 1962-63; 15: 297-306, 343-351, 385-394 (in German).

[9] Volenec V, Beban-Brkić J, Šimić M. The cyclic quadrangle in the isotropic plane. Sarajevo J Math 2011; 7: 265-275.