

An explicit formula of the intrinsic metric on the Sierpinski gasket via code representation

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Abstract: The computation of the distance between any two points of the Sierpinski gasket with respect to the intrinsic metric has already been investigated by several authors. However, to the best of our knowledge, in the literature there is not an explicit formula obtained by using the code set of the Sierpinski gasket. In this paper, we obtain an explicit formula for the intrinsic metric on the Sierpinski gasket via the code representations of its points. We finally give an important geometrical property of the Sierpinski gasket with regard to the intrinsic metric by using its code representation.

Key words: Sierpinski gasket, code representation, intrinsic metric

1. Introduction

The Sierpinski gasket was described by Sierpinski in 1915 and then it became one of the typical examples of fractals. This set, which can be written as a finite union of its scaled copies (see Figure 1), is a quite simple but amazing self-similar set in fractal geometry. In various mathematical studies, especially in fractal geometry, the Sierpinski gasket is often considered or used as a test model. Thus, the Sierpinski Gasket, which we will denote by S , has been studied in fractal geometry for years (see, for example, [1, 6, 7, 10, 13, 14] and references therein). It is well known that S is the attractor of the iterated function system $\{\blacktriangle; f_0, f_1, f_2\}$ such that

$$\begin{aligned}f_0(x, y) &= \left(\frac{1}{2}x, \frac{1}{2}y\right) \\f_1(x, y) &= \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right) \\f_2(x, y) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)\end{aligned}$$

(\blacktriangle is the filled-in convex hull of the three points $\{P_0, P_1, P_2\}$, where $P_0 = (0, 0)$, $P_1 = (1, 0)$ and $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$).

S can be defined in different ways. In [8, 9], the authors defined S as follows: let $P_0 = (0, 0)$, $P_1 = (1, 0)$, and $P_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Assume that $i_1 i_2 \dots i_n$ is the word of length n over the alphabet $X = \{0, 1, 2\}$ for any $i_1, i_2, \dots, i_n \in X$. For every such word, the elementary subtriangle of level n with vertices $(f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n})(P_0)$,

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$(f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n})(P_1)$, and $(f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n})(P_2)$ is denoted by $T_{i_1 i_2 \dots i_n}$. Then they define the Sierpinski gasket as

$$S = \bigcap_{n \geq 0} T_n \text{ where } T_n = \bigcup_{s \in \{0,1,2\}^n} T_s.$$

It is well known that different metrics can be defined on the same set. However, the interesting and natural one of these metrics is the one that reflects the internal structure of the set. For example, consider the restriction of the Euclidean metric to S . According to this metric, the distance between a and b is l (see Figure 2). However, there is not any path between a and b on S with length l . For this reason, this metric is not meaningful on this special set.

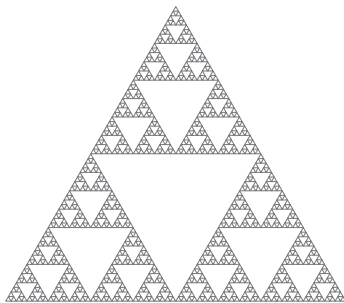


Figure 1. The Sierpinski gasket as an attractor of an IFS.

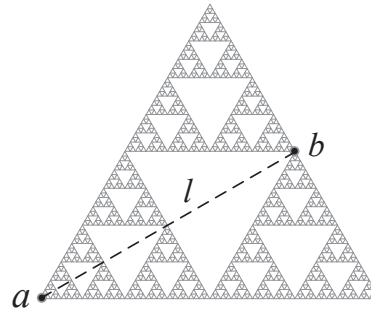


Figure 2. Distance between two points on S with respect to the Euclidean metric.

A more suitable metric on S is the intrinsic metric, which is defined as follows:

$$d_{int}(x, y) = \inf\{\delta \mid \delta \text{ is the length of a rectifiable curve in } S \text{ joining } x \text{ and } y\} \tag{1}$$

for $x, y \in S$ (for details, see [2]). The intrinsic metric, which is obtained by taking into account the paths on the structure, eliminates this discrepancy.

