Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2018) 42: 487 – 501

doi:10.3906/mat-1611-57

On a biharmonic equation involving slightly supercritical exponent

Kamal OULD BOUH*
Department of Mathematics, Taibah University, Medina, Saudi Arabia

Received: 14.11.2016 • **Accepted/Published Online:** 19.05.2017 • **Final Version:** 24.03.2018

Abstract: We consider the biharmonic equation with supercritical nonlinearity (P_ε) : $\Delta^2 u = K|u|^{8/(n-4)+\varepsilon}u$ in Ω, $\Delta u = u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, K is a C^3 positive function, and ε is a positive real parameter. In contrast with the subcritical case, we prove the nonexistence of sign-changing solutions of (P_ε) that blow up at two near points. We also show that (P_ε) has no bubble-tower sign-changing solutions.

Key words: Sign-changing solutions, bubble-tower solution, fourth-order equation, supercritical exponent

1. Introduction

The study of concentration phenomena for second-order elliptic equations involving a nearly critical exponent has attracted considerable attention in the last decades. In this paper, we are concerned with the concentration phenomena of the following biharmonic equation under the Navier boundary condition:

$$(P_\varepsilon) \qquad \begin{cases} \Delta^2 u = K|u|^{p-1+\varepsilon}u & \text{in } \Omega, \\ \quad \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, $p+1 = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding of $H^2(\Omega) \times H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, K is a C^3 positive function, and ε is a small positive parameter.

This type of equation naturally arises from the study of conformal geometry. A well-known example is the problem of prescribing the Paneitz curvature: given a function K defined in compact Riemannian manifold (M, g) of dimension $n \geq 5$, we ask whether there exists a metric $\overline{g}$ conformal to g such that K is the Paneitz curvature of the new metric $\overline{g}$ (for details one can see [6, 10, 13, 15] and the references therein).

The concentration phenomena for second-order elliptic equations (P_ε) involving a nearly subcritical exponent ($\varepsilon \in (1-p, 0)$) were studied in [11, 15, 17] for $K \not\equiv 1$ and [5, 8] for $K \equiv 1$ only.

In the critical case (when $\varepsilon = 0$), the limiting problem exhibits a lack of compactness. In fact, van der Vorst showed in [22, 23] that (P_0) has no positive solutions if Ω is a star-shaped domain, whereas Ebobisse and Ould Ahmedou proved in [14] that (P_0) has a positive solution provided that some homology group of Ω is nontrivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a positive solution exists shows [16].

*Correspondence: hbouh@taibahu.edu.sa
2000 *AMS Mathematics Subject Classification:* 35J20, 35J60

For the supercritical case, $\varepsilon > 0$ and K is a constant, it was proved in [19] that for ε small, (P_ε) has no sign-changing solutions that blow up at two points. This result shows that the situation is different from the subcritical one. In this paper, we consider the case in which K is a nonconstant function and we seek to understand the influence of the function K in the study of the sign-changing solutions of (P_ε). We note that, when the biharmonic operator in (P_ε) is replaced by the Laplacian one, there are many works devoted to the study of the solutions of the counterpart of (P_ε), for example [3, 4, 7, 9, 12, 18].

To state our results, we need to introduce some notations and assumptions that we are using. We denote by G the Green function of Δ^2, that is,

$$\forall x \in \Omega, \quad \Delta^2 G(x,.) = c_n \delta_x \quad \text{in} \quad \Omega, \quad \Delta G(x,.) = G(x,.) = 0 \quad \text{on} \quad \partial\Omega,$$

where δ_x is the Dirac mass at x and $c_n = (n-4)(n-2)w_n$, with w_n the area of the unit sphere of $\mathbb{R}^n$. We denote by H the regular part of G, that is,

$$H(x_1, x_2) = |x_1 - x_2|^{4-n} - G(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \Omega^2.$$

Let

$$\delta_{(a,\mu)}(x) = c_0 \frac{\mu^{(n-4)/2}}{(1+\mu^2|x-a|^2)^{(n-4)/2}}, \quad c_0 = \Big(n(n-4)(n^2-4)\Big)^{(n-4)/8}, \ \mu > 0, \ a \in \mathbb{R}^n, \tag{1.1}$$

and $P\delta_{(a,\mu)}$ denotes the projection of the $\delta_{(a,\mu)}$s onto $H^2(\Omega) \cap H_0^1(\Omega)$. It is defined by

$$\Delta^2 P\delta_{(a,\mu)} = \Delta^2 \delta_{(a,\mu)} \quad \text{in } \Omega; \qquad \Delta P\delta_{(a,\mu)} = P\delta_{(a,\mu)} = 0 \quad \text{on } \partial\Omega.$$

Notice that the family $\delta_{(a,\mu)}$ comprises the only minimizers of the Sobolev inequality on the whole space, that is,

$$S = \Big\{ \|\Delta u\|^2_{L^2(\mathbb{R}^n)} \|u\|^{-2}_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}, \text{ such that } \Delta u \in L^2, u \in L^{\frac{2n}{n-4}}, u \neq 0 \Big\}.$$

The space $H^2(\Omega) \cap H_0^1(\Omega)$ is equipped with the norm $\|.\|$ and its corresponding inner product $\langle ., . \rangle$ defined by

$$\|u\| = \left(\int_\Omega |\Delta u|^2 \right)^{1/2} \qquad \text{and} \qquad \langle u, v \rangle = \int_\Omega \Delta u \Delta v, \qquad u, v \in H^2(\Omega) \times H_0^1(\Omega). \tag{1.2}$$

Let:

(H_1) For each critical point y of K, we have

$$c_1 H(y,y) - \frac{c_3 \Delta K(y)}{36 K(y)} \neq 0 \quad \text{if } n = 6 \quad \text{and} \quad \Delta K(y) \neq 0 \quad \text{if } n \geq 7,$$

where c_1, c_3 are positive constants defined in Proposition 2.4.

