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Applications of extended Watson's summation theorem

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Abstract: In this research paper, several interesting applications of the extended classical summation theorem are given. As special cases, we recover several known results available in the literature.

Key words: Generalized hypergeometric function, Gauss quadratic transformation formula, beta integral method

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by (see, for example [19, Chapter 4]; see also [25, pp. 71-72])

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\; z\right] &= {}_pF_q[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z] \\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

$$\Big(p \leq q \text{ and } |z|<\infty;\; p=q+1 \text{ and } |z|<1;\; p=q+1,\; |z|=1 \text{ and } \Re(\omega)>0\Big)$$

where

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of the gamma function by

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1) & (n\in\mathbb{N};\alpha\in\mathbb{C})\\ 1 & (n=0;\alpha\in\mathbb{C}\setminus\{0\}).\end{cases}$$

It is interesting to mention here that whenever a hypergeometric function ${}_2F_1$ or a generalized hypergeometric function ${}_pF_q$ reduces to gamma functions, the results are very important from the applications point

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2010 *AMS Mathematics Subject Classification:* Primary 33C90, 33C05; Secondary 33C70 of view. Thus, the classical summation theorems such as those of Gauss, Kummer, and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple, and Saalschütz for the series ${}_3F_2$; and others play a key role in the theory of generalized hypergeometric series. Until 1991, only a few summation theorems for the series ${}_2F_1$ and ${}_3F_2$ were available.

Applications of the above mentioned classical summation theorems are well known now.

During 1992–1996, in a series of three research papers, Lavoie et al. [11–13] established the generalizations of the above mentioned classical summation theorems and obtained a large number of special cases and limiting cases of their findings. Later on, Lewanowicz [14] and Vidunas [26] obtained further generalizations of Watson's and Kummer's summation theorems, respectively.

In 2010–2011, Rakha and Rathie [20] and Kim et al. [9] generalized and extended the above mentioned classical summation theorems in the most general form.

In our present investigation, we are interested in Gauss's second summation theorem and Watson's summation theorem [2], given respectively by the following:

$$ {}_2F_1\left[\begin{array}{l} a,\ b; \\ \frac{1}{2}(a+b+1); \end{array} \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.2} $$

and

$$ {}_3F_2\left[\begin{array}{l} a,\ b,\ c; \\ \frac{1}{2}(a+b+1),\ 2c; \end{array} 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.3} $$

provided that $\Re(2c-a-b) > -1$ as well as their extensions [9] given by

$$ \begin{aligned} {}_3F_2\left[\begin{array}{l} a,\ b,\ d+1; \\ \frac{1}{2}(a+b+3),\ d; \end{array} \frac{1}{2}\right] =& \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2}\right)} \\ & \cdot\left(\frac{\frac{1}{2}(a+b-1)-\frac{ab}{d}}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} + \frac{\frac{a+b+1}{d}-2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right) \end{aligned} \tag{1.4} $$

provided that $d \neq 0, -1, -2, \cdots$, and

$$ \begin{aligned} {}_4F_3\left[\begin{array}{l} a,\ b,\ c,\ d+1; \\ \frac{1}{2}(a+b+1),\ 2c+1,\ d; \end{array} 1\right] =& \frac{2^{a+b-2}\,\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\ & \cdot\left(\frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} + \left(\frac{2c-d}{d}\right)\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\right) \end{aligned} \tag{1.5} $$

provided that $\Re(2c-a-b) > -1$ and $d \neq 0, -1, -2, \cdots$.

Remark 1 *For different proofs of Watson's theorems, we refer the reader to [4, 15, 23, 27, 28].*

Remark 2 *Kim et al. [9] established the extension (1.5) of the classical Watson's theorem by using the contiguous functions relation together with contiguous summation theorems.*

Now we shall mention some of the useful applications of the classical Watson's summation theorem (1.3).

In 1928, by employing the classical Watson's summation theorem (1.3), Bailey [1] established the following interesting result involving the product of a hypergeometric function:

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] \times {}_1F_1\left[\begin{matrix}\beta; \\ 2\beta;\end{matrix}\; -x\right] = {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1); \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta;\end{matrix}\; \frac{x^2}{4}\right]. \tag{1.6} $$

For $\beta = \alpha$, the last result reduces to the following well-known Preece's identity [17]:

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] \times {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; -x\right] = {}_1F_2\left[\begin{matrix}\alpha; \\ \alpha+\frac{1}{2},\ 2\alpha;\end{matrix}\; \frac{x^2}{4}\right]. \tag{1.7} $$

For a short proof of (1.6), (1.7), and other interesting contiguous results, we refer the reader to [21, 22]. Also, from (1.7), it is easy to deduce that

$$ \mathrm{e}^{-\frac{x}{2}}\ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix}-\,; \\ \alpha+\frac{1}{2};\end{matrix}\; \frac{x^2}{16}\right], \tag{1.8} $$

which is Kummer's second theorem [19].

In the same paper, using Watson's summation theorem, Bailey [1] obtained the following quadratic transformation due to Gauss [3, 19]:

$$ (1+x)^{-2a}\ {}_2F_1\left[\begin{matrix}a, \quad b; \\ 2b;\end{matrix}\; \frac{4x}{(1+x)^2}\right] = {}_2F_1\left[\begin{matrix}a, \quad a-b+\frac{1}{2}; \\ b+\frac{1}{2};\end{matrix}\; x^2\right]. \tag{1.9} $$

From (1.9), the following results can be obtained and are recorded in [19]:

$$ {}_2F_1\left[\begin{matrix}\gamma,\ \gamma+\frac{1}{2}; \\ 2\gamma;\end{matrix}\; z\right] = (1-z)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\gamma-1} \tag{1.10} $$

and

$$ {}_2F_1\left[\begin{matrix}\gamma,\ \gamma-\frac{1}{2}; \\ 2\gamma;\end{matrix}\; z\right] = \left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}. \tag{1.11} $$

Using (1.10) and (1.11), Saad and Hall [24] obtained the following elementary but interesting and useful results:

$$ \int_0^\infty t^{a-1}\,\mathrm{e}^{-ht}\ {}_1F_1\left[\begin{matrix}a+\frac{1}{2}; \\ 2a;\end{matrix}\; kt\right]\mathrm{d}t = \frac{\Gamma(a)}{h^a}\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}, \tag{1.12} $$

$$ \int_0^\infty t^{a-\frac{1}{2}}\,\mathrm{e}^{-ht}\ {}_1F_1\left[\begin{matrix}a; \\ 2a;\end{matrix}\; kt\right]\mathrm{d}t = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}, \tag{1.13} $$

$$ \int_0^\infty t^{a}\,\mathrm{e}^{-ht}\ {}_1F_1\left[\begin{matrix}a+\frac{1}{2}; \\ 2a+1;\end{matrix}\; kt\right]\mathrm{d}t = \frac{\Gamma\left(a+1\right)}{h^{a+1}}\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}, \tag{1.14} $$

$$\int_0^\infty t^{a-\frac{1}{2}}\, \mathrm{e}^{-ht}\, {}_1F_1\left[\begin{array}{c} a+1; \\ 2a+1; \end{array} kt\right] \mathrm{dt} = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}, \tag{1.15}$$

$$\int_0^\infty t^{a-1}\, \mathrm{e}^{-ht}\, {}_1F_1\left[\begin{array}{c} a-\frac{1}{2}; \\ 2a; \end{array} kt\right] \mathrm{dt} = \frac{\Gamma\left(a\right)}{h^{a}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}, \tag{1.16}$$

$$\int_0^\infty t^{a-\frac{3}{2}}\, \mathrm{e}^{-ht}\, {}_1F_1\left[\begin{array}{c} a; \\ 2a; \end{array} kt\right] \mathrm{dt} = \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}, \tag{1.17}$$

$$\int_0^\infty t^{a-1}\, \mathrm{e}^{-ht}\, {}_1F_1\left[\begin{array}{c} a+\frac{1}{2}; \\ 2a+1; \end{array} kt\right] \mathrm{dt} = \frac{\Gamma\left(a\right)}{h^{a}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}, \tag{1.18}$$

and

$$\int_0^\infty t^{a-\frac{1}{2}}\, \mathrm{e}^{-ht}\, {}_1F_1\left[\begin{array}{c} a; \\ 2a+1; \end{array} kt\right] \mathrm{dt} = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}, \tag{1.19}$$

where $|k| < |h|$.