In several works the intrinsic metric on the Sierpinski gasket was constructed and defined in different ways since there exist different ways to construct (or define) the Sierpinski gasket (for details, see [3, 5, 8, 9, 14]). For example, in [8], an alternative definition of the intrinsic metric on S is given as follows: let $x, y \in S$ and let $\Delta_n(x), \Delta_n(y)$ be two elementary subtriangles of level n where $x \in \Delta_n(x)$ and $y \in \Delta_n(y)$ for all $n \geq 0$. For every $n \geq 0$, let x_n and y_n be the left lower vertices of $\Delta_n(x)$ and $\Delta_n(y)$, respectively. Then the authors define the intrinsic metric as

$$d_{int}(x, y) = \lim_{n \rightarrow \infty} \frac{d_n(x_n, y_n)}{2^n}$$

where $x, y \in S$ and d_n is the minimal length of a chain connecting x_n and y_n (for details, see [8]).

Strichartz also defined the intrinsic metric in a different way by using barycentric coordinates (for details, see [12]).

In [11], Romik tackled the discrete Sierpinski gasket and defined the metric giving the shortest distance on the points of this set using the code spaces. Romik then computed the average distance between points on the Sierpinski gasket using the connection between the tower of Hanoi problem and the discrete Sierpinski gasket.

In this paper, we use code representations of the points of the Sierpinski gasket to define the intrinsic metric. We note that the junction points of the Sierpinski gasket have two different code representations. In this work, we give an explicit formula for the intrinsic metric on S such that the formula does not depend on the choice of the representations of the junction points as mentioned in Proposition 3.3.

2. The code representation on S

We first give brief information about the coding process.

Let us denote the left-bottom part, the right-bottom part, and the upper part of the Sierpinski gasket by S_0, S_1 , and S_2 , respectively.

As shown in Figure 3, $S = S_0 \cup S_1 \cup S_2$, $S_0 \cap S_1 = \{p\}$, $S_1 \cap S_2 = \{q\}$, and $S_0 \cap S_2 = \{r\}$. Let $a_1 \in \{0, 1, 2\}$. Now similarly we denote the left-bottom part, the right-bottom part, and the upper part of S_{a_1} by $S_{a_1 0}, S_{a_1 1}$, and $S_{a_1 2}$, respectively.

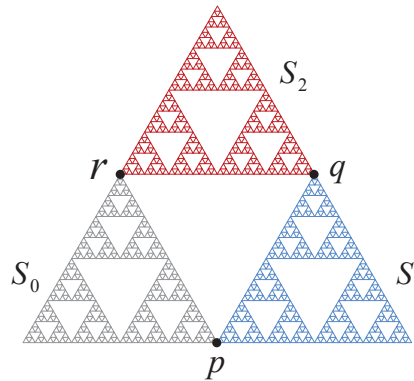


Figure 3. The subtriangles S_0, S_1 , and S_2 of S .

Following the same argument, let $S_{a_1 a_2 \dots a_k}$ denote the smaller triangular pieces of S where $a_i \in \{0, 1, 2\}$ and $i = 1, 2, \dots, k$. For the sequence

$$S_{a_1}, S_{a_1 a_2}, S_{a_1 a_2 a_3}, \dots, S_{a_1 a_2 \dots a_n}, \dots,$$

it is obvious that $S_{a_1} \supset S_{a_1 a_2} \supset S_{a_1 a_2 a_3} \supset \dots \supset S_{a_1 a_2 \dots a_n} \supset \dots$ and the infinite intersection

$$\bigcap_{k=1}^{\infty} S_{a_1 a_2 \dots a_k}$$

is a singleton, say $\{a\}$, where $a \in S$. We denote the point $a \in S$ by $a_1 a_2 \dots a_n \dots$ where $a_n \in \{0, 1, 2\}$ and $n = 1, 2, \dots$. Note that, if $a \in S$ is the intersection point of any two subtriangles of $S_{a_1 a_2 \dots a_k}$ (such a point is called a junction point of S), then a has two different representations such that $a_1 a_2 \dots a_k \beta \alpha \alpha \alpha \dots$ and $a_1 a_2 \dots a_k \alpha \beta \beta \beta \dots$ where $\alpha, \beta \in \{0, 1, 2\}$. Otherwise, a has a unique representation (for an alternative code representation of the points of S , see [4]).