(H_2) All the critical points of K are in Ω, i.e. there exists positive constant c so that $|\nabla K(y)| \geq c$, for each $y \in \partial\Omega$.

In the first result, we prove that there are no sign-changing solutions of (P_ε) with $\varepsilon > 0$ that have two peaks concentrated at the same point. More precisely, we have:

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$, and the assumptions ($H_1 - H_2$) hold. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε that satisfy*

$$u_\varepsilon = P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})} - P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})} + v_\varepsilon, \qquad \text{with } |u_\varepsilon|_\infty^\varepsilon \text{ is bounded,} \tag{1.3}$$

$$\begin{cases} a_{\varepsilon,i} \in \Omega,\ \mu_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to \infty \text{ for } i = 1, 2, \\ \langle P\delta_{(a_{\varepsilon,1},\mu_{\varepsilon,1})}, P\delta_{(a_{\varepsilon,2},\mu_{\varepsilon,2})}\rangle \to 0\ ,\ ||v_\varepsilon|| \to 0 \text{ as } \varepsilon \to 0, \end{cases} \tag{1.4}$$

and $|a_{\varepsilon,1} - a_{\varepsilon,2}| < \sigma$ where σ is a positive constant such that $\sigma < (1/2)\inf\{|y_i - y_j|,\ i \neq j,\ \nabla K(y_l) = 0,\ l = i, j\}$.

In the following result, we give a sufficient condition on the function K to ensure the nonexistence of sign-changing solutions of (P_ε) with $\varepsilon > 0$.

Theorem 1.2 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$, and the assumptions ($H_1 - H_2$) hold. Assume that there exists at most one critical point y of K satisfying*

$$c_1 H(y, y) - \frac{c_3 \Delta K(y)}{16 K(y)} < 0 \quad \text{if } n = 6, \quad \text{and} \quad \Delta K(y) > 0 \quad \text{if } n \geq 7. \tag{1.5}$$

Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε that satisfy (1.3) and (1.4).

Remark 1.3 *We notice that Theorems 1.1, 1.2 of [19], which are proved in the case $K \equiv 1$, are also true for all C^3 function K in dimension 5.*

Observe that, in the case of the Laplacian operator, all positive solutions blow up with comparable speeds, but for sign-changing solutions, Pistoia and Weth [20] constructed solutions (u_ε) with many bubbles blowing up at the same point, "*bubble-tower solutions*" ($\mu_i/\mu_j \to \infty$ or 0), which cannot appear in the case of the positive solutions (by using the Harnack inequalities). This is a new phenomenon for sign-changing solutions compared with the positive one. In our case, we prove that this phenomenon cannot appear when $\varepsilon > 0$. In fact, we prove that:

Theorem 1.4 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$, and the assumptions ($H_1 - H_2$) hold. Then there exists $\varepsilon_0 > 0$, such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε of the form*

$$u_\varepsilon = \sum_{i=1}^{p} \gamma_i P\delta_{(a_{\varepsilon,i},\mu_{\varepsilon,i})} + v_\varepsilon, \qquad \text{with } \gamma_i \in \{1, -1\}, |u_\varepsilon|_\infty^\varepsilon \text{ is bounded,} \tag{1.6}$$

$$\begin{cases} p \geq 2,\ \mu_{\varepsilon,i}/\mu_{\varepsilon,i+1} \to 0,\ a_{\varepsilon,i} \in \Omega,\ \mu_{\varepsilon,i} d(a_{\varepsilon,i}, \partial\Omega) \to \infty \text{ for } 1 \leq i \leq p, \\ |a_{\varepsilon,i} - a_{\varepsilon,j}| < \sigma,\ \langle P\delta_{(a_{\varepsilon,i},\mu_{\varepsilon,i})}, P\delta_{(a_{\varepsilon,j},\mu_{\varepsilon,j})}\rangle \to 0,\ i \neq j,\ ||v_\varepsilon|| \to 0 \text{ as } \varepsilon \to 0, \end{cases} \tag{1.7}$$

and $\exists\, q_1, q_2$ ($q_1 < q_2$), $\mu_{\varepsilon,q_1}|a_{\varepsilon,q_1} - a_{\varepsilon,q_2}|$ is bounded, $\gamma_i \neq \gamma_p$ for $q_1 \leq i < p$.

Theorem 1.5 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 6$, and the assumptions ($H_1 - H_2$) hold. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε that satisfy (1.6), (1.7), and there exists q, $\mu_{\varepsilon,q}|a_{\varepsilon,q} - a_{\varepsilon,q+1}|$ is bounded, $\gamma_q \neq \gamma_{q+1}$, and $\mu_{\varepsilon,q}|a_{\varepsilon,q} - a_{\varepsilon,i}| \to +\infty$ for $i > q + 1$.*

We notice that if $\mu_{\varepsilon,i}|a_{\varepsilon,i}-a_{\varepsilon,j}|$ is bounded for $i<j$, then q in the previous theorem is equal to $p-1$ and the set $\{i>q+1\}$ is empty in this case. Furthermore, if $\gamma_{p-1}=\gamma_p$, Theorem 1.5 also holds in this case by adding the following assumption : $\mu_{\varepsilon,l}|a_{\varepsilon,l}-a_{\varepsilon,l+1}|$ is very small if $l\neq p-2$ where $l=\min\{r:\ \gamma_r=...=\gamma_p\}$

The remainder of the present paper is organized as follows. In Section 2, we assume that there exist solutions u_ε of (P_ε) that satisfy (1.3) and (1.4). Using some information about this solution, we derive some useful estimates. In Section 3, combining the estimates obtained in the last section, we prove sign-changing solution results by contradiction. In Section 4, we prove bubble-tower solution results (Theorems 1.4, 1.5).