Next, in 2003, Krattenthaler and Rao [10] discovered the following new hypergeometric identity by employing (1.9) and by making use of the so-called Beta integral method:

$${}_4F_3\left[\begin{array}{c} a,\ b,\ e,\ 1-f; \\ 2b,\ \frac{1}{2}-\frac{1}{2}f+\frac{1}{2}e,\ 1-\frac{1}{2}f+\frac{1}{2}e; \end{array} 1\right] = \frac{\Gamma(f)\Gamma(f-e+2a)}{\Gamma\left(f-e\right)\Gamma\left(f+2a\right)}\, {}_4F_3\left[\begin{array}{c} a,\ a-b+\frac{1}{2},\ \frac{1}{2}e,\ \frac{1}{2}e+\frac{1}{2}; \\ b+\frac{1}{2},\ \frac{1}{2}f+a,\ \frac{1}{2}f+a+\frac{1}{2}; \end{array} 1\right], \tag{1.20}$$

provided that a or e be a nonpositive integer.

On the other hand, it is worth noting that for every hypergeometric identity, we can evaluate a number of integrals involving the hypergeometric function and the logarithmic function. In this sequel, among other results, the classical Watson's summation theorem (1.3) can be employed. Brychkov [5], in 2002, evaluated several interesting finite integrals involving the hypergeometric function and the logarithmic function.

Very recently, Choi and Rathie [6] evaluated some single integrals involving the hypergeometric function and logarithmic function (including those obtained by Brychkov [5]) in terms of psi and zeta functions suitable for numerical computations. Here are two of the results they obtained:

$$\int_0^1 t^{c-1}(1-t)^{c-1}\, {}_2F_1\left[\begin{array}{c} a,\ b; \\ \frac{1}{2}(a+b+1); \end{array} t\right] \mathrm{dt} = \frac{2^{1-2c}\,\pi\,\Gamma(c)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.21}$$

provided that $\Re(c) > 0$ and $\Re(2c - a - b) > -1$ and

$$\int_0^1 t^{c-1}(1-t)^{c-1}\ \ln^n\left(t-t^2\right)\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ t\right]\mathrm{d}t$$

$$= \frac{2\,\pi\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial c^r}\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{1.22}$$

where

$$A = \frac{2^{-2c}\ \Gamma(c)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.23}$$

$$\frac{\partial}{\partial c}A = A\cdot B, \tag{1.24}$$

$$B = -2\ln 2 + \Psi(c) + \Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \Psi\left(c-\frac{1}{2}a+\frac{1}{2}\right) - \Psi\left(c-\frac{1}{2}b+\frac{1}{2}\right), \tag{1.25}$$

$$\frac{\partial^n}{\partial c^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{1.26}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B =& (-1)^{n-r}(n-r-1)!\Big\{\zeta(n-r,c) + \zeta\left(n-r, c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) \\ & - \zeta\left(n-r, c-\frac{1}{2}a+\frac{1}{2}\right) - \zeta\left(n-r, c-\frac{1}{2}b+\frac{1}{2}\right)\Big\}.\end{aligned} \tag{1.27}$$

Following on similar lines, Gaboury and Rathie [8], with the help of Watson's summation theorem, obtained eight results related to double finite integrals involving the hypergeometric function and logarithmic function. Here are two of their results:

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$

$$= \frac{\pi\ 2^{1-2c}\ \Gamma(c)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.28}$$

provided that $\Re(c) > 1$ and $\Re(2c - a - b) > -1$, and

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ln^n\left(z-z^2\right)\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$

$$= \frac{2\,\pi\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{1.29}$$

where

$$A=\frac{2^{-2c}\,\Gamma(c)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.30}$$

$$\frac{\partial}{\partial c}A=A\cdot B, \tag{1.31}$$

$$B=-2\ln 2+\Psi(c)+\Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)-\Psi\left(c-\frac{1}{2}a+\frac{1}{2}\right)-\Psi\left(c-\frac{1}{2}b+\frac{1}{2}\right), \tag{1.32}$$

$$\frac{\partial^n}{\partial c^n}A=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{1.33}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B=&(-1)^{n-r}(n-r-1)!\Big\{\zeta(n-r,c)+\zeta\left(n-r,c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\\&-\zeta\left(n-r,c-\frac{1}{2}a+\frac{1}{2}\right)-\zeta\left(n-r,c-\frac{1}{2}b+\frac{1}{2}\right)\Big\}.\end{aligned} \tag{1.34}$$

For other integrals of this type, see [8].

The remainder of this paper is organized as follows. In Section 2, we establish a natural generalization of the Bailey's hypergeometric identity (1.6) involving the product of two generalized hypergeometric functions and we discuss some interesting special cases. In Section 3, we aim to provide the extension of the quadratic transformation (1.9) due to Bailey and to give a large number of special cases and applications. In this section, we also establish an extension of an hypergeometric identity (1.20) given by Krattenthaler and Rao [10]. Section 4 is devoted to the obtainment of three new classes of integrals involving the generalized hypergeometric function. Next, in Section 5, these new classes of integrals are used to evaluate integrals involving the generalized hypergeometric function and logarithmic function in terms of psi and Hurwitz zeta functions. Finally, in Sections 6 and 7, we obtain similar results for double integrals using an interesting result due to Edwards [7]. We conclude this section by remarking that the results established in this paper are simple, interesting, and easily obtainable and may be potentially useful.

2. Generalization of Bailey's identity for the product of two generalized hypergeometric functions

In this section, we shall prove a natural generalization of Bailey's identity (1.6) and give some special cases.

Theorem 1 *The following natural generalization of Bailey's identity (1.6) holds true:*

$$\begin{aligned}{}_1F_1\left[\begin{matrix}\alpha;\\2\alpha;\end{matrix}\;x\right]\times{}_2F_2\left[\begin{matrix}\beta,\ d+1;\\2\beta+1,\ d;\end{matrix}\;-x\right]=&{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1);\\\alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta;\end{matrix}\;\frac{x^2}{4}\right]\\&-\frac{x\left(\frac{2\beta}{d}-1\right)}{2(2\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2);\\\alpha+\frac{1}{2},\ \beta+\frac{3}{2},\ \alpha+\beta+1;\end{matrix}\;\frac{x^2}{4}\right].\end{aligned} \tag{2.1}$$

Proof In order to prove (2.1), we proceed as follows. Let

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] \times {}_2F_2\left[\begin{matrix}\beta,\ d+1; \\ 2\beta+1,\ d;\end{matrix}\; -x\right] = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}. \tag{2.2} $$

Now, in the product

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] \times {}_2F_2\left[\begin{matrix}\beta,\ d+1; \\ 2\beta+1,\ d;\end{matrix}\; -x\right], \tag{2.3} $$

it is not difficult to see that the coefficient a_n of x^n in the product, after some simplifications, is obtained as

$$ a_n = \frac{(\alpha)_n}{(2\alpha)_n\ n!}\ {}_4F_3\left[\begin{matrix}-n,\ 1-n-2\alpha,\ \beta,\ d+1; \\ 1-n-\alpha,\ 2\beta+1,\ d;\end{matrix}\; 1\right]. \tag{2.4} $$

Changing n to $2n$ and using (1.5), we get after simple simplifications

$$ a_{2n} = \frac{\left(\frac{1}{2}(\alpha+\beta)\right)_n \left(\frac{1}{2}(\alpha+\beta+1)\right)_n}{\left(\alpha+\frac{1}{2}\right)_n \left(\beta+\frac{1}{2}\right)_n (\alpha+\beta)_n\ 2^{2n}\ n!}. \tag{2.5} $$

Similarly, in (2.4), changing n to $2n+1$ and using (1.5) again gives

$$ a_{2n+1} = -\frac{\left(\frac{2\beta}{d}-1\right)}{2(2\beta+1)} \cdot \frac{\left(\frac{1}{2}(\alpha+\beta+1)\right)_n \left(\frac{1}{2}(\alpha+\beta+2)\right)_n}{\left(\alpha+\frac{1}{2}\right)_n \left(\beta+\frac{3}{2}\right)_n (\alpha+\beta+1)_n\ 2^{2n}\ n!}. \tag{2.6} $$

Finally, substituting the values of a_{2n} and a_{2n+1} in (2.2) and summing up the two series, we easily arrive at the right-hand side of (2.1). $\square$

Let us give two special cases of the generalization of Bailey's identity (2.1). If we set $d = 2\beta$ in (2.1), we at once get Bailey's identity (1.6). Setting $\beta = \alpha$ in (2.1), we obtain the following presumably new identity:

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ 2\alpha;\end{matrix}\; x\right] \times {}_2F_2\left[\begin{matrix}\alpha,\ d+1; \\ 2\alpha+1,\ d;\end{matrix}\; -x\right] = {}_1F_2\left[\begin{matrix}\alpha; \\ \alpha+\frac{1}{2},\ 2\alpha;\end{matrix}\; \frac{x^2}{4}\right] - \frac{x\left(\frac{2\alpha}{d}-1\right)}{2(2\alpha+1)}\ {}_1F_2\left[\begin{matrix}\alpha+1; \\ \alpha+\frac{3}{2},\ 2\alpha+1;\end{matrix}\; \frac{x^2}{4}\right]. \tag{2.7} $$

Furthermore, if we put $d = 2\alpha$ in (2.7), we obtain Preece's identity (1.7).