3. The construction of the intrinsic metric on S

Let a and b be two different points of S whose representations are $a = a_1a_2 \dots a_n \dots$ and $b = b_1b_2 \dots b_n \dots$, respectively. Then there exists a natural number s such that $a_s \neq b_s$. Let

$$k = \min\{s \mid a_s \neq b_s, s = 1, 2, 3, \dots\}. \tag{2}$$

We then have $a \in S_{a_1a_2 \dots a_{k-1}a_k}$ and $b \in S_{a_1a_2 \dots a_{k-1}b_k}$. Without loss of generality, we assume that $a_k = 0$ and $b_k = 1$, which means $a \in S_{a_1a_2 \dots a_{k-1}0}$ and $b \in S_{a_1a_2 \dots a_{k-1}1}$ as seen in Figure 4 (in what follows we use the abbreviation $\sigma = a_1a_2 \dots a_{k-1}$ for simplicity). Note also that, in the other cases, i.e. a and b are in another subtriangle of $S_{a_1a_2 \dots a_{k-1}}$, similar procedures would be valid.

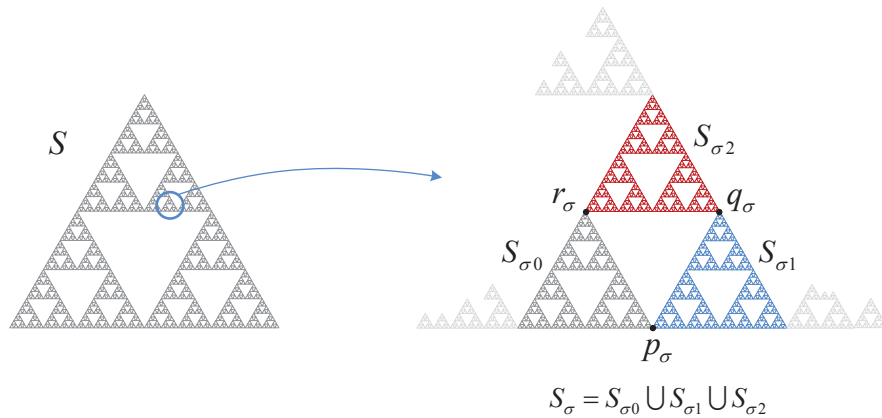


Figure 4. The subtriangle S_σ where $\sigma = a_1a_2 \dots a_{k-1}$ and the points $a \in S_{\sigma 0}$ and $b \in S_{\sigma 1}$.

Let $p_\sigma, r_\sigma, q_\sigma$ be the intersection points of the subtriangles $S_{\sigma 0}$ and $S_{\sigma 1}$, $S_{\sigma 0}$ and $S_{\sigma 2}$, and $S_{\sigma 1}$ and $S_{\sigma 2}$, respectively. The shortest paths between a and b must pass through either the point p_σ or the line $r_\sigma q_\sigma$ (see Figure 4).

We now investigate these two different ways as follows:

Case 1: First consider the shortest path passing through the point p_σ . Any path between a and b can be expressed as the union of a path between a and p_σ and a path between p_σ and b . We first look at the shortest paths between a and p_σ (the paths between p_σ and b can be obtained using a similar argument).

- If $a \in S_{a_1a_2 \dots a_{k-1}00}$ or $a \in S_{a_1a_2 \dots a_{k-1}02}$, then we must compute the length of the line segment $p_\sigma p_\sigma$ or the length of the line segment $q_\sigma p_\sigma$ where p_σ, q_σ are the intersection points of the subtriangles $S_{\sigma'0}$ and $S_{\sigma'1}$, $S_{\sigma'1}$ and $S_{\sigma'2}$ respectively where $\sigma' = a_1a_2 \dots a_{k-1}0$. In both cases, the length of the shortest paths between a and p_σ is

$$\mu = \frac{1}{2^{k+1}} + \varepsilon,$$

for some $\varepsilon \geq 0$.

For the case $a = r_\sigma$, where r_σ is the intersection point of the subtriangles $S_{\sigma'0}$ and $S_{\sigma'2}$, there obviously exist two shortest paths between a and p_σ . These paths are the union of the line segments $r_\sigma p_\sigma$ and $p_\sigma p_\sigma$ or the union of the line segments $r_\sigma q_\sigma$ and $q_\sigma p_\sigma$. The length of these paths can be easily computed as $\mu = \frac{1}{2^k}$.

- Suppose that $a \in S_{a_1a_2 \dots a_{k-1}01}$. If $a \in S_{a_1a_2 \dots a_{k-1}010}$ or $a \in S_{a_1a_2 \dots a_{k-1}012}$, then we must compute the

length of the line segment $p_{\sigma''}p_{\sigma}$ or the length of the line segment $q_{\sigma''}p_{\sigma}$ where $p_{\sigma''}, q_{\sigma''}$ are the intersection points of the subtriangles $S_{\sigma''_0}$ and $S_{\sigma''_1}, S_{\sigma''_1}$ and $S_{\sigma''_2}$ respectively where $\sigma'' = a_1a_2 \dots a_{k-1}01$.

