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An application of q-Sumudu transform for fractional q-kinetic equation

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Abstract: The aim of this paper is to give an alternative solution for the q-kinetic equation involving the Riemann–Liouville fractional q-integral operator. The solution is obtained in terms of the q-Mittag–Leffler functions using inverse q-Sumudu transform. As applications, some corollaries are presented to illustrate the main results.

Key words: Riemann–Liouville fractional q-integrals, q-kinetic equation, q-Sumudu transforms

1. Introduction

Fractional differential equations appear more and more frequently for modeling of relevant systems in several fields of applied sciences. These equations play important roles not only in mathematics but also in physics, dynamical systems, control systems, and engineering to create the mathematical models of many physical phenomena. In particular, the kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and natural science. Therefore, in the literature we found several papers that analyze extensions and generalizations of these equations involving various fractional calculus operators. One may be referred, for example, to such works as those of Zaslavsky [23]; Saichev and Zaslavsky [16]; Saxena et al. [17], [18], [19]; Haubold et al. [11]; and Chouhan et al. [6].

The fractional q-calculus is the extension of the ordinary fractional calculus in q-theory. Recently there was a significant increase of activity in the area of q-calculus due to the applications of q-calculus in mathematics, statistics, and physics. Particularly, q-analysis has found many applications in the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra, geometric function theory, optimal control problems, q-difference and q-integral equations, and q-transform analysis. A detailed account of q-analysis can be seen in the works of Slater [20], Exton [7], Gasper and Rahman [9], Kac and Cheung [13], and Annaby and Mansour [5]. The Sumudu transform was introduced by Watugala [21]. For more details, one can be referred to Abdi [1] and Hahn [10] for applications of the q-Laplace transform in solving q-difference equations. Recently, Albayrak et al. [3] introduced q-analogues of the Sumudu transform and derived certain fundamental properties of q-Sumudu transforms like linearity, shifting theorems, differentiation, integration, and certain interesting connection theorems involving q-Laplace transforms.

Recently, Garg and Chanchlani [8] solved the following fractional q-kinetic equation by applying the method of q-Laplace transform and its inverse to obtain the solution in the following closed form:

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$$N_q(t) - N_0 f_q(t) = -c I_q^{\alpha} N_q(t), \quad c > 0, \alpha > 0, 0 < |q| < 1. \tag{1.1}$$

The aim of this paper, due to the usefulness and importance of the fractional differential equations in certain physical problems, is to give an alternative solution method, using q-Sumudu transform, for the q-kinetic equation (1.1) involving the Riemann–Liouville fractional q-integral operator.

2. Preliminaries and definitions

For convenience, we give here the basic definitions and related details of q-calculus. Throughout this paper, we will assume that q satisfies the condition $0 < |q| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q \to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

For any real number α,

$$[\alpha]_q := \frac{q^{\alpha} - 1}{q - 1}.$$

In particular, if $n \in \mathbb{N}$, we denote

$$[n]_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1,$$

and q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$

where $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$. Here and in the following, let $\mathbb{C}, \mathbb{R}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, and positive integers, respectively. For $x, y, \nu, a, t \in \mathbb{R}$, we have following usual notations:

$$(a;q)_n = \begin{cases} \prod_{k=0}^{n-1} \left(1 - aq^k\right) & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \end{cases},$$

$$(a;q)_{\infty} = \prod_{k=0}^{\infty} \left(1 - aq^k\right),$$

$$(a;q)_t = \frac{(a;q)_{\infty}}{(aq^t;q)_{\infty}},$$

$$(x-y)_q^{\nu} = x^{\nu} (y/x;q)_{\nu} = x^{\nu} \frac{(y/x;q)_{\infty}}{(y/xq^{\nu};q)_{\infty}}.$$

The q-analogues of the classical exponential function are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}, |t| < 1, \tag{2.1}$$

$$E_q(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n}{(q;q)_n} = (t;q)_\infty. \tag{2.2}$$

q-Analogues of generalized Mittag–Leffler type functions are defined by Prabhakar [15] and by Wiman [22] respectively as follows:

$$E^{\gamma}_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{(q^\gamma;q)_n \; z^n}{\Gamma_q(\alpha n+\beta)\,[n]_q!}, \tag{2.3}$$

and

$$E_{\alpha,\beta}(z;q) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n+\beta)}, \tag{2.4}$$

where $\alpha > 0; \beta, \gamma, z \in \mathbb{C}$.