3. Extension of Bailey's quadratic transformation

In this section, we shall prove an extension of Bailey's quadratic transformation (1.9) and we derive several consequences and special cases.

Theorem 2 *The following transformation formula holds true:*

$$ (1-x)^{-2a}\ {}_3F_2\left[\begin{matrix}a, & b, & d+1; \\ 2b+1, & d;\end{matrix}\; -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}a, & a-b+\frac{1}{2}; \\ b+\frac{1}{2};\end{matrix}\; x^2\right] - \frac{2a(2b-d)x}{d(2b+1)}\ {}_2F_1\left[\begin{matrix}a+1, & a-b+\frac{1}{2}; \\ b+\frac{3}{2};\end{matrix}\; x^2\right]. \tag{3.1} $$

Proof Let

$$(1-x)^{-2a}\,{}_3F_2\left[\begin{array}{ccc} a, & b, & d+1; \\ 2b+1, & d; & \end{array} -\frac{4x}{(1-x)^2}\right] = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}. \tag{3.2}$$

In the product

$$(1-x)^{-2a}\,{}_3F_2\left[\begin{array}{ccc} a, & b, & d+1; \\ 2b+1, & d; & \end{array} -\frac{4x}{(1-x)^2}\right], \tag{3.3}$$

it is not difficult to see that the coefficient a_n of x^n in the product, after some simplifications, is obtained as

$$a_n = \frac{(2a)_n}{n!}\,{}_4F_3\left[\begin{array}{c} -n,\ n+2a,\ b,\ d+1; \\ a+\frac{1}{2},\ 2b+1,\ d; \end{array} 1\right]. \tag{3.4}$$

Now, changing n to $2n$ and using (1.5), we find, after simple simplifications, that

$$a_{2n} = \frac{(a)_n\left(a-b+\frac{1}{2}\right)_n}{\left(b+\frac{1}{2}\right)_n\ n!}. \tag{3.5}$$

Similarly, in (3.4), changing n to $2n+1$ and using (1.5) again yields

$$a_{2n+1} = -\frac{2a(2b-d)}{d(2b+1)}\cdot\frac{(a+1)_n\left(a-b+\frac{1}{2}\right)_n}{\left(b+\frac{3}{2}\right)_n\ n!}. \tag{3.6}$$

Substituting the values of a_{2n} and a_{2n+1} in (3.2) and summing up the two series, we easily arrive at the right-hand side of (3.1). □

It is quite easy to see that setting $d = 2b$ in (3.1) leads us to the quadratic transformation (1.9).

Replacing x by $\frac{x}{a}$ in (3.1) and taking the limit as $a\to\infty$, we obtain

$$\mathrm{e}^{-2x}\,{}_2F_2\left[\begin{array}{cc} b, & d+1; \\ 2b+1, & d; \end{array} 4x\right] = {}_0F_1\left[\begin{array}{c} -\,; \\ b+\frac{1}{2}; \end{array} x^2\right] + \frac{2(2b-d)x}{2b+1}\,{}_0F_1\left[\begin{array}{c} -\,; \\ b+\frac{3}{2}; \end{array} x^2\right], \tag{3.7}$$

which is a known result obtained by Rathie and Pogany [16] by other means.

Further, if we take $d = 2b$ in (3.7) and if we replace x by $\frac{x}{4}$, we recover Kummer's second theorem (1.8). Thus, equation (3.7) can be regarded as the generalization of Kummer's second theorem.

Let us deduce some interesting special cases of (3.1). First of all, changing x to $-x$ in (3.1), we have

$$\begin{aligned}(1+x)^{-2a}\,{}_3F_2\left[\begin{array}{ccc} a, & b, & d+1; \\ 2b+1, & d; & \end{array} \frac{4x}{(1+x)^2}\right] &= {}_2F_1\left[\begin{array}{cc} a, & a-b+\frac{1}{2}; \\ b+\frac{1}{2}; & \end{array} x^2\right] \\ &\quad + \frac{2a(2b-d)x}{2b+1}\,{}_2F_1\left[\begin{array}{cc} a+1, & a-b+\frac{1}{2}; \\ b+\frac{3}{2}; & \end{array} x^2\right].\end{aligned} \tag{3.8}$$

Now, let $z = \frac{4x}{(1+x)^2}$, $a = \gamma+\frac{1}{2}$ and $b = \gamma$. Then, after some simplifications, we find

$$\begin{aligned}{}_3F_2\left[\begin{array}{ccc} \gamma, & \gamma+\frac{1}{2}, & d+1; \\ 2\gamma+1, & d; & \end{array} z\right] =&(2\gamma-d)(1-z)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma} \\ &+ (1-2\gamma+d)(1-z)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}.\end{aligned} \tag{3.9}$$

If we set $d = 2\gamma$ in (3.9), we recover (1.10) and, thus, equation (3.9) can be viewed as the generalization of (1.10).

Also, if we put $z = \frac{4x}{(1+x)^2}$, $a = \gamma - \frac{1}{2}$ and $b = \gamma$ in (3.8), then we obtain

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\gamma, & \gamma-\frac{1}{2}, & d+1; \\ 2\gamma+1, & d; & \end{matrix}\; z\right] &= \left(\frac{2\gamma-1}{2\gamma+1}\right)(2\gamma-d)\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma} \\
&+ \left[1-(2\gamma-d)\left(\frac{2\gamma-1}{2\gamma+1}\right)\right]\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}. \qquad (3.10)
\end{aligned}
$$

Letting $d = 2\gamma$ in (3.10) leads us to (1.11). Thus, equation (3.10) can be viewed as the generalization of (1.11).

Here, as an application of our new identity (3.1), we shall establish the following result, which is given in the form of a theorem.

Theorem 3 *Let a or e be a nonpositive integer. Then the following identity holds true:*

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}a, & b, & d+1, & e, & 1-f; \\ 2b+1, & d, & \frac{1}{2}-\frac{1}{2}f+\frac{1}{2}e, & 1-\frac{1}{2}f+\frac{1}{2}e; & \end{matrix}\; 1\right] = \frac{\Gamma(f)}{\Gamma(e)\Gamma(f-e)}\left\{\frac{\Gamma(e)\Gamma(f-e+2a)}{\Gamma(f+2a)}\right. \\
&\cdot {}_4F_3\left[\begin{matrix}a, & a-b+\frac{1}{2}, & \frac{1}{2}e, & \frac{1}{2}e+\frac{1}{2}; \\ b+\frac{1}{2}, & \frac{1}{2}f+a, & \frac{1}{2}f+a+\frac{1}{2}; & \end{matrix}\; 1\right] - \frac{2a(2b-d)}{d(2b+1)}\frac{\Gamma(e+1)\Gamma(f-e+2a)}{\Gamma(f+1+2a)} \\
&\left.\cdot {}_4F_3\left[\begin{matrix}a+1, & a-b+\frac{1}{2}, & \frac{1}{2}e+\frac{1}{2}, & \frac{1}{2}e+1; \\ b+\frac{3}{2}, & \frac{1}{2}f+a+\frac{1}{2}, & \frac{1}{2}f+a+1; & \end{matrix}\; 1\right]\right\}. \qquad (3.11)
\end{aligned}
$$

Proof In order to prove the result (3.11), we proceed as follows. First of all, assume that a is a nonpositive integer. Now multiply the left-hand side of (3.1) by $x^{e-1}(1-x)^{f-e-1}$, where we temporarily suppose that $\Re(f) > \Re(e) > 0$, and integrating the resulting equation with respect to x from 0 to 1, expressing the involved ${}_3F_2$ as a series, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series, and denoting it by S_1, we have

$$
S_1 = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(d+1)_n(-1)^n\, 2^{2n}}{(2b+1)_n(d)_n\, n!}\int_0^1 x^{e+n-1}(1-x)^{f-e-2n-1}\mathrm{d}x. \qquad (3.12)
$$

Evaluating the beta integral and using the identity

$$
\Gamma(\alpha-n) = \frac{(-1)^n\,\Gamma(\alpha)}{(1-\alpha)_n}, \qquad (3.13)
$$

we find after simple calculations

$$
S_1 = \frac{\Gamma(e)\Gamma(f-e)}{\Gamma(f)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(d+1)_n(e)_n(1-f)_n}{(2b+1)_n(d)_n\left(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}f\right)_n\left(1+\frac{1}{2}e-\frac{1}{2}f\right)_n\; n!}, \qquad (3.14)
$$

which can be written in the form

$$
S_1 = \frac{\Gamma(e)\Gamma(f-e)}{\Gamma(f)}\,{}_5F_4\left[\begin{matrix}a, & b, & d+1, & e, & 1-f; \\ 2b+1, & d, & \frac{1}{2}-\frac{1}{2}f+\frac{1}{2}e, & 1-\frac{1}{2}f+\frac{1}{2}e; & \end{matrix}\; 1\right]. \qquad (3.15)
$$