In both cases, we get

$$\mu = \frac{1}{2^{k+2}} + \varepsilon,$$

for some $\varepsilon \geq 0$.

For the case $a = r_{\sigma''}$, where $r_{\sigma''}$ is the intersection point of the subtriangles $S_{\sigma''_0}$ and $S_{\sigma''_2}$, there are two paths giving the distance of the shortest paths between a and p_{σ} as before. These paths are the union of the line segments $r_{\sigma''}p_{\sigma''}$ and $p_{\sigma''}p_{\sigma}$ or the union of the line segments $r_{\sigma''}q_{\sigma''}$ and $q_{\sigma''}p_{\sigma}$. The length of these two paths is $\mu = \frac{1}{2^{k+1}}$.

Using a similar procedure for smaller triangles, we can determine the shortest paths between a and b and the length of these paths. Similarly, one can determine the shortest paths between p_{σ} and b . Then, by splicing these shortest paths between “ a and p_{σ} ” and “ p_{σ} and b ”, one can compute the length of the shortest paths between a and b passing through the point p_{σ} .

Case 2: Let us consider the shortest paths passing through the line segment $r_{\sigma}q_{\sigma}$. In a similar way, we can obtain the shortest paths (thus the corresponding length) between “ a and r_{σ} ” and between “ b and q_{σ} ”. As we add $\frac{1}{2^k}$ (that is, the length of the path $r_{\sigma}q_{\sigma}$) to these lengths, we obtain the length of the shortest path passing through $r_{\sigma}q_{\sigma}$.

Consequently, the length of the shortest paths between a and b is the minimum of the lengths obtained from Case 1 and Case 2. We can formulate this length, and hence the metric d , as follows.

Definition 3.1 Let $a_1a_2 \dots a_{k-1}a_ka_{k+1} \dots$ and $b_1b_2 \dots b_{k-1}b_kb_{k+1} \dots$ be representations of the points $a \in S$ and $b \in S$, respectively. Suppose that $a_i = b_i$ for $i = 1, 2, \dots, k - 1$ and $a_k \neq b_k$. We define the metric $d : S \times S \rightarrow R$ by

$$d(a, b) = \min \left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i} \right\}$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \quad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \text{ and } a_i \neq b_k \\ 1, & \text{otherwise} \end{cases}, \quad \delta_i = \begin{cases} 0, & b_i \neq b_k \text{ and } b_i \neq a_k \\ 1, & \text{otherwise} \end{cases}.$$

Remark 3.2 Note that the first value $\sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}$ is the length of the shortest paths passing through the point p_{σ} and the second value $\frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i}$ is the length of the shortest paths passing through the line segment $r_{\sigma}q_{\sigma}$ where $\frac{1}{2^k}$ is the length of the line segment $r_{\sigma}q_{\sigma}$.

It is obvious from the construction above that $d(a, b)$ is defined as the minimum of the lengths of the admissible paths connecting the points a and b in S .

Conclusion 3.2 *The metric in Definition 3.1 is equivalent to the metric given in (1).*

Proposition 3.3 *The metric d defined in Definition 3.1 does not depend on the choice of the code representations of the points.*

Proof Let a be a junction point whose code representations are of the form $a_1a_2a_2\dots a_2a_2a_2\dots$ and $a_2a_1a_1\dots a_1a_1a_1\dots$ such that $a_1 \neq a_2$ (in the general case, i.e. if the code representation of a is of the form

$$a_1a_2\dots a_{k-1}a_ka_{k+1}a_{k+1}a_{k+1}\dots,$$

the claim can be proven similarly).

Let x be an arbitrary point of S that has the code representation

$$x_1x_2\dots x_{k-1}x_kx_{k+1}x_{k+2}x_{k+3}\dots$$

Assume that $x_1 \neq a_1$. We consider the following two cases: $x_1 \neq a_2$ and $x_1 = a_2$.