2. Preliminary results

In this section, we assume that there exist solutions (u_ε) of (P_ε) that satisfy

$$u_\varepsilon = P\delta_{(a_{(\varepsilon,1)},\mu_{(\varepsilon,1)})} - P\delta_{(a_{(\varepsilon,2)},\mu_{(\varepsilon,2)})} + v_\varepsilon, \tag{2.1}$$

with $|u_\varepsilon|^\varepsilon_\infty$ is bounded, $a_{\varepsilon,i}\in\Omega$, for $i=1,2$, and $\|v_\varepsilon\|\to 0$, $\langle P\delta_{(a_{(\varepsilon,1)},\mu_{(\varepsilon,1)})}, P\delta_{(a_{(\varepsilon,2)},\mu_{(\varepsilon,2)})}\rangle \to 0$, $\mu_{(\varepsilon,i)}d(a_{(\varepsilon,i)},\partial\Omega)\to+\infty$ as $\varepsilon\to 0$.

First, arguing as in [2, 21], we see that for u_ε satisfying (2.1), there is a unique way to choose α_i, a_i, μ_i, and v such that

$$u_\varepsilon = \alpha_1 P\delta_{(a_1,\mu_1)} - \alpha_2 P\delta_{(a_2,\mu_2)} + v, \tag{2.2}$$

with as $\varepsilon\to 0$

$$\begin{cases} \alpha_i\in\mathbb{R}, \quad \frac{\alpha_i^{8/(n-4)}K(a_i)}{\alpha_j^{8/(n-4)}K(a_j)}\to 1,\\ a_i\in\Omega, \quad \mu_i\in\mathbb{R}_+^*, \quad \mu_i d(a_i,\partial\Omega)\to+\infty,\\ v\to 0 \quad \text{in } H^2(\Omega)\times H_0^1(\Omega), \quad v\in E, \end{cases} \tag{2.3}$$

where E denotes the subspace of $H_0^1(\Omega)$ defined by

$$E := \left\{ w : \langle w,\varphi\rangle = 0 \quad \forall\varphi\in\{P\delta_i, \partial P\delta_i/\partial\mu_i, \partial P\delta_i/\partial a_i^j,\ i\leq 2; j\leq n\}\right\}. \tag{2.4}$$

Here, a_i^j denotes the jth component of a_i and in the sequel, in order to simplify the notations, we set

$$a_{(\varepsilon,i)} = a_i\,,\ \mu_{(\varepsilon,i)}=\mu_i\,,\ \delta_{(a_i,\mu_i)}=\delta_i,\ \text{and}\ P\delta_{(a_i,\mu_i)} = P\delta_i.$$

In the following, we always assume that u_ε (which satisfies (2.1)) is written as in (2.2) and (2.3) holds.

Lemma 2.1 *Let u_ε satisfy the assumption of the theorems. μ_i occurring in (2.2) satisfies*

$$\mu_i^\varepsilon\to 1 \quad \textit{as } \varepsilon\to 0, \qquad \textit{for each} \quad i=1,2.$$

Proof The proof is the same as that of Lemma 2.2 of [19], so we omit it. □

Remark 2.2 *From Lemma 2.1, we remark that:*

(i) Since Ω is bounded and $\mu_i^\varepsilon\to 1$ as $\varepsilon\to 0$ it is easy to derive that $\varepsilon\log(1+\mu_i^2|x-a_i|^2)$ tends to 0 as $\varepsilon\to 0$ and therefore we get:

$$\delta_i^\varepsilon(x) - c_0^\varepsilon\mu_i^{\varepsilon(n-4)/2} = O\left(\varepsilon\log(1+\mu_i^2|x-a_i|^2)\right) \qquad \textit{in } \Omega.$$

(ii) We also point out that it follows from assumption that $|u_\varepsilon|_\infty^\varepsilon$ *is bounded and* $\mu_i^\varepsilon \to 1$ *as* $\varepsilon \to 0$ *that* $|v_\varepsilon|_\infty^\varepsilon$ *is bounded, a fact that is used in the proof of Lemma 2.3 and Proposition 2.4 below.*

As usual in these types of problems, we first deal with the v-part of u in order to show that it is negligible with respect to the concentration phenomenon. Namely, we have the following estimate:

Lemma 2.3 *The function* v *defined in* (2.2) *satisfies the following estimate:*

$$||v|| \leq c\varepsilon + c \begin{cases} \sum_i \left(\frac{|\nabla K(a_i)|}{\mu_i} + \frac{1}{\mu_i^2} + \frac{1}{(\mu_i d_i)^{n-4}} \right) + \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{n-4}{n}} & \text{if } n < 12, \\ \sum_i \left(\frac{|\nabla K(a_i)|}{\mu_i} + \frac{1}{\mu_i^2} + \frac{1}{(\mu_i d_i)^{(n+4)/2 - \varepsilon(n-4)}} \right) + \varepsilon_{12}^{\frac{n+4}{2(n-4)}} (\log \varepsilon_{12}^{-1})^{\frac{n+4}{2n}} & \text{if } n \geq 12, \end{cases}$$

where $d_i := d(a_i, \partial\Omega)$ *and* ε_{12} *is defined by*

$$\varepsilon_{12} = \left(\frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1} + \mu_1\mu_2|a_1 - a_2|^2 \right)^{(4-n)/2}. \tag{2.5}$$

Proof Since $u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ is a solution of (P_ε) and $v \in E$ (see (2.4)), we obtain

$$\int_\Omega -\Delta u_\varepsilon v = \|v\|^2 = \int_\Omega K|u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon v = \int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)v$$
$$+ p \int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v^2 + o(\|v\|^2).$$