Now, multiplying the right-hand side of (3.1) by $x^{e-1}(1-x)^{f-e-1}$, and proceeding essentially in the way as above and denoting it by S_2, gives after simple calculations

$$\begin{aligned}S_2 = & \frac{\Gamma(e)\Gamma(f-e+2a)}{\Gamma(f+2a)}\, {}_4F_3\left[\begin{matrix} a, & a-b+\frac{1}{2}, & \frac{1}{2}e, & \frac{1}{2}e+\frac{1}{2}; \\ b+\frac{1}{2}, & \frac{1}{2}f+a, & \frac{1}{2}f+a+\frac{1}{2}; & \end{matrix}\; 1\right] \\ & - \frac{2a(2b-d)}{d(2b+1)}\frac{\Gamma(e+1)\Gamma(f-e+2a)}{\Gamma(f+1+2a)} \\ & \cdot\, {}_4F_3\left[\begin{matrix} a+1, & a-b+\frac{1}{2}, & \frac{1}{2}e+\frac{1}{2}, & \frac{1}{2}e+1; \\ b+\frac{3}{2}, & \frac{1}{2}f+a+\frac{1}{2}, & \frac{1}{2}f+a+1; & \end{matrix}\; 1\right]. \end{aligned} \tag{3.16}$$

Finally, equating (3.15) and (3.16), the asserted result (3.11) follows. □

Note that setting $d = 2b$ in (3.11), we recover the known result (1.20) due to Krattenthaler and Rao [10].

We conclude this section by giving some applications of results (3.9) and (3.10). To this end, let us first state an integral formula that is not difficult to prove, that is,

$$\int_0^\infty t^{d-1}e^{-ht}\, {}_2F_2\left[\begin{matrix} a, & e+1; \\ b+1, & e; \end{matrix}\; kt\right] dt = \frac{\Gamma(d)}{h^d}\, {}_3F_2\left[\begin{matrix} a, & d, & e+1; \\ b+1, & e; & \end{matrix}\; \frac{k}{h}\right], \tag{3.17}$$

provided that $\Re(d) > 0$, $|k| < |h|$ and $e \neq 0, -1, -2, \cdots$.

Upon taking suitable values and applying the result (3.9), it is quite easy to see that the following relations can be obtained from equation (3.17):

$$\begin{aligned}\int_0^\infty t^{a-1}e^{-ht}\, {}_2F_2\left[\begin{matrix} a+\frac{1}{2}, & e+1; \\ 2a+1, & e; \end{matrix}\; kt\right] dt = & \frac{\Gamma(a)}{h^a}\left[(2a-e)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}\right. \\ & \left. + (1-2a+e)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}\right], \end{aligned} \tag{3.18}$$

$$\begin{aligned}\int_0^\infty t^{a-\frac{1}{2}}e^{-ht}\, {}_2F_2\left[\begin{matrix} a, & e+1; \\ 2a+1, & e; \end{matrix}\; kt\right] dt = & \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\left[(2a-e)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}\right. \\ & \left. + (1-2a+e)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}\right], \end{aligned} \tag{3.19}$$

$$\begin{aligned}\int_0^\infty t^{a}e^{-ht}\, {}_2F_2\left[\begin{matrix} a+\frac{1}{2}, & e+1; \\ 2a+2, & e; \end{matrix}\; kt\right] dt = & \frac{\Gamma\left(a+1\right)}{h^{a+1}}\left[(e-2a)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}\right. \\ & \left. + (2a+1-e)\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a+1}\right], \end{aligned} \tag{3.20}$$

and

$$\int_0^\infty t^{a-\frac{1}{2}} \mathrm{e}^{-ht} \, {}_2F_2 \left[\begin{matrix} a+1, & e+1; \\ 2a+2, & e; \end{matrix} \; kt \right] \mathrm{d}t = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}} \Bigg[(e-2a) \left(1-\frac{k}{h}\right)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a} + (2a+1-e) \left(1-\frac{k}{h}\right)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a+1} \Bigg], \tag{3.21}$$

provided that the conditions easily obtainable from (3.17) are satisfied.

Similarly, in (3.17), upon taking suitable values and applying the result (3.10), we find the following relations:

$$\int_0^\infty t^{a-1} \mathrm{e}^{-ht} \, {}_2F_2 \left[\begin{matrix} a-\frac{1}{2}, & e+1; \\ 2a+1, & e; \end{matrix} \; kt \right] \mathrm{d}t = \frac{\Gamma(a)}{h^a} \Bigg[(2a-e) \left(\frac{2a-1}{2a+1} \right) \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a} + \left[1-(2a-e) \left(\frac{2a-1}{2a+1} \right) \right] \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a-1} \Bigg], \tag{3.22}$$

$$\int_0^\infty t^{a-\frac{3}{2}} \mathrm{e}^{-ht} \, {}_2F_2 \left[\begin{matrix} a, & e+1; \\ 2a+1, & e; \end{matrix} \; kt \right] \mathrm{d}t = \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}} \Bigg[(2a-e) \left(\frac{2a-1}{2a+1} \right) \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a} + \left[1-(2a-e) \left(\frac{2a-1}{2a+1} \right) \right] \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a-1} \Bigg], \tag{3.23}$$

$$\int_0^\infty t^{a-1} \mathrm{e}^{-ht} \, {}_2F_2 \left[\begin{matrix} a+\frac{1}{2}, & e+1; \\ 2a+2, & e; \end{matrix} \; kt \right] \mathrm{d}t = \frac{\Gamma(a)}{h^a} \Bigg[(2a+1-e) \left(\frac{a}{a+1} \right) \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a+1} + \left[1-(2a+1-e) \left(\frac{a}{a+1} \right) \right] \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a} \Bigg], \tag{3.24}$$

and

$$\int_0^\infty t^{a-\frac{1}{2}} \mathrm{e}^{-ht} \, {}_2F_2 \left[\begin{matrix} a, & e+1; \\ 2a+2, & e; \end{matrix} \; kt \right] \mathrm{d}t = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}} \Bigg[(2a+1-e) \left(\frac{a}{a+1} \right) \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a+1} + \left[1-(2a+1-e) \left(\frac{a}{a+1} \right) \right] \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}} \right)^{2a} \Bigg], \tag{3.25}$$

provided that the conditions easily obtainable from (3.17) are satisfied.

Remark 3 *In (3.18) to (3.21), if we set respectively $e=2a$, $e=2a$, $e=2a+1$, and $e=2a+1$, we recover (1.12) to (1.15), respectively. Also, in (3.22) to (3.25), if we set respectively $e=2a$, $e=2a$, $e=2a+1$, and $e=2a+1$, we recover (1.16) to (1.19), respectively.*

4. New class of finite integrals involving generalized hypergeometric functions

This section aims to provide a new class of finite integrals involving generalized hypergeometric functions. Let us give, in the form of a theorem, the three integral formulas that we shall establish.

Theorem 4 *The following finite integrals formulas holds true:*

$$\begin{aligned}
&\int_0^1 t^{b-1}(1-t)^{\frac{1}{2}(1+a-b)-1}\,{}_3F_2\left[\begin{matrix} a, & c, & d+1; \\ 2c+1, & d; & \end{matrix}\; t\right]\mathrm{d}t \\
&= \Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right) \\
&\cdot\left[\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\right. \\
&\left.+\frac{\left(\frac{2c}{d}-1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\right],
\end{aligned}\tag{4.1}$$

provided that $\Re(b)>0$, $\Re(1+a-b)>0$, $\Re(2c-a-b)>-1$ *and* $d\neq 0,-1,-2,\cdots$,

$$\begin{aligned}
&\int_0^1 t^{c-1}(1-t)^{c}\,{}_3F_2\left[\begin{matrix} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & d; & \end{matrix}\; t\right]\mathrm{d}t \\
&= 2^{-2c}\,\pi\,\Gamma(c)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) \\
&\cdot\left[\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\right. \\
&\left.+\frac{\left(\frac{2c}{d}-1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\right],
\end{aligned}\tag{4.2}$$

provided that $\Re(c)>0$, $\Re(2c-a-b)>-1$ *and* $d\neq 0,-1,-2,\cdots$,
and

$$\begin{aligned}
&\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_3F_2\left[\begin{matrix} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & 2c+1; & \end{matrix}\; t\right]\mathrm{d}t \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(d-c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(d)} \\
&\cdot\left[\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\right. \\
&\left.+\frac{\left(\frac{2c}{d}-1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\right],
\end{aligned}\tag{4.3}$$

provided that $\Re(c)>0$, $\Re(d-c)>0$, $\Re(2c-a-b)>-1$ *and* $d\neq 0,-1,-2,\cdots$.