Case 1: Suppose that $x_1 \neq a_2$. We now investigate the distance between the points

$$x_1x_2\dots x_kx_{k+1}x_{k+2}x_{k+3}\dots \text{ and } a_1a_2a_2\dots a_2a_2a_2\dots$$

Due to the definition of d , we have the following equations:

$$\alpha_i = \begin{cases} 0, & x_i = a_1 \\ 1, & x_i \neq a_1 \end{cases},$$

$$\beta_i = \begin{cases} 0, & a_2 = x_1 \\ 1, & a_2 \neq x_1 \end{cases},$$

$$\gamma_i = \begin{cases} 0, & x_i \neq x_1 \text{ and } x_i \neq a_1 \\ 1, & \text{otherwise} \end{cases},$$

$$\delta_i = \begin{cases} 0, & a_2 \neq a_1 \text{ and } a_2 \neq x_1 \\ 1, & \text{otherwise} \end{cases}.$$

We thus get $\beta_i = 1$ for all $i \geq 2$ owing to the fact that $x_1 \neq a_2$. Moreover, α_i can change according to the value of x_i and a_1 for each $i \geq 2$. It is also easily seen that $\delta_i = 0$ for every $i \geq 2$ since $a_2 \neq a_1$ and $a_2 \neq x_1$. It follows that

$$\sum_{i=2}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\alpha_i}{2^i}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i}{2^i}.$$

Now we compute the distance between the points

$$x_1x_2\dots x_kx_{k+1}x_{k+2}x_{k+3}\dots \text{ and } a_2a_1a_1\dots a_1a_1a_1\dots$$

Owing to the definition of d , we have the following equations:

$$\alpha'_i = \begin{cases} 0, & x_i = a_2 \\ 1, & x_i \neq a_2 \end{cases},$$

$$\beta'_i = \begin{cases} 0, & a_1 = x_1 \\ 1, & a_1 \neq x_1 \end{cases},$$

$$\gamma'_i = \begin{cases} 0, & x_i \neq x_1 \text{ and } x_i \neq a_2 \\ 1, & \text{otherwise} \end{cases},$$

$$\delta'_i = \begin{cases} 0, & a_2 \neq a_1 \text{ and } a_1 \neq x_1 \\ 1, & \text{otherwise} \end{cases}.$$

Similarly, we have $\beta'_i = 1$ for all $i \geq 2$ as a result of the fact that $x_1 \neq a_1$. Moreover, α'_i can change according to the value of x_i and a_2 for each $i \geq 2$. We also have that $\delta'_i = 0$ for every $i \geq 2$ since $a_1 \neq a_2$ and $a_1 \neq x_1$. This shows that

$$\sum_{i=2}^{\infty} \frac{\alpha'_i + \beta'_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\alpha'_i}{2^i}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma'_i + \delta'_i}{2^i} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma'_i}{2^i}.$$

Finally we show that $\alpha_i = \gamma'_i$ and $\alpha'_i = \gamma_i$ for all $i \geq 2$, respectively. We already know that $a_1 \neq a_2$, $x_1 \neq a_1$, and $x_1 \neq a_2$.

Assume that $\gamma'_i = 0$ for a fixed i . In this case, we have $x_i \neq a_2$ and $x_i \neq x_1$. We thus have $x_i = a_1$. Namely, it is $\alpha_i = 0$. Let $\gamma'_i = 1$ for a fixed i . Hence, it must be $x_i = a_2$ or $x_i = x_1$. This shows that $x_i \neq a_1$. That is, we obtain $\alpha_i = 1$.

Suppose that $\gamma_i = 0$ for a fixed i . We thus have $x_i \neq x_1$ and $x_i \neq a_1$ and this shows that $x_i = a_2$, so we get $\alpha'_i = 0$. Let $\gamma'_i = 1$ for a fixed i . Therefore, it must be $x_i = x_1$ or $x_i = a_1$. It follows that $x_i \neq a_1$ and thus we get $\alpha'_i = 1$.

This concludes the proof in Case 1.

Case 2: Let $x_1 = a_2$. The assertion can be proved similarly. □

4. Illustrative examples

In this section we give two examples where we compute the distance between two kinds of pairs of points in S .

Example 4.1 Let a and b be the points in S whose representations are $\overline{01\bar{2}} = 012012012 \dots$ and $\bar{1} = 111 \dots$ respectively.