The Jackson definite integral of the function f is defined by (see [12])

$$\int_0^x f(t) d_q t = a(1-q) \sum_{k=0}^{\infty} q^k f(aq^k), a \in \mathbb{R}. \tag{2.5}$$

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} = \frac{(q;q)_{x-1}}{(1-q)^{x-1}}, \tag{2.6}$$

and it is well known that

$$\lim_{q \to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha).$$

The q-gamma and the q-beta function have the following properties (see Kac and Sole [14]):

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\, d_q x \quad (\alpha > 0), \tag{2.7}$$

$$B_q(t;s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x \;\; (t,s>0), \tag{2.8}$$

$$B_q(t;s) = \frac{\Gamma_q(t)\,\Gamma_q(s)}{\Gamma_q(t+s)} \;\; (t,s>0). \tag{2.9}$$

The Riemann–Liouville fractional q-integral operator was introduced by Al-Salam [4] and also given independently by Agarwal [2] (for more details, see [5]):

$$I_q^\alpha f(t) := \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qz/t;q)_{\alpha-1} f(z)\, d_q z, \quad \mathrm{Re}(\alpha) > 0, \tag{2.10}$$

where $\alpha \notin \{-1,-2,\dots\}$.

Definition 2.1 (see [3]): *The first type of q-analogue of the Sumudu transform will be denoted by S_q and defined as follows:*

$$S_q\{f(t);s\} = \frac{1}{(1-q)s}\int_0^s E_q\left(\frac{q}{s}t\right)f(t)\,d_qt, \qquad s\in(-\tau_1,\tau_2) \tag{2.11}$$

$$= (q;q)_\infty \sum_{k=0}^{\infty}\frac{q^k f\left(sq^k\right)}{(q;q)_k} \tag{2.12}$$

over the set of functions

$$A = \left\{f(t)|\exists M,\tau_1,\tau_2>0,\ |f(t)|<ME_q\left(|t|/\tau_j\right),\ t\in(-1)^j\times[0,\infty)\right\}.$$

The q-Sumudu transforms of some special functions that we use in the sequel are given as follows:

$$S_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\Gamma_q(\alpha), \quad (\mathrm{Re}(\alpha)>0) \tag{2.13}$$

and

$$S_q\left\{\frac{1-q}{t}E_{\alpha,0}\left(-ct^\alpha;q\right);s\right\} = \frac{1}{s\left(1+cs^\alpha(1-q)^\alpha\right)}. \tag{2.14}$$

Albayrak et al. [3] gave the q-Sumudu convolution theorem as follows:

$$S_q\{(f*_q g)(x);s\} = sS_q\{f(x);s\}S_q\{g(x);s\} \tag{2.15}$$

where $f*_q g$ is the q-convolution of two analytic functions $f(t)$ and $g(t)$ defined as

$$(f*_q g)(t) = \frac{t}{1-q}\int_0^1 f(tx)\,g[t(1-qx)]\,d_qx \tag{2.16}$$

where $g(t)=\sum\limits_{k=0}^{\infty}a_nt^n$ and

$$g[t-x] = \sum_{k=0}^{\infty}a_n(t-x)_q^n.$$

3. Main theorem

We start this section with a lemma that gives a functional relation involving the Riemann–Liouville fractional q-integral operator and the q-Sumudu transforms.

Lemma 3.1 *The q-Sumudu transform of the Riemann–Liouville fractional q-integral operator:*

$$S_q\{I_q^\alpha f(t);s\} = s^\alpha(1-q)^\alpha S_q\{f(t);s\}, \tag{3.1}$$

where the Riemann–Liouville fractional q-integral operator is as in (2.10).

Proof Applying the q-Sumudu transform to both sides of (2.10) and then making use of (2.15) and the known formula (2.13), one can easily find

$$S_q\{I_q^\alpha f(t);s\} = s^\alpha(1-q)^\alpha S_q\{f(t);s\}.$$

Now we give our main theorem, which involves an analytical solution of the fractional q-kinetic equation. □

Theorem 3.1 *The solution of the q-kinetic equation (1.1) involving the Riemann–Liouville fractional q-integral operator is*

$$N_q(t) = N_0 \int_0^t f_q[t-qu]\, u^{-1} E_{\alpha,0}(-cu^\alpha; q)\, d_q u, \tag{3.2}$$

where $c > 0, \alpha > 0, 0 < |q| < 1$.