Proof In order to prove (4.1), we proceed as follows. Denoting the left-hand side of (4.1) by I, expressing the ${}_3F_2$ function as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series involved), evaluating the beta-integral, and after some simplifications, we obtain

$$I = \frac{\Gamma(b)\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n (d+1)_n}{\left(\frac{1}{2}(a+b+1)\right)_n (2c+1)_n (d)_n \, n!}. \tag{4.4}$$

Rewriting the series (4.4) in terms of the generalized hypergeometric function, we have

$$I = \frac{\Gamma(b)\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)} \, {}_4F_3 \left[\begin{matrix} a, & b, & c, & d+1; \\ \frac{1}{2}(a+b+1), & 2c+1, & d; & \end{matrix} \; 1 \right]. \tag{4.5}$$

Observing that the ${}_4F_3$ function can be evaluated by using the extended Watson's summation theorem (1.5) and after making some simplifications, we obtain the result (4.1).

In exactly the same manner, the integrals (4.2) and (4.3) can be established. □

Let us now give some special cases of the three integrals formulas (4.1), (4.2), and (4.3).

In (4.1), if we let $a = -2n$ or $a = -2n-1$, where n is zero or a positive integer, in each case, one of the two terms in the right-hand side of (4.1) vanishes, and after some calculations, we get the following interesting results:

$$\int_0^1 t^{b-1}(1-t)^{\frac{1}{2}(1-b-2n)-1} \, {}_3F_2 \left[\begin{matrix} -2n, & c, & d+1; \\ 2c+1, & d; & \end{matrix} \; t \right] \mathrm{d}t = \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2}b\right)\Gamma(b)}{\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \cdot \frac{\left(\frac{1}{2}\right)_n \left(c - \frac{1}{2}b + \frac{1}{2}\right)_n}{\left(c + \frac{1}{2}\right)_n \left(\frac{1}{2}b + \frac{1}{2}\right)_n} \tag{4.6}$$

and

$$\int_0^1 t^{b-1}(1-t)^{-\frac{1}{2}(b+2n)-1} \, {}_3F_2 \left[\begin{matrix} -2n-1, & c, & d+1; \\ 2c+1, & d; & \end{matrix} \; t \right] \mathrm{d}t = \frac{\left(\frac{2c}{d} - 1\right)}{2c+1} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b\right)} \cdot \frac{\left(\frac{3}{2}\right)_n \left(c - \frac{1}{2}b + 1\right)_n}{\left(c + \frac{3}{2}\right)_n \left(1 + \frac{1}{2}b\right)_n}. \tag{4.7}$$

In particular, if we set $d = 2c$ in (4.7), we obtain

$$\int_0^1 t^{b-1}(1-t)^{-\frac{1}{2}(b+2n)-1} \, {}_2F_1 \left[\begin{matrix} -2n-1, & c; \\ 2c; & \end{matrix} \; t \right] \mathrm{d}t = 0. \tag{4.8}$$

In (4.2), if we let $b = -2n$ and replace a by $a+2n$ or if we let $b = -2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer, in each case, one of the two terms in the right-hand side of (4.2) vanishes, and after some elementary algebra, we find the following results:

$$\int_0^1 t^{c-1}(1-t)^{c} \, {}_3F_2 \left[\begin{matrix} -2n, & a+2n, & d+1; \\ \frac{1}{2}(a+1), & d; & \end{matrix} \; t \right] \mathrm{d}t = \frac{\sqrt{\pi}\, 2^{-2c}\, \Gamma(c)}{\Gamma\left(c + \frac{1}{2}\right)} \cdot \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a + \frac{1}{2} - c\right)_n}{\left(c + \frac{1}{2}\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n} \tag{4.9}$$

and

$$\int_0^1 t^{c-1}(1-t)^{c} \, {}_3F_2 \left[\begin{matrix} -2n-1, & a+2n+1, & d+1; \\ \frac{1}{2}(a+1), & d; & \end{matrix} \; t \right] \mathrm{d}t = \left(1 - \frac{2c}{d}\right) \cdot \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+2)} \cdot \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a + \frac{1}{2} - c\right)_n}{\left(c + \frac{3}{2}\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n}. \tag{4.10}$$

In particular, putting $d = 2c$ in (4.10) yields

$$\int_0^1 t^{c-1}(1-t)^c \, {}_3F_2\left[\begin{matrix} -2n-1, & a+2n+1, & 2c+1; \\ \frac{1}{2}(a+1), & 2c; & \end{matrix}\; t\right] \mathrm{d}t = 0. \tag{4.11}$$

Finally, in (4.3), if we let $b = -2n$ and replace a by $a+2n$ or if we let $b = -2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer, in each case, one of the two terms in the right-hand side of (4.3) vanishes, and after some elementary algebra, we find the following results:

$$\int_0^1 t^{c-1}(1-t)^{d-c-1} \, {}_3F_2\left[\begin{matrix} -2n, & a+2n, & d+1; \\ \frac{1}{2}(a+1), & 2c+1; & \end{matrix}\; t\right] \mathrm{d}t = \frac{\Gamma(c)\Gamma(d-c)}{\Gamma(d)} \cdot \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}-c\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n} \tag{4.12}$$

and

$$\int_0^1 t^{c-1}(1-t)^{d-c-1} \, {}_3F_2\left[\begin{matrix} -2n-1, & a+2n+1, & d+1; \\ \frac{1}{2}(a+1), & 2c+1; & \end{matrix}\; t\right] \mathrm{d}t = \frac{\left(1-\frac{2c}{d}\right)}{2c+1} \cdot \frac{\Gamma(c)\Gamma(d-c)}{\Gamma(d)} \cdot \frac{\left(\frac{3}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}-c\right)_n}{\left(c+\frac{3}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n}. \tag{4.13}$$

If we further set $d = 2c$ in (4.13), we get

$$\int_0^1 t^{c-1}(1-t)^{c-1} \, {}_2F_1\left[\begin{matrix} -2n-1, & a+2n+1; \\ \frac{1}{2}(a+1); & \end{matrix}\; t\right] \mathrm{d}t = 0. \tag{4.14}$$

Many other results can be obtained by specializing the different parameters appearing in (4.1) to (4.3).

Remark 4 *The integrals* (4.6) *and* (4.9) *are curious as the right-hand side of these are independent of d.*

5. New class of finite integrals involving the generalized hypergeometric function and logarithmic function

In this section, we shall establish a new class of integrals involving the generalized hypergeometric function and logarithmic function.

Theorem 5 *The following integral formula holds true:*

$$\begin{aligned} &\int_0^1 t^{b-1}(1-t)^{\frac{1}{2}(1+a-b)-1} \ln^n\left(\frac{t}{\sqrt{1-t}}\right) {}_3F_2\left[\begin{matrix} a, & c, & d+1; \\ 2c+1, & d; & \end{matrix}\; t\right] \mathrm{d}t \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r A}{\partial b^r} \cdot \frac{\partial^{n-r-1} B}{\partial b^{n-r-1}} \\ &\quad + \left(\frac{2c}{d}-1\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}a+1\right)} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r C}{\partial b^r} \cdot \frac{\partial^{n-r-1} D}{\partial b^{n-r-1}} \end{aligned} \tag{5.1}$$

where

$$A = \frac{\Gamma(b)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{5.2}$$

$$\frac{\partial\, A}{\partial b} = A \cdot B, \tag{5.3}$$

$$B = \Psi(b) - \frac{1}{2}\Psi\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2}\Psi\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b + \frac{1}{2}\right) + \frac{1}{2}\Psi\left(c - \frac{1}{2}b + \frac{1}{2}\right), \tag{5.4}$$

$$\frac{\partial^n\, A}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r\, A}{\partial b^r} \cdot \frac{\partial^{n-r-1}\, B}{\partial b^{n-r-1}}, \tag{5.5}$$

$$\begin{aligned}\frac{\partial^{n-r-1}\, B}{\partial b^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) \\ &- \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}b + \frac{1}{2}\right) \\ &+ \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c - \frac{1}{2}b + \frac{1}{2}\right)\Bigg\},\end{aligned} \tag{5.6}$$

$$C = \frac{\Gamma(b)\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c - \frac{1}{2}b + 1\right)}, \tag{5.7}$$

$$\frac{\partial\, C}{\partial b} = C \cdot D, \tag{5.8}$$

$$D = \Psi(b) - \frac{1}{2}\Psi\left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2}\Psi\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b\right) + \frac{1}{2}\Psi\left(c - \frac{1}{2}b + 1\right), \tag{5.9}$$

$$\frac{\partial^n\, C}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r\, C}{\partial b^r} \cdot \frac{\partial^{n-r-1}\, D}{\partial b^{n-r-1}}, \tag{5.10}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}\, D}{\partial b^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) \\ &- \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \frac{1}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}b\right) \\ &+ \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c - \frac{1}{2}b + 1\right)\Bigg\}.\end{aligned} \tag{5.11}$$