Hence, we have

$$Q(v, v) = f(v) + o(\|v\|^2), \tag{2.6}$$

where

$$Q(v, v) = \|v\|^2 - p \int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v^2,$$
$$f(v) = \int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)v.$$

Using Remark 2.2 and according to [1], it is easy to see that

$$Q(v, v) = \|v\|^2 - p \sum_{i=1,2} \alpha_i^{p-1+\varepsilon} K(a_i) \int_\Omega (P\delta_i)^{p-1+\varepsilon} v^2 + o(\|v\|^2)$$

is positive definite, that is, there exists $c > 0$ independent of ε, satisfying $Q(v, v) \geq c\|v\|^2$, for each $v \in E$. Then, from (2.6), we get

$$\|v\|^2 = O(\|f(v)\|).$$

Now, using Lemma 2.1, we obtain

$$f(v) = \int_\Omega K((\alpha_1 P\delta_1)^{p+\varepsilon} - (\alpha_2 P\delta_2)^{p+\varepsilon})v + O\Big(\int_\Omega (\delta_i\delta_j)^{\frac{p}{2}} |v| + \sum_{i \neq j} \int_\Omega \delta_i^{p-1}\delta_j |v| (\text{if } n < 6) \Big). \tag{2.7}$$

Using Remark 2.2 and the fact that $v \in E$, we get

$$
\begin{aligned}
|\int_\Omega KP\delta_i^{p+\varepsilon} v| &= |\int K\delta_i^{p+\varepsilon} v| + O\Big(\int \delta_i^{p-1+\varepsilon}\theta_i|v|\Big)\\
&\le \int K\Big(c_0^\varepsilon \mu_i^\varepsilon + O(\varepsilon \log(1+\mu_i^2|x-a_i|^2))\Big)\delta_i^p|v| + c|\theta_i|_{L^\infty}\int \delta_i^{p-1+\varepsilon}|v|\\
&\le c\|v\|\Big(\varepsilon + \frac{|\nabla K(a_i)|}{\mu_i} + \frac{1}{\mu_i^2} + \frac{1}{(\mu_i d_i)^{n-4}}(\text{if } n<12) + \frac{1}{(\mu_i d_i)^{n+4/2+\varepsilon(n-4)}}(\text{if } n \ge 12)\Big), \qquad (2.8)
\end{aligned}
$$

where $\theta_i := \theta_{a_i,\mu_i} := \delta_i - P\delta_i$.

For the other integrals of (2.7), we use Holder's inequality and we obtain for $i \neq j$

$$
\int_\Omega (\delta_i\delta_j)^{p/2} \mid v \mid \le c\|v\|\Big(\int_\Omega (\delta_i\delta_j)^{n/(n-4)}\Big)^{(n+4)/2n} \le c\|v\|\varepsilon_{ij}^{(n+4)/2(n-4)}(\log \varepsilon_{ij}^{-1})^{(n+4)/2n} \qquad (2.9)
$$

and if $n < 12$, we have $p-1 = 8/(n-4) > 1$; therefore,

$$
\int_\Omega \delta_i^{p-1}\delta_j \mid v \mid \le c\|v\|\Big(\int_\Omega (\delta_i\delta_j)^{n/(n-4)}\Big)^{(n-4)/n} \le c\|v\|\varepsilon_{ij}(\log \varepsilon_{ij}^{-1})^{(n-4)/n}. \qquad (2.10)
$$

Combining (2.7), ..., (2.10), the proof follows. □

Now we are able to obtain the following result, which is a crucial point in the proof of our theorems.

Proposition 2.4 *Assume that $n \ge 5$ and let α_i, a_i, and μ_i be the variables defined in (2.2). We have*

$$
1-\alpha_i^{8/(n-4)+\varepsilon}K(a_i) = O\Big(\varepsilon \log \mu_i + \sum_i \frac{1}{(\mu_i d_i)^{n-4}} + \varepsilon_{12} + \|v\|^2\Big), \qquad (2.11)
$$

$$
\begin{aligned}
&\Big|\alpha_i c_1 \frac{H(a_i,a_i)}{\mu_i^{n-4}} - \alpha_i \frac{c_3}{n^2}\frac{\Delta K(a_i)}{K(a_i)\mu_i^2} - \alpha_j c_1\Big(\frac{2\mu_i}{n-4}\frac{\partial \varepsilon_{12}}{\partial \mu_i} + \frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) + \alpha_i c_2 \varepsilon\Big|\\
&\le c\Big(\varepsilon^2 + \frac{1}{\mu_i^3} + \|v\|^2\Big) + c\begin{cases} \sum_k \frac{\log(\mu_k d_k)}{(\mu_k d_k)^{n-1}} + \varepsilon_{12}^{\frac{n}{n-4}}\log \varepsilon_{12}^{-1} (\text{if } n \ge 6),\\ \sum_k \frac{1}{(\mu_k d_k)^2} + \varepsilon_{12}^2(\log \varepsilon_{12}^{-1})^{\frac{2}{5}} (\text{if } n = 5),\end{cases} \qquad (2.12)
\end{aligned}
$$

where $i, j \in \{1,2\}$ with $i \neq j$ and c_1, c_2, c_3 are positive constants defined by

$$
c_1 = c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{(n+4)/2}}, \quad c_2 = \frac{n-4}{2}c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\log(1+|x|^2)\frac{|x|^2-1}{(1+|x|^2)^{n+1}}dx
$$
$$
\text{and} \quad c_3 = c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{|x|^2}{(1+|x|^2)^n}dx.
$$