To compute $d(a, b)$ we need the natural number k defined in (2). Since the first terms of the representations are different, we get $k = 1$. Straightforward calculations give us $\beta_i = 1$, $\delta_i = 1$,

$$\alpha_i = \begin{cases} 0 & ; \quad i \equiv 2 \pmod{3} \\ 1 & ; \quad \text{otherwise} \end{cases}$$

and

$$\gamma_i = \begin{cases} 0 & ; \quad i \equiv 0 \pmod{3} \\ 1 & ; \quad \text{otherwise} \end{cases}$$

for all $i \geq k + 1 = 2$, from which we conclude

$$\sum_{i=2}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \sum_{m=1}^{\infty} \left(\frac{1}{2^{3m-1}} + \frac{2}{2^{3m}} + \frac{2}{2^{3m+1}} \right) = \frac{5}{7}$$

and

$$\frac{1}{2} + \sum_{i=2}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{2} + \sum_{m=1}^{\infty} \left(\frac{2}{2^{3m-1}} + \frac{1}{2^{3m}} + \frac{2}{2^{3m+1}} \right) = \frac{1}{2} + \frac{6}{7}$$

and hence $d(a, b) = \frac{5}{7}$.

Example 4.2 Let a and b be the points in S whose representations are $000\bar{2} = 000222222 \dots$ and $0122\bar{0} = 0122000000 \dots$ respectively.

Since the second terms of the representations are different, we get $k = 2$. One can obtain $\alpha_i = 1$ for $i \geq k + 1 = 3$, $\beta_3 = \beta_4 = 1$ and $\beta_i = 0$ for $i \geq 5$, $\gamma_3 = 1$ and $\gamma_i = 0$ for $i \geq 4$, $\delta_3 = \delta_4 = 0$ and $\delta_i = 1$ for $i \geq 5$. We then obtain

$$\sum_{i=3}^{\infty} \frac{\alpha_i + \beta_i}{2^i} = \frac{2}{2^3} + \frac{2}{2^4} + \sum_{i=5}^{\infty} \frac{1}{2^i} = \frac{7}{16}$$

and

$$\frac{1}{2^2} + \sum_{i=3}^{\infty} \frac{\gamma_i + \delta_i}{2^i} = \frac{1}{4} + \frac{1}{2^3} + \sum_{i=5}^{\infty} \frac{1}{2^i} = \frac{7}{16},$$

which says that $d(a, b)$ is the value $\frac{7}{16}$. Notice that two values are equal and it means that there exist at least two shortest paths between the points.

Indeed, since it is a junction point, the point $000\bar{2}$ has two code representations and one can take the representation of this point as $002\bar{0}$. In this case the computation yields $k = 2$, $\alpha_i = 1$ for $i \geq 3$, $\beta_3 = \beta_4 = 1$ and $\beta_i = 0$ for $i \geq 5$, $\gamma_3 = 0$ and $\gamma_i = 1$ for $i \geq 4$, $\delta_3 = \delta_4 = 0$ and $\delta_i = 1$ for $i \geq 5$. This together with an elementary calculation gives that $d(a, b) = \frac{7}{16}$.

5. A geometrical property of the geodesic metric

In this section, we give a remarkable geometrical property with respect to the intrinsic metric on S . For any $P \in S$, Cristea and Steinsky showed that

$$d(P, P_0) + d(P, P_1) + d(P, P_2) = 2$$

by Proposition 12 in [3] and Viviani's theorem. In the following proposition, we prove the general case in a different way.

Proposition 5.1 *Let S_σ be a subtriangle of S and let P_{σ_0} , P_{σ_1} , and P_{σ_2} be vertices of S_σ where $\sigma = a_1 a_2 \dots a_n$ for any $n \in \mathbb{N}$. If P_σ is an arbitrary point of S_σ then*

$$d(P_\sigma, P_{\sigma_0}) + d(P_\sigma, P_{\sigma_1}) + d(P_\sigma, P_{\sigma_2}) = \frac{1}{2^{n-1}}.$$

Proof Let us first denote the vertices of S_σ as follows:

$$\begin{aligned} P_{\sigma_0} &= a_1 a_2 a_3 \dots a_n 000 \dots \\ P_{\sigma_1} &= a_1 a_2 a_3 \dots a_n 111 \dots \\ P_{\sigma_2} &= a_1 a_2 a_3 \dots a_n 222 \dots \end{aligned}$$