Proof Differentiating n times both sides of (4.1) with respect to b yields

$$\int_0^1 t^{b-1}(1-t)^{\frac{1}{2}(1+a-b)-1}\ \ln^n\left(\frac{t}{\sqrt{1-t}}\right)\ {}_3F_2\left[\begin{matrix} a, & c, & d+1; \\ 2c+1, & d; & \end{matrix}\ t\right]\mathrm{d}t$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)}\frac{\partial^n\ A}{\partial b^n}$$
$$+\left(\frac{2c}{d}-1\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}a+1\right)}\frac{\partial^n\ C}{\partial b^n}, \tag{5.12}$$

where A and C are the same as given in (5.2) and (5.7), respectively.

Now it easy to see that

$$\frac{\partial\ A}{\partial b} = A\frac{\partial\ \ln A}{\partial b} = A\cdot B \tag{5.13}$$

and

$$\frac{\partial\ C}{\partial b} = C\frac{\partial\ \ln C}{\partial b} = C\cdot D, \tag{5.14}$$

where B and D are the same as given in (5.4) and (5.9), respectively. From (5.13), we have

$$\frac{\partial^n\ A}{\partial b^n} = \frac{\partial^{n-1}}{\partial b^{n-1}}\left(\frac{\partial\ A}{\partial b}\right) = \frac{\partial^{n-1}\ A\cdot B}{\partial b^{n-1}}, \tag{5.15}$$

which upon using the Leibniz theorem becomes

$$\frac{\partial^n\ A}{\partial b^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial b^r}\cdot\frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}} \tag{5.16}$$

and similarly

$$\frac{\partial^n\ C}{\partial b^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ C}{\partial b^r}\cdot\frac{\partial^{n-r-1}\ D}{\partial b^{n-r-1}}, \tag{5.17}$$

where $\frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}}$ and $\frac{\partial^{n-r-1}\ D}{\partial b^{n-r-1}}$ are the same as given in (5.6) and (5.11), respectively.

Finally, substituting the values of $\frac{\partial^n\ A}{\partial b^n}$ and $\frac{\partial^n\ C}{\partial b^n}$ from (5.15) and (5.17) into (5.12) leads us to the asserted result (5.1). □

Theorem 6 *The following integral formula holds true:*

$$\int_0^1 t^{c-1}(1-t)^{c}\ \ln^n\left(t-t^2\right)\ {}_3F_2\left[\begin{matrix} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & d; & \end{matrix}\ t\right]\mathrm{d}t$$
$$= \frac{\pi\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial c^r}\cdot\frac{\partial^{n-r-1}\ B}{\partial c^{n-r-1}}$$
$$+\frac{\pi\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ C}{\partial c^r}\cdot\frac{\partial^{n-r-1}\ D}{\partial c^{n-r-1}} \tag{5.18}$$

where

$$A = \frac{2^{-2c}\,\Gamma(c)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c - \frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}b + \frac{1}{2}\right)}, \tag{5.19}$$

$$\frac{\partial\, A}{\partial c} = A \cdot B, \tag{5.20}$$

$$B = -2\ln 2 + \Psi(c) + \Psi\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \Psi\left(c - \frac{1}{2}a + \frac{1}{2}\right) - \Psi\left(c - \frac{1}{2}b + \frac{1}{2}\right), \tag{5.21}$$

$$\frac{\partial^n\, A}{\partial c^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\, A}{\partial c^r}\cdot\frac{\partial^{n-r-1}\, B}{\partial c^{n-r-1}}, \tag{5.22}$$

$$\begin{aligned}\frac{\partial^{n-r-1}\, B}{\partial b^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Big\{\zeta(n-r,c) + \zeta\left(n-r, c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)\\ &- \zeta\left(n-r, c - \frac{1}{2}a + \frac{1}{2}\right) - \zeta\left(n-r, c - \frac{1}{2}b + \frac{1}{2}\right)\Big\},\end{aligned} \tag{5.23}$$

$$C = \frac{\left(\frac{2c}{d} - 1\right)\,2^{-2c}\,\Gamma(c)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c - \frac{1}{2}a + 1\right)\Gamma\left(c - \frac{1}{2}b + 1\right)}, \tag{5.24}$$

$$\frac{\partial\, C}{\partial c} = C \cdot D, \tag{5.25}$$

$$D = \frac{2}{2c-d} - 2\ln 2 + \Psi(c) + \Psi\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \Psi\left(c - \frac{1}{2}a + 1\right) - \Psi\left(c - \frac{1}{2}b + 1\right), \tag{5.26}$$

$$\frac{\partial^n\, C}{\partial c^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\, C}{\partial c^r}\cdot\frac{\partial^{n-r-1}\, D}{\partial c^{n-r-1}}, \tag{5.27}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}\, D}{\partial c^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Big\{-2^{n-r}(2c-d)^{-(n-r)} + \zeta(n-r,c)\\ &+ \zeta\left(n-r, c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right) - \zeta\left(n-r, c - \frac{1}{2}a + 1\right)\\ &- \zeta\left(n-r, c - \frac{1}{2}b + 1\right)\Big\}.\end{aligned} \tag{5.28}$$

Theorem 7 *The following integral formula holds true:*

$$\int_0^1 t^{c-1}(1-t)^{d-c-1}\ \ln^n\left(\frac{t}{1-t}\right)\ {}_3F_2\left[\begin{array}{ccc} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & d; & \end{array} t\right]\mathrm{d}t$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(d\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial c^r}\cdot\frac{\partial^{n-r-1}\ B}{\partial c^{n-r-1}}$$

$$+\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(d\right)\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ C}{\partial c^r}\cdot\frac{\partial^{n-r-1}\ D}{\partial c^{n-r-1}} \tag{5.29}$$

where

$$A = \frac{\Gamma(c)\Gamma(d-c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{5.30}$$

$$\frac{\partial\ A}{\partial c} = A\cdot B, \tag{5.31}$$

$$B = \Psi(c)+\Psi\left(c+\frac{1}{2}\right)-\Psi(d-c)+\Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)$$
$$-\Psi\left(c-\frac{1}{2}a+\frac{1}{2}\right)-\Psi\left(c-\frac{1}{2}b+\frac{1}{2}\right), \tag{5.32}$$

$$\frac{\partial^n\ A}{\partial c^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial c^r}\cdot\frac{\partial^{n-r-1}\ B}{\partial c^{n-r-1}}, \tag{5.33}$$

$$\frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}} = (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,c)+\zeta\left(n-r,c+\frac{1}{2}\right)$$
$$-(-1)^{n-r-1}\zeta\left(n-r,d-c\right)+\zeta\left(n-r,c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)$$
$$-\zeta\left(n-r,c-\frac{1}{2}a+\frac{1}{2}\right)-\zeta\left(n-r,c-\frac{1}{2}b+\frac{1}{2}\right)\Bigg\}, \tag{5.34}$$

$$C = \frac{\left(\frac{2c}{d}-1\right)\ \Gamma(c)\Gamma(d-c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}, \tag{5.35}$$

$$\frac{\partial\ C}{\partial c} = C\cdot D, \tag{5.36}$$

$$D = \frac{2}{2c-d}+\Psi(c)+\Psi(d-c)+\Psi\left(c+\frac{1}{2}\right)+\Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)$$
$$-\Psi\left(c-\frac{1}{2}a+1\right)-\Psi\left(c-\frac{1}{2}b+1\right), \tag{5.37}$$

$$\frac{\partial^n C}{\partial c^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r C}{\partial c^r} \cdot \frac{\partial^{n-r-1} D}{\partial c^{n-r-1}}, \tag{5.38}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1} D}{\partial c^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Bigg\{2^{n-r+1}(2c-d)^{-(n-r)} + \zeta(n-r,c) \\ &+ \zeta\left(n-r, c+\frac{1}{2}\right) - (-1)^{n-r-1}\zeta(n-r, d-c) \\ &+ \zeta\left(n-r, c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \zeta\left(n-r, c-\frac{1}{2}a+1\right) \\ &- \zeta\left(n-r, c-\frac{1}{2}b+1\right)\Bigg\}. \end{aligned} \tag{5.39}$$

The proofs of (5.18) and (5.29) are omitted since they are very similar to the one of (5.1).