Proof It suffices to prove the proposition for $i = 1$. Multiplying (P_ε) by $\mu_1\partial P\delta_1/\partial\mu_1$ and integrating on Ω, we obtain

$$
\alpha_1\int_\Omega \delta_1^p \mu_1\frac{\partial P\delta_1}{\partial \mu_1} - \alpha_2\int_\Omega \delta_2^p\mu_1\frac{\partial P\delta_2}{\partial\mu_2} = \int_\Omega K|u_\varepsilon|^{p-1+\varepsilon}u_\varepsilon \mu_1\frac{\partial P\delta_1}{\partial\mu_1}. \qquad (2.13)
$$

From [1], we know that

$$\int_\Omega \delta_1^p \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} = \frac{n-4}{2} c_1 \frac{H(a_1,a_1)}{\mu_1^{n-4}} + O\Big(\frac{\log(\mu_1 d_1)}{(\mu_1 d_1)^{n-1}}\Big), \tag{2.14}$$

$$\int_\Omega \delta_2^p \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} = c_1 \Big(\mu_1 \frac{\partial \varepsilon_{12}}{\partial \mu_1} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) + R, \tag{2.15}$$

where R satisfies

$$R = O\Big(\sum_{k=1,2} \frac{\log(\mu_k d_k)}{(\mu_k d_k)^{n-1}} + \varepsilon_{12}^{\frac{n}{n-4}} \log \varepsilon_{12}^{-1}\Big). \tag{2.16}$$

For the other term of (2.13), we have

$$\int_\Omega K|u_\varepsilon|^{p-1+\varepsilon} u_\varepsilon \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} = \int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)\mu_1 \frac{\partial P\delta_1}{\partial \mu_1}$$
$$+ (p+\varepsilon)\int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} + O\Big(\|v\|^2 + \varepsilon_{12}^{\frac{n}{n-4}} \log \varepsilon_{12}^{-1}\Big). \tag{2.17}$$

Concerning the last integral, it can be written as

$$\int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon} v \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} = \int_\Omega K(\alpha_1 P\delta_1)^{p-1+\varepsilon} v\mu_1 \frac{\partial P\delta_1}{\partial \mu_1}$$
$$+ O\Big(\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v|\Big), \tag{2.18}$$

where $A = \{x : 2\alpha_2 P\delta_2 \le \alpha_1 P\delta_1\}$.

Observe that, for $n \ge 12$, we have $p-1 = 8/(n-4) \le 1$; thus,

$$\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v| \le c\int_\Omega |v|(\delta_1\delta_2)^{(n+4)/2(n-4)}$$
$$\le c\|v\| \varepsilon_{12}^{(n+4)/2(n-4)} (\log \varepsilon_{12}^{-1})^{(n+4)/2n}. \tag{2.19}$$

However, for $n \le 11$, we have

$$\int_{\Omega\setminus A} P\delta_2^{p-1} P\delta_1 |v| + \int_A P\delta_1^{p-1} P\delta_2 |v| \le c\varepsilon_{12} (\log \varepsilon_{12}^{-1})^{(n-4)/n} ||v||. \tag{2.20}$$

For the other integral in (2.18), using [1, 21] and Remark 2.2, we get

$$\int_\Omega K P\delta_1^{p-1+\varepsilon} v \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} = O\Big(\|v\| \Big[\varepsilon + \Big(\frac{1}{(\mu_1 d_1)^{\inf(n-4,(n+4)/2)}}(if n \ne 12) + \frac{\log(\mu_1 d_1)}{(\mu_1 d_1)^4}(if n = 12)\Big)\Big]\Big).$$

It remains to estimate the second integral of (2.17). We have

$$\int_\Omega K|\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{p-1+\varepsilon}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)\mu_1 \frac{\partial P\delta_1}{\partial \mu_1}$$
$$= \int_\Omega K(\alpha_1 P\delta_1)^{p+\varepsilon} \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} - \int_\Omega K(\alpha_2 P\delta_2)^{p+\varepsilon}\mu_1 \frac{\partial P\delta_1}{\partial \mu_1} \tag{2.21}$$
$$- (p+\varepsilon)\int_\Omega K\alpha_2 P\delta_2 (\alpha_1 P\delta_1)^{p-1+\varepsilon} \mu_1 \frac{\partial P\delta_1}{\partial \mu_1} + O\Big(\varepsilon_{12}^{\frac{n}{n-4}} \log \varepsilon_{12}^{-1}\Big).$$

Now, using Remark 2.2 and [1], we have

$$\int_\Omega KP\delta_1^{p+\varepsilon}\mu_1\frac{\partial P\delta_1}{\partial\mu_1} = \frac{n-4}{2}\Big(K(a_1)c_2\varepsilon - c_3\frac{\Delta K(a_1)}{n^2\mu_1^2} + 2c_1K(a_1)\frac{H(a_1,a_1)}{\mu_1^{n-4}}\Big) + O\Big(\varepsilon^2 + \frac{1}{\mu_1^3} + \frac{\log(\mu_1 d_1)}{(\mu_1 d_1)^{n-1}} + \frac{1}{(\mu_1 d_1)^2}(\text{if } n=5)\Big), \tag{2.22}$$

$$\int_\Omega KP\delta_2^{p+\varepsilon}\mu_1\frac{\partial P\delta_1}{\partial\mu_1} = c_1K(a_2)\Big(\mu_1\frac{\partial\varepsilon_{12}}{\partial\mu_1} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) + R_1, \tag{2.23}$$

$$p\int_\Omega KP\delta_2 P\delta_1^{p-1+\varepsilon}\mu_1\frac{\partial P\delta_1}{\partial\mu_1} = c_1K(a_1)\Big(\mu_1\frac{\partial\varepsilon_{12}}{\partial\mu_1} + \frac{n-4}{2}\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) + R_1, \tag{2.24}$$

where $R_1 = R + O\Big(\varepsilon\varepsilon_{12}(\log\varepsilon_{12}^{-1})^{(n-4)/n}\Big)$, and R is defined by (2.16).