Given an arbitrary point $P_\sigma = a_1 a_2 a_3 \dots a_n x_{n+1} x_{n+2} x_{n+3} \dots$ of S_σ , notice that x_{n+1} is equal to one of the elements of the set $\{0, 1, 2\}$. Suppose that $x_{n+1} = 0$ (the other cases are done similarly). In this case, we have the following inequalities:

$$d(P_\sigma, P_{\sigma_1}) \geq \frac{1}{2^{n+1}} \quad \text{and} \quad d(P_\sigma, P_{\sigma_2}) \geq \frac{1}{2^{n+1}}$$

owing to the fact that the the terms $(n+i)$ th (for $i = 1, 2, \dots$) of P_{σ_1} and P_{σ_2} are different from the term x_{n+1} of P_σ . Let us now consider the term x_{n+2} of P_σ . In a similar way, if $x_{n+2} = 0$ then we obtain the inequalities as follows:

$$d(P_\sigma, P_{\sigma_1}) \geq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} \quad \text{and} \quad d(P_\sigma, P_{\sigma_2}) \geq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}$$

since the term x_{n+2} of P_σ is different from the terms $(n+1)$ th of P_{σ_1} and P_{σ_2} . If we continue this way, namely $x_{n+i} = 0$ for $i = 1, 2, 3, 4, \dots$, then we have the following equalities:

$$\begin{aligned} d(P_\sigma, P_{\sigma_0}) &= 0, \\ d(P_\sigma, P_{\sigma_1}) &= \frac{1+1}{2^{n+2}} + \frac{1+1}{2^{n+3}} + \frac{1+1}{2^{n+4}} + \dots = \frac{1}{2^n}, \\ d(P_\sigma, P_{\sigma_2}) &= \frac{1+1}{2^{n+2}} + \frac{1+1}{2^{n+3}} + \frac{1+1}{2^{n+4}} + \dots = \frac{1}{2^n} \end{aligned}$$

and we thus obtain the desired result. Assume that there exists at least one s such that $x_{n+s} \neq 0$ for $s = 1, 2, 3, \dots$. Without loss of generality, we can choose $x_{n+s} = 1$. Obviously, the term x_{n+s} of P_σ is different from the terms $(n+s+i)$ th of P_{σ_0} for all $i = 1, 2, 3, \dots$. In this case, we get

$$\begin{aligned} d(P_\sigma, P_{\sigma_0}) &\geq \frac{1}{2^{n+s+1}} + \frac{1}{2^{n+s+2}} + \frac{1}{2^{n+s+3}} + \dots = A, \\ d(P_\sigma, P_{\sigma_1}) &\geq \frac{1+1}{2^{n+2}} + \frac{1+1}{2^{n+3}} + \dots + \frac{1+1}{2^{n+s-1}} + \frac{0+1}{2^{n+s}} + \frac{1}{2^{n+s+1}} + \frac{1}{2^{n+s+2}} + \dots = B, \\ d(P_\sigma, P_{\sigma_2}) &\geq \frac{1+1}{2^{n+2}} + \frac{1+1}{2^{n+3}} + \dots + \frac{1+1}{2^{n+s-1}} + \frac{1+1}{2^{n+s}} + \frac{1}{2^{n+s+1}} + \frac{1}{2^{n+s+2}} + \dots = C. \end{aligned}$$

From now on, for every index $n+s+i$, exactly two terms of P_{σ_0} , P_{σ_1} , and P_{σ_2} are different from the term x_{n+s+i} of P_σ where $i = 1, 2, 3, \dots$. To give an example, let us take $x_{n+s+1} = 2$. Since the terms

$(n + s + 1)$ th of P_{σ_0} and P_{σ_1} are 0 and 1, respectively, we add $\frac{1}{2^{n+s+1}}$ to A , $\frac{1}{2^{n+s+1}}$ to B , and 0 to C . The computation is similar for $x_k \in \{0, 1, 2\}$ where $k \geq n + s + 1$. It follows that $d(P_\sigma, P_{\sigma_0}) + d(P_\sigma, P_{\sigma_1}) + d(P_\sigma, P_{\sigma_2})$ is the sum of A , B , C and

$$\frac{1+1}{2^{n+s+1}} + \frac{1+1}{2^{n+s+2}} + \frac{1+1}{2^{n+s+3}} + \dots$$

With a simple calculation, we get

$$d(P_\sigma, P_{\sigma_0}) + d(P_\sigma, P_{\sigma_1}) + d(P_\sigma, P_{\sigma_2}) = \frac{1}{2^{n-1}}$$

and thus the proof is completed. \square

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