We now shift our focus to some special cases of the first integral formula (5.1). Several interesting special cases could also be obtained from (5.18) and (5.29). For brevity, we decided to restrict ourselves to special cases of the first integral formula.

Case 1. If we set $a = c = \frac{1}{2}$, $d = 2$ in (5.1), and making use of the following known result [18, p. 513, Eq. (252)]:

$${}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & 3; \\ 2, & 2; & \end{matrix}\; x\right] = \frac{1}{\pi}\left[3K(\sqrt{x}) - 2\left(\frac{K(\sqrt{x}) - E(\sqrt{x})}{x}\right)\right], \tag{5.40}$$

where $K(x)$ holds for the complete elliptic integral of the first kind and $E(x)$ denotes the complete elliptic integral of the second kind, we obtain

$$\begin{aligned}&\int_0^1 t^{b-1}(1-t)^{\frac{1}{2}\left(\frac{3}{2}-b\right)-1} \ln^n\left(\frac{t}{\sqrt{1-t}}\right)\left[3K(\sqrt{t}) - 2\left(\frac{K(\sqrt{t}) - E(\sqrt{t})}{t}\right)\right] \mathrm{d}t \\ &= \frac{\pi^{\frac{3}{2}}}{\Gamma^2\left(\frac{3}{4}\right)} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r A}{\partial b^r} \cdot \frac{\partial^{n-r-1} B}{\partial b^{n-r-1}} \\ &\quad - \frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^{\frac{1}{2}}} \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r C}{\partial b^r} \cdot \frac{\partial^{n-r-1} D}{\partial b^{n-r-1}}, \end{aligned} \tag{5.41}$$

where

$$A = \frac{\Gamma(b)\Gamma^2\left(\frac{3}{4} - \frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}b\right)}, \tag{5.42}$$

$$\frac{\partial A}{\partial b} = A \cdot B, \tag{5.43}$$

$$B = \Psi(b) - \Psi\left(\frac{3}{4} - \frac{1}{2}b\right) - \Psi\left(\frac{1}{2}b + \frac{1}{2}\right) + \frac{1}{2}\Psi\left(1 - \frac{1}{2}b\right), \tag{5.44}$$

$$\frac{\partial^n A}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r A}{\partial b^r} \cdot \frac{\partial^{n-r-1} B}{\partial b^{n-r-1}}, \tag{5.45}$$

$$\begin{aligned}\frac{\partial^{n-r-1} B}{\partial b^{n-r-1}} =&(-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r-1}}\zeta\left(n-r,\frac{3}{4}-\frac{1}{2}b\right)\\ &- \frac{1}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}b+\frac{1}{2}\right) + \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,1-\frac{1}{2}b\right)\Bigg\},\end{aligned} \tag{5.46}$$

$$C = \frac{\Gamma(b)\Gamma^2\left(\frac{3}{4}-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}b\right)}, \tag{5.47}$$

$$\frac{\partial C}{\partial b} = C \cdot D, \tag{5.48}$$

$$D = \Psi(b) - \Psi\left(\frac{3}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b\right) + \frac{1}{2}\Psi\left(\frac{3}{2}-\frac{1}{2}b\right), \tag{5.49}$$

$$\frac{\partial^n C}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r C}{\partial b^r} \cdot \frac{\partial^{n-r-1} D}{\partial b^{n-r-1}}, \tag{5.50}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1} D}{\partial b^{n-r-1}} =&(-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r-1}}\zeta\left(n-r,\frac{3}{4}-\frac{1}{2}b\right)\\ &- \frac{1}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}b\right) + \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{3}{2}-\frac{1}{2}b\right)\Bigg\}.\end{aligned} \tag{5.51}$$

Case 2. Putting $a = c = 1$ and $d = 1$ in (5.1) and with the help of the following result [18, p. 519, Eq. (384)]:

$${}_3F_2\left[\begin{matrix}1, & 1, & \frac{3}{2}; \\ 3, & \frac{1}{2}; & \end{matrix}\; x\right] = -\frac{2}{x}\left[3 + \frac{3-x}{x}\ln(1-x)\right], \tag{5.52}$$

we obtain

$$\begin{aligned}&\int_0^1 t^{b-2}(1-t)^{-\frac{1}{2}b}\ \ln^n\left(\frac{t}{\sqrt{1-t}}\right)\ \left[3 + \frac{(3-t)}{t}\ln(1-t)\right]\mathrm{d}t\\ &= -\frac{\pi}{4}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r A}{\partial b^r}\cdot\frac{\partial^{n-r-1} B}{\partial b^{n-r-1}} - \frac{3}{2}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r C}{\partial b^r}\cdot\frac{\partial^{n-r-1} D}{\partial b^{n-r-1}},\end{aligned} \tag{5.53}$$

where

$$A = \frac{\Gamma(b)\Gamma^2\left(1-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}b\right)}, \tag{5.54}$$

$$\frac{\partial\ A}{\partial b} = A \cdot B, \tag{5.55}$$

$$B = \Psi(b) - \Psi\left(1 - \frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b + \frac{1}{2}\right) + \frac{1}{2}\Psi\left(\frac{3}{2} - \frac{1}{2}b\right), \tag{5.56}$$

$$\frac{\partial^n\ A}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r\ A}{\partial b^r} \cdot \frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}}, \tag{5.57}$$

$$\begin{aligned}\frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r-1}}\zeta\left(n-r, 1 - \frac{1}{2}b\right) \\ & - \frac{1}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}b + \frac{1}{2}\right) + \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, \frac{3}{2} - \frac{1}{2}b\right)\Bigg\},\end{aligned} \tag{5.58}$$

$$C = \frac{\Gamma(b)\Gamma^2\left(1 - \frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(2 - \frac{1}{2}b\right)}, \tag{5.59}$$

$$\frac{\partial\ C}{\partial b} = C \cdot D, \tag{5.60}$$

$$D = \Psi(b) - \Psi\left(1 - \frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b\right) + \frac{1}{2}\Psi\left(2 - \frac{1}{2}b\right), \tag{5.61}$$

$$\frac{\partial^n\ C}{\partial b^n} = \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r\ C}{\partial b^r} \cdot \frac{\partial^{n-r-1}\ D}{\partial b^{n-r-1}}, \tag{5.62}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}\ D}{\partial b^{n-r-1}} =& (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r-1}}\zeta\left(n-r, 1 - \frac{1}{2}b\right) \\ & - \frac{1}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}b\right) + \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, 2 - \frac{1}{2}b\right)\Bigg\}.\end{aligned} \tag{5.63}$$

Case 3. Setting $a = \frac{5}{2},\ c = 1,\ d = \frac{1}{2}$ in (5.1) and making use of the following result [18, p. 521, Eq. (410)]:

$${}_3F_2\left[\begin{matrix} \frac{5}{2}, & 1, & \frac{3}{2}; \\ 3, & \frac{1}{2}; & \end{matrix}\ x\right] = \frac{4}{9x^2}\left[6 + x - 2\frac{(3-4x)}{\sqrt{1-x}}\right], \tag{5.64}$$

we have

$$\begin{aligned}\int_0^1 t^{b-3}(1-t)^{\frac{1}{2}\left(\frac{7}{2}-b\right)-1}\ \ln^n\left(\frac{t}{\sqrt{1-t}}\right)\ \left[6 + t - 2\frac{(3-4t)}{\sqrt{1-t}}\right] \mathrm{d}t =& \frac{1}{2\sqrt{2}}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ A}{\partial b^r} \cdot \frac{\partial^{n-r-1}\ B}{\partial b^{n-r-1}} \\ & + \frac{9}{2\sqrt{2}}\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\ C}{\partial b^r} \cdot \frac{\partial^{n-r-1}\ D}{\partial b^{n-r-1}}\end{aligned} \tag{5.65}$$

where

$$A = \frac{\Gamma(b)\Gamma\left(\frac{7}{4}-\frac{1}{2}b\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\frac{1}{2}b\right)}, \tag{5.66}$$

$$\frac{\partial\, A}{\partial b} = A \cdot B, \tag{5.67}$$

$$B = \Psi(b) - \frac{1}{2}\Psi\left(\frac{7}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b+\frac{1}{2}\right) + \frac{1}{2}\Psi\left(\frac{3}{2}-\frac{1}{2}b\right), \tag{5.68}$$

$$\frac{\partial^n\, A}{\partial b^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\, A}{\partial b^r}\cdot\frac{\partial^{n-r-1}\, B}{\partial b^{n-r-1}}, \tag{5.69}$$