Proof Let us denote $S_q\{f_q(t), s\} = F(s)$ and $S_q\{N_q(t), s\} = N(s)$. Applying the q-Sumudu transform to both sides of the equation (1.1) and then using identity (3.1), we get

$$N(s) - N_0 F(s) = -cs^\alpha (1-q)^\alpha N(s)$$

and therefore

$$N(s) = \frac{N_0 F(s)}{1 + cs^\alpha (1-q)^\alpha}. \tag{3.3}$$

If we take the inverse q-Sumudu transform of (3.3) and then make use of (2.14) and the convolution theorem for S_q-transform (2.15), we get

$$S_q^{-1}\{N(s); t\} = S_q^{-1}\left\{N_0 s S_q\{f_q(t); s\} S_q\left\{(1-q)\, t^{-1} E_{\alpha,0}(-ct^\alpha; q); s\right\}; t\right\},$$

$$N_q(t) = N_0 \int_0^t f_q[t-qu]\, u^{-1} E_{\alpha,0}(-cu^\alpha; q)\, d_q u.$$

□

4. Special cases

We give some examples as the special cases of our result given in the previous section.

Corollary 4.1 *For $\mu > 0$, the solution of the fractional q-kinetic equation*

$$N_q(t) - N_0 t^{\mu-1} = -cI_q^\alpha N_q(t) \tag{4.1}$$

is given by

$$N_q(t) = N_0 \Gamma_q(\mu)\, t^{\mu-1} E_{\alpha,\mu}(-ct^\alpha; q). \tag{4.2}$$

Proof If we set $f(t) = t^{\mu-1}\ (\mu > 0)$ in the solution (3.2), then we have

$$N_q(t) = N_0 \int_0^t (t-qu)_q^{\mu-1}\, u^{-1} E_{\alpha,0}(-cu^\alpha; q)\, d_q u.$$

Using the definition (2.4) of the Mittag–Leffler type function, we find

$$N_q(t) = N_0 \int_0^t (t-qu)_q^{\mu-1}\, u^{-1} \sum_{n=0}^\infty \frac{(-cu^\alpha)^n}{\Gamma(\alpha n)} d_q u.$$

Interchanging the order of integration and summation, we have

$$N_q(t) = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\alpha n)} \int_0^t (t-qu)_q^{\mu-1} u^{\alpha n-1} d_q u.$$

If we make the change of variable $u = tx$, then we obtain

$$\begin{aligned} N_q(t) &= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\alpha n)} \int_0^1 (t-qtx)_q^{\mu-1} (tx)^{\alpha n-1} t d_q x \\ &= N_0 t^{\mu-1} \sum_{n=0}^{\infty} \frac{(-ct^{\alpha})^n}{\Gamma(\alpha n)} \int_0^1 (1-qx)_q^{\mu-1} x^{\alpha n-1} d_q x. \end{aligned}$$

Hence, we find from (2.8) and (2.9)

$$\begin{aligned} N_q(t) &= N_0 t^{\mu-1} \sum_{n=0}^{\infty} \frac{(-ct^{\alpha})^n}{\Gamma(\alpha n)} \beta(\mu; \alpha n) \\ &= N_0 \Gamma(\mu) t^{\mu-1} \sum_{n=0}^{\infty} \frac{(-ct^{\alpha})^n}{\Gamma(\alpha n + \mu)}, \end{aligned}$$

and using the definition (2.4) of the Mittag–Leffler type function, we get

$$N_q(t) = N_0 \Gamma(\mu) t^{\mu-1} E_{\alpha,\mu}(-ct^{\alpha}; q).$$

□

Corollary 4.2 *If we choose* $f_q(t) = t^{\mu-1} E_{\alpha,\mu}^{\gamma}(-ct^{\alpha}; q)$ *in* (1.1)*, then the solution of the fractional* q*-kinetic equation*

$$N_q(t) - N_0 t^{\mu-1} E_{\alpha,\mu}^{\gamma}(-ct^{\alpha}; q) = -c I_q^{\alpha} N_q(t) \tag{4.3}$$

is given by

$$N_q(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^{\gamma}; q)_k (-ct^{\alpha})^k}{[k]_q!} E_{\alpha,\alpha k+\mu}(-ct^{\alpha}; q). \tag{4.4}$$