Therefore, combining (2.13), ..., (2.24), and Lemma 2.3, the proof of Proposition 2.4 follows. □

Proposition 2.5 *Let α_i, a_i, and μ_i be the variables defined in (2.2). Then, for $n \geq 6$, we have*

$$c_1\alpha_i\frac{1}{\mu_i^{n-3}}\frac{\partial H(a_i,a_i)}{\partial a_i} - c_4\alpha_i\frac{\nabla K(a_i)}{\mu_i} + 2c_1\frac{\alpha_j}{\mu_i}\Big(\frac{\partial\varepsilon_{12}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_1,a_2)\frac{1}{(\mu_1\mu_2)^{(n-4)/2}}\Big) = O\Big(\sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-2}} + \varepsilon_{12}^{\frac{n}{n-4}}\log\varepsilon_{12}^{-1} + \varepsilon\varepsilon_{12}(\log\varepsilon_{12}^{-1})^{\frac{n-4}{n}} + \frac{\varepsilon}{(\mu_i d_i)^{n-3}} + \frac{1}{\mu_i^3} + \|v\|^2\Big),$$

where $i, j \in \{1,2\}$, $j \neq i$.

Proof The proof is similar to the proof of Proposition 2.4, but there exist some integrals that have different estimates. In fact, the equations (2.13), (2.17), ..., (2.21) are also true if we change $\mu_1\partial\delta_1/\partial\mu_1$ by $\mu_1^{-1}\partial\delta_1/\partial a_1$. For the other integrals, we use Remark 2.2 and [1] and Proposition 2.5 follows. □

3. Proof of sign-changing solution results

We remark that, if we suppose that the problem (P_ε) has a solution u_ε as stated in (1.3) and (1.4), then this solution has to satisfy (2.2), and from Proposition 2.4, we have

$$(E_i)\quad c_1\frac{H(a_i,a_i)}{\mu_i^{n-4}} - \frac{c_3}{n^2}\frac{\Delta K(a_i)}{K(a_i)\mu_i^2} - c_1\Big(\frac{2\mu_i}{n-4}\frac{\partial\varepsilon_{12}}{\partial\mu_i} + \frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) + c_2\varepsilon = O\Big(\frac{|\nabla K(a_i)|^2}{\mu_i^2}\Big) + o\Big(\varepsilon + \frac{1}{\mu_i^2} + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}} + \varepsilon_{12}\Big), \quad \text{for } i = 1,2.$$

From Proposition 2.5, we get

$$c_1\frac{1}{\mu_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i} - c_4\frac{\nabla K(a_i)}{\mu_i} + 2c_1\frac{1}{\mu_i}\Big(\frac{\partial\varepsilon_{12}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_1,a_2)\frac{1}{(\mu_1\mu_2)^{(n-2)/2}}\Big) = o\Big(\sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-3}} + \varepsilon_{12}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}} + \frac{1}{\mu_i}\Big).$$

Thus,

$$(F_i) \qquad c_4 \frac{|\nabla K(a_i)|}{\mu_i} + O\Big(\frac{1}{(\mu_i d_i)^{n-3}} + \varepsilon_{12}\Big) = o\Big(\sum_{k=1,2} \frac{1}{(\mu_k d_k)^{n-3}} + \varepsilon_{12}^{\frac{n-3}{n-4}} + \varepsilon^{\frac{n-3}{n-4}} + \frac{1}{\mu_i}\Big).$$

Furthermore, an easy computation shows that

$$\mu_i \frac{\partial \varepsilon_{12}}{\partial \mu_i} = -\frac{n-4}{2} \varepsilon_{12}\Big(1 - 2\frac{\mu_j}{\mu_i}\varepsilon_{12}^{2/n-4}\Big), \qquad \text{for } i,j = 1,2, \ j \neq i. \tag{3.1}$$

Before starting the proof of the other results, we give the following crucial proposition, which is a key point in the proof of Theorems 1.1 and 1.2.

Proposition 3.1 *Let $n \geq 6$ and the assumptions $(H_1 - H_2)$ hold. Let (u_ε) be a family of sign-changing solutions of (P_ε) that satisfy (1.3), (1.4). Then, for $i = 1,2$, we have $a_{\varepsilon,i} \to y_{j_i}$ as $\varepsilon \to 0$, such that y_{j_i} is a critical point of K. Moreover, (1.5) is satisfied for $i = 1,2$.*

Proof Let (u_ε) be a family of sign-changing solutions of (P_ε) as stated in Proposition 3.1. Without loss of generality, we can assume that $\mu_1 \leq \mu_2$. We distinguish two cases.

Case 1. $M\mu_1 < \mu_2$, where M is a large positive constant.

Since $H(a_1,a_2) \leq c d_1^{4-n}$, then

$$\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}} = o\Big(\frac{1}{(\mu_1 d_1)^{n-4}}\Big). \tag{3.2}$$

Furthermore, using (3.1) and the fact that $\mu_2 \geq \mu_1$, an easy computation shows that

$$\pm\mu_1 \frac{\partial \varepsilon_{12}}{\partial \mu_1} - 2\mu_2 \frac{\partial \varepsilon_{12}}{\partial \mu_2} \geq \frac{n-4}{2}\varepsilon_{12}. \tag{3.3}$$

Arguing by contradiction, first, we suppose that $|\nabla K(a_1)| \geq c > 0$.