$$\begin{aligned}\frac{\partial^{n-r-1}\, B}{\partial b^{n-r-1}} =&(-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{7}{4}-\frac{1}{2}b\right)\\ &- \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{1}{4}-\frac{1}{2}b\right) - \frac{1}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}b+\frac{1}{2}\right)\\ &+ \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{3}{2}-\frac{1}{2}b\right)\Bigg\},\end{aligned} \tag{5.70}$$

$$C = \frac{\Gamma(b)\Gamma\left(\frac{7}{4}-\frac{1}{2}b\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(2-\frac{1}{2}b\right)}, \tag{5.71}$$

$$\frac{\partial\, C}{\partial b} = C \cdot D, \tag{5.72}$$

$$D = \Psi(b) - \frac{1}{2}\Psi\left(\frac{7}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b\right) + \frac{1}{2}\Psi\left(2-\frac{1}{2}b\right), \tag{5.73}$$

$$\frac{\partial^n\, C}{\partial b^n} = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r\, C}{\partial b^r}\cdot\frac{\partial^{n-r-1}\, D}{\partial b^{n-r-1}}, \tag{5.74}$$

and

$$\begin{aligned}\frac{\partial^{n-r-1}\, D}{\partial b^{n-r-1}} =&(-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{7}{4}-\frac{1}{2}b\right)\\ &- \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{1}{4}-\frac{1}{2}b\right) - \frac{1}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}b\right)\\ &+ \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,2-\frac{1}{2}b\right)\Bigg\}.\end{aligned} \tag{5.75}$$

Obviously, using other known results such as [18, p. 519, Eq. (364)] and [18, p. 519, Eq. (380)], that is,

$$ {}_3F_2\left[\begin{matrix}1, & 1, & \frac{5}{2}; \\ 3, & \frac{3}{2}; & \end{matrix}\; x\right] = -\frac{2}{3x}\left[1+\frac{(1-x)}{x}\ln(1-x)\right] \tag{5.76}$$

and

$$ {}_3F_2\left[\begin{matrix}2, & 1, & \frac{3}{2}; \\ 3, & \frac{1}{2}; & \end{matrix}\; x\right] = \frac{2}{x}\left[\frac{3-x}{1-x}+\frac{3}{x}\ln(1-x)\right], \tag{5.77}$$

we can obtain other interesting special cases of (5.1).

6. New class of finite double integrals involving the generalized hypergeometric function

In this section, we shall establish a new class of double integrals involving the generalized hypergeometric function. We first begin by giving six integrals formulas in the form of six theorems. The proofs of these double integrals are essentially the same as those given in Section 4, so we omit the details. Note that while deriving the integrals, we shall use the following well-known double integrals due to Edwards [7]:

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\mathrm{d}x\,\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{6.1}$$

provided that $\Re(\alpha)>0$ and $\Re(\beta)>0$.

Theorem 8 *Under the conditions given in (4.1), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{b}(1-x)^{b-1}(1-y)^{\frac{1}{2}(1+a-b)-1}(1-xy)^{1-\frac{1}{2}(1+a+b)}\cdot\, {}_3F_2\left[\begin{matrix}a, & c, & d+1; \\ 2c+1, & d; & \end{matrix}\; \frac{y(1-x)}{1-xy}\right] dx\, dy = U_1, \tag{6.2}$$

where U_1 is the right-hand side of (4.1).

Theorem 9 *Under the conditions given in (4.2), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{c}(1-x)^{c-1}(1-y)^{c}(1-xy)^{-2c}\, {}_3F_2\left[\begin{matrix}a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & d; & \end{matrix}\; \frac{y(1-x)}{1-xy}\right] dx\, dy = U_2, \tag{6.3}$$

where U_2 is the right-hand side of (4.2).

Theorem 10 *Under the conditions given in (4.3), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{c}(1-x)^{c-1}(1-y)^{d-c-1}(1-xy)^{1-d}\cdot\, {}_3F_2\left[\begin{matrix}a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & 2c+1; & \end{matrix}\; \frac{y(1-x)}{1-xy}\right] dx\, dy = U_3, \tag{6.4}$$

where U_3 is the right-hand side of (4.3).

Theorem 11 *Under the conditions given in (4.1), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{\frac{1}{2}(1+a-b)}(1-x)^{\frac{1}{2}(1+a-b)-1}(1-y)^{b-1}(1-xy)^{1-\frac{1}{2}(1+a+b)} \cdot {}_3F_2\left[\begin{array}{lll} a, & c, & d+1; \\ 2c+1, & d; & \end{array} \frac{1-y}{1-xy}\right] dx\, dy = U_1, \tag{6.5}$$

where U_1 is the right-hand side of (4.1).

Theorem 12 *Under the conditions given in (4.2), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{c+1}(1-x)^{c}(1-y)^{c-1}(1-xy)^{-2c} \cdot {}_3F_2\left[\begin{array}{lll} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & d; & \end{array} \frac{1-y}{1-xy}\right] dx\, dy = U_2, \tag{6.6}$$

where U_2 is the right-hand side of (4.2).

Theorem 13 *Under the conditions given in (4.3), the following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{d-c}(1-x)^{d-c-1}(1-y)^{c-1}(1-xy)^{1-d} \cdot {}_3F_2\left[\begin{array}{lll} a, & b, & d+1; \\ \frac{1}{2}(a+b+1), & 2c+1; & \end{array} \frac{1-y}{1-xy}\right] dx\, dy = U_3, \tag{6.7}$$

where U_3 is the right-hand side of (4.3).

Remark 5 *A large number of special cases of our double integrals (6.2) to (6.7) can be obtained in a way similar to that in Sections 4 and 5, so we again prefer to omit the details.*

7. New class of finite double integrals involving the generalized hypergeometric function and logarithmic function

We end this paper by presenting a new class of double integrals involving the generalized hypergeometric function and logarithmic function. Since the proofs of these integrals are similar to those of the previous section, we choose to omit the details. We simply state our double integrals in the form of theorems.

Theorem 14 *The following double integrals formula holds true:*

$$\int_0^1\int_0^1 y^{b}(1-x)^{b-1}(1-y)^{\frac{1}{2}(1+a-b)-1}(1-xy)^{1-\frac{1}{2}(1+a+b)} \cdot \ln^n\left(\frac{y(1-x)}{\sqrt{(1-y)(1-xy)}}\right) {}_3F_2\left[\begin{array}{lll} a, & c, & d+1; \\ 2c+1, & d; & \end{array} \frac{y(1-x)}{1-xy}\right] dx\, dy = V_1, \tag{7.1}$$

where V_1 is the right-hand side of (5.1).

Theorem 15 *The following double integrals formula holds true:*

$$\begin{aligned}
&\int_0^1\int_0^1 y^{\frac{1}{2}(1+a-b)}(1-x)^{\frac{1}{2}(1+a-b)-1}(1-y)^{b-1}(1-xy)^{1-\frac{1}{2}(1+a+b)} \\
&\qquad \cdot \ln^n\left(\frac{(1-y)}{\sqrt{y(1-x)(1-xy)}}\right) {}_3F_2\left[\begin{matrix} a, & c, & d+1; & \dfrac{1-y}{1-xy} \\ 2c+1, & d; & & \end{matrix}\right] dx\, dy \\
&= V_1, \qquad (7.2)
\end{aligned}$$

where V_1 is the right-hand side of (5.1).

Theorem 16 *The following double integrals formula holds true:*

$$\begin{aligned}
&\int_0^1\int_0^1 y^{c}(1-x)^{c-1}(1-y)^{c}(1-xy)^{-2c} \\
&\qquad \cdot \ln^n\left(\frac{y(1-x)(1-y)}{(1-xy)^2}\right) {}_3F_2\left[\begin{matrix} a, & c, & d+1; & \dfrac{y(1-x)}{1-xy} \\ \frac{1}{2}(a+b+1), & d; & & \end{matrix}\right] dx\, dy \\
&= V_2, \qquad (7.3)
\end{aligned}$$

where V_2 is the right-hand side of (5.18).

Theorem 17 *The following double integrals formula holds true:*

$$\begin{aligned}
&\int_0^1\int_0^1 y^{c+1}(1-x)^{c}(1-y)^{c-1}(1-xy)^{-2c} \\
&\qquad \cdot \ln^n\left(\frac{y(1-x)(1-y)}{(1-xy)^2}\right) {}_3F_2\left[\begin{matrix} a, & c, & d+1; & \dfrac{1-y}{1-xy} \\ \frac{1}{2}(a+b+1), & d; & & \end{matrix}\right] dx\, dy \\
&= V_2, \qquad (7.4)
\end{aligned}$$

where V_2 is the right-hand side of (5.18).

8. Competing interests

The authors declare that they have no competing interests.

9. Author's contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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