Proof If we set

$$\begin{aligned} f_q(t) &= t^{\mu-1} E_{\alpha,\mu}^{\gamma}(-ct^{\alpha}; q) \\ &= \sum_{k=0}^{\infty} \frac{(q^{\gamma}; q)_k (-c)^k}{\Gamma_q(\alpha k + \mu) [k]_q!} t^{\mu+\alpha k-1} \end{aligned}$$

in the solution (3.2), then we have

$$N_q(t) = N_0 \int_0^t \sum_{k=0}^{\infty} \frac{(q^{\gamma}; q)_k (-c)^k}{\Gamma_q(\alpha k + \mu) [k]_q!} (t-qu)_q^{\mu+\alpha k-1} u^{-1} E_{\alpha,0}(-cu^{\alpha}; q) d_q u.$$

Making use of (2.4) we obtain,

$$N_q(t) = N_0 \int_0^t \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-c)^k}{\Gamma_q(\alpha k+\mu)[k]_q!} (t-qu)_q^{\mu+\alpha k-1} u^{-1} \sum_{n=0}^{\infty} \frac{(-cu^\alpha)^n}{\Gamma_q(\alpha n)} d_q u.$$

Interchanging the order of integration and summations, we have

$$N_q(t) = N_0 \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-c)^k}{\Gamma_q(\alpha k+\mu)[k]_q!} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_q(\alpha n)} \int_0^t (t-qu)_q^{\mu+\alpha k-1} u^{-1+\alpha n} d_q u.$$

If we make the change of variable $u = tx$, then we obtain

$$\begin{aligned} N_q(t) &= N_0 \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-c)^k}{\Gamma_q(\alpha k+\mu)[k]_q!} \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_q(\alpha n)} \int_0^1 (t-qtx)_q^{\mu+\alpha k-1} (tx)^{-1+\alpha n} t d_q x \\ &= N_0 t^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-ct^\alpha)^k}{\Gamma_q(\alpha k+\mu)[k]_q!} \sum_{n=0}^{\infty} \frac{(-ct^\alpha)^n}{\Gamma_q(\alpha n)} \int_0^1 (1-qx)_q^{\mu+\alpha k-1} x^{\alpha n-1} d_q x. \end{aligned}$$

Hence, we find from (2.8) and (2.9)

$$N_q(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-ct^\alpha)^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(-ct^\alpha)^n}{\Gamma_q(\alpha k+\mu+\alpha n)}.$$

Finally, in view of the definition of the q-analogue of the generalized Mittag–Leffler function defined by (2.3), we have

$$N_q(t) = N_0 t^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^\gamma;q)_k (-ct^\alpha)^k}{[k]_q!} E_{\alpha,\alpha k+\mu}(-ct^\alpha;q).$$

□

Corollary 4.3 *If we choose $\alpha = 1$ in (1.1), then the solution of the fractional q-kinetic equation*

$$N_q(t) - N_0 f_q(t) = -c I_q N_q(t) \tag{4.5}$$

is given by

$$N_q(t) = -cN_0 \int_0^t f_q[t-qu]\, e_q(-(1-q)u)\, d_q u. \tag{4.6}$$

Proof If we set $\alpha = 1$ in the solution (3.2), then we have

$$N_q(t) = N_0 \int_0^t f_q[t-qu]\, u^{-1} E_{1,0}(-cu;q)\, d_q u.$$

Using the definition of the q-analogue of the generalized Mittag–Leffler function and the q-gamma function defined by (2.4) and (2.6), we have

$$N_q(t) = N_0 \int_0^t f_q[t-qu]\, u^{-1} \left(\sum_{n=1}^{\infty} \frac{(-cu)^n}{\Gamma_q(n)} \right) d_q u$$

$$= N_0 \int_0^t f_q[t-qu]\, u^{-1} \left(\sum_{n=1}^{\infty} \frac{(-cu)^n (1-q)^{n-1}}{(q;q)_{n-1}} \right) d_q u$$

$$= -cN_0 \int_0^t f_q[t-qu] \left(\sum_{n=0}^{\infty} \frac{(-u)^n (1-q)^n}{(q;q)_n} \right) d_q u.$$

Thus, from the definition of the q-analogue of the classical exponential function given by (2.1), we find

$$N_q(t) = -cN_0 \int_0^t f_q[t-qu]\, e_q(-(1-q)u)\, d_q u.$$

□

5. Conclusions

Although the q-Sumudu transform is the theoretical dual of the q-Laplace transform, it has many applications in sciences and engineering for its special fundamental properties. This paper infers that the q-Sumudu transform has a very interesting property with the Riemann–Liouville fractional q-integral operator, which makes it easy to visualize. Therefore, we conclude this paper by remarking that the q-Sumudu transform method is a very effective and convenient alternative method for solving fractional q-differential equations.

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