Multiplying (F_i) for $i = 1$ by a small positive constant m and adding to (E_1) and (E_2): $(mF_1 + E_1 + 2E_2)$, using (3.2) and (3.3), we get

$$c_1\Big(\frac{H(a_1,a_1)}{\mu_1^{n-4}} + 2\frac{H(a_2,a_2)}{\mu_2^{n-4}}\Big) + c_1'\varepsilon_{12} + 3c_2\varepsilon + c_4'\frac{|\nabla K(a_1)|}{\mu_1} = o\Big(\varepsilon + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}} + \varepsilon_{12} + \frac{1}{\mu_1}\Big).$$

Then we derive a contradiction. Hence, $a_1 \to y_{j_1}$, where y_{j_1} is a critical point of K and therefore $d_1 = d(a_1, \partial\Omega) \geq c > 0$.

If now $|\nabla K(a_2)| \geq c > 0$, then multiplying (F_2) by a small positive constant m and adding to (E_2), using (3.1) and (3.2), we obtain

$$c_1\frac{H(a_2,a_2)}{\mu_2^{n-2}} + c_1'\varepsilon_{12} + c_2\varepsilon + c_4'\frac{|\nabla K(a_2)|}{\mu_2} = o\Big(\varepsilon + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-2}} + \varepsilon_{12} + \frac{1}{\mu_2}\Big).$$

Thus,

$$\varepsilon = o\Big((\mu_1 d_1)^{4-n}\Big) \quad \text{ and } \varepsilon_{12} = o\Big((\mu_1 d_1)^{4-n}\Big). \tag{3.4}$$

Using (E_1), we derive

$$c_1\frac{H(a_1,a_1)}{\mu_1^{n-4}}-\frac{c_3}{n^2}\frac{\Delta K(a_1)}{K(a_1)\mu_1^2}=o\Big(\frac{1}{\mu_1^2}+\frac{1}{(\mu_1 d_1)^{n-4}}\Big),$$

which is a contradiction with the assumption (H_1). Then $a_2\to y_{j_2}$, where y_{j_2} is a critical point of K and therefore $d_2=d(a_2,\partial\Omega)\geq c>0$.

For the other part of the claim, let us suppose that y_{j_2} does not satisfy (1.5). From (E_2), it is easy to obtain (3.4), which gives a contradiction in (E_1).

Finally, assume that y_{j_1} does not satisfy (1.5). Using (H_1) and the fact that $a_i\to y_{j_i}$, we get

$$\frac{\Delta K(a_2)}{\mu_2^2}=o\Big(\Big|\frac{c_1H(a_1,a_1)}{\mu_1^2}-\frac{c_3\Delta K(a_1)}{36K(a_1)\mu_1^2}\Big|\Big)\ \text{ if }\ n=6 \text{ and } \frac{\Delta K(a_2)}{\mu_2^2}=o\Big(\frac{|\Delta K(a_1)|}{\mu_1^2}\Big)\quad\text{if}\quad n>6. \tag{3.5}$$

Now, using (3.2), (3.3) and adding (E_1) to $2(E_2)$, we obtain

$$2c_1\frac{H(a_2,a_2)}{\mu_2^{n-4}}+\Big(c_1\frac{H(a_1,a_1)}{\mu_1^{n-4}}-\frac{c_3}{n^2}\frac{\Delta K(a_1)}{K(a_1)\mu_1^2}\Big)+c_1\varepsilon_{12}+3c_2\varepsilon=o\Big(\varepsilon+\frac{1}{\mu_1^2}+\sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}}+\varepsilon_{12}\Big).$$

Hence, we get a contradiction. Thus, case 1 cannot occur.

Case 2. $\mu_2<M\mu_1$. In this case, it is easy to show that

$$\varepsilon_{12}=\frac{1}{(\mu_1\mu_2|a_1-a_2|^2)^{(n-4)/2}}+o(\varepsilon_{12}), \tag{3.6}$$

which implies that

$$\mu_i\frac{\partial\varepsilon_{12}}{\partial\mu_i}=-\frac{n-4}{2}\frac{1}{(\mu_1\mu_2|a_1-a_2|^2)^{(n-4)/2}}+o(\varepsilon_{12})\quad\text{for}\quad i=1,2. \tag{3.7}$$

Then (E_i) becomes

$$c_1\frac{H(a_i,a_i)}{\mu_i^{n-4}}-\frac{c_3}{n^2}\frac{\Delta K(a_i)}{K(a_i)\mu_i^2}+c_2\varepsilon+c_1\Big(\varepsilon_{12}-\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{\frac{n-4}{2}}}\Big)=o\Big(\varepsilon+\frac{1}{\mu_i^2}+\sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}}+\varepsilon_{12}\Big). \tag{3.8}$$

If $|\nabla K(a_i)|\geq c>0$, multiplying (F_i) by a small positive constant m and adding to (E_i) for $i=1,2$: $(mF_i+E_1+E_2)$, we get

$$c_1\Big(\frac{H(a_1,a_1)}{\mu_1^{n-4}}+\frac{H(a_2,a_2)}{\mu_2^{n-4}}\Big)+c_1'\frac{G(a_1,a_2)}{(\mu_1\mu_2)^{\frac{n-4}{2}}}+2c_2\varepsilon+c_4'\frac{|\nabla K(a_i)|}{\mu_i}=o\Big(\varepsilon+\sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}}+\varepsilon_{12}+\frac{1}{\mu_i}\Big). \tag{3.9}$$

Using the fact that $G(a_1,a_2):=|a_1-a_2|^{4-n}-H(a_1,a_2)>0$ and

$$\varepsilon_{12}=O\Big(\frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}+\frac{G(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big), \tag{3.10}$$

(3.9) gives a contradiction. Hence, each concentration point converges to a critical point.

For the other part of the claim, let us suppose that there exists i that does not satisfy (1.5). From (E_i), we have

$$\varepsilon = o\Big(\frac{1}{(\mu_i d_i)^{n-4}}\Big) \quad \text{and} \quad \frac{G(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}} = o\Big(\frac{1}{(\mu_i d_i)^{n-4}}\Big),$$

and then using (E_j), we also derive a contradiction with the assumption (H_1). Hence, the proof of our theorem is thereby completed. □

Proof of Theorem 1.1 Arguing by contradiction, let us suppose that the problem (P_ε) has a solution u_ε as stated in Theorem 1.1. From Proposition 3.1, we deduce that, for $i=1,2$, we have $a_i \to y_{j_i}$, such that y_{j_i} is a critical point of K that satisfies (1.5). From the definition of σ the a_i points have to converge to the same critical point.

Without loss of generality, we can assume that $\mu_2 \geq \mu_1$. Two cases may occur.

Case 1. $M\mu_1 < \mu_2$, where M is a large positive constant.

Multiplying (E_2) by 2 and adding to $(-E_1)$: $(2E_2 - E_1)$, we obtain:

$$\begin{aligned} &2\Big(c_1\frac{H(a_2,a_2)}{\mu_2^{n-4}} - \frac{c_3}{n^2}\frac{\Delta K(a_2)}{K(a_2)\mu_2^2}\Big) - \Big(c_1\frac{H(a_1,a_1)}{\mu_1^{n-4}} - \frac{c_3}{n^2}\frac{\Delta K(a_1)}{K(a_1)\mu_1^2}\Big) \\ &\quad + \frac{2c_1}{n-4}\Big(\mu_1\frac{\partial\varepsilon_{12}}{\partial\mu_1} - 2\mu_2\frac{\partial\varepsilon_{12}}{\partial\mu_2}\Big) - \frac{c_1 H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}} + c_2\varepsilon \\ &\quad = o\Big(\varepsilon + \sum_{k=1,2}\frac{1}{\mu_k^2} + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}} + \varepsilon_{12}\Big). \end{aligned} \tag{3.11}$$

Now, combining (3.2), (3.3), (3.5), and (3.11), we derive a contradiction.

Case 2. $M\mu_1 \geq \mu_2$. In this case, we see that ε_{12} is written as (3.6) and therefore (E_i) becomes

$$c_1\frac{H(a_i,a_i)}{\mu_i^{n-4}} - \frac{c_3}{n^2}\frac{\Delta K(a_i)}{K(a_i)\mu_i^2} + c_2\varepsilon + c_1\Big(\varepsilon_{12} - \frac{H(a_1,a_2)}{(\mu_1\mu_2)^{(n-4)/2}}\Big) = o\Big(\varepsilon + \frac{1}{\mu_i^2} + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}} + \varepsilon_{12}\Big). \tag{3.12}$$

Since $\mu_i|a_1 - a_2| \to \infty$ for $i=1,2$ and $|a_1-a_2| < \sigma$, it is easy to show that there is at least i such that $\mu_i^{-2} = o(|\nabla K(a_i)|/\mu_i)$. Multiplying (F_i) for $i=1,2$ by a small positive constant m and adding to (E_i) for $i=1,2$: $(m(F_1+F_2)+E_1+E_2)$, we get

$$\begin{aligned} &\frac{c_1}{2}\Big(\frac{H(a_i,a_i)}{\mu_i^{n-4}} + \frac{H(a_i,a_i)}{\mu_i^{n-4}}\Big) + \frac{c_1'}{(\mu_1\mu_2)^{(n-4)/2}}\Big(\frac{1}{|a_1-a_2|^{(n-4)/2}} - H(a_1,a_2)\Big) + c_2\varepsilon \\ &\quad + c_4'\sum_{k=1,2}\frac{|\nabla K(a_i)|}{\mu_i} = o\Big(\varepsilon + \sum_{k=1,2}\frac{1}{(\mu_k d_k)^{n-4}} + \varepsilon_{12} + \sum_{k=1,2}\frac{1}{\mu_i}\Big). \end{aligned}$$

Finally, using the fact that $G(a_1,a_2) > 0$ and (3.10), we derive a contradiction in this case. Our proof is thereby completed. □

Proof of Theorem 1.2 Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.2. By Theorem 1.1, we deduce that, $|a_1 - a_2| > \sigma$. Proposition 3.1 implies that (1.5) holds for $i=1,2$, which is a contradiction with the assumption of Theorem 1.2. □

4. Proof of bubble-tower solution results

In this section, we assume that problem (P_ε) has solutions (u_ε) that satisfy (1.6) and (1.7), which means u_ε is written as

$$u_\varepsilon = \sum_{i=1}^{p} \gamma_i P\delta_{(a_{\varepsilon,i},\mu_{\varepsilon,i})} + v_\varepsilon, \qquad \text{with} \qquad \gamma_i \in \{-1,1\}.$$

Observe that, as in Section 2, there is a unique way to choose a_i and μ_i such that v_ε is orthogonal to each $P\delta_{(a_i,\mu_i)}$ and their derivatives with respect to μ_i and $(a_i)_j$, where $(a_i)_j$ denotes the jth component of a_i. As in Lemma 2.1, we get $\mu_i^\varepsilon \to 1$ as $\varepsilon \to 0$ for each $i = 1,\dots,p$.

Arguing also as in the proof of Propositions 2.4 and 2.5, we have, for each $i = 1,...,p$,

$$(E_i) \qquad c_1 \frac{H(a_i,a_i)}{\mu_i^{n-4}} - \frac{c_3}{n^2}\frac{\Delta K(a_i)}{K(a_i)\mu_i^2} + c_1 \sum_{j\neq i}\gamma_i\gamma_j \Big(\frac{2\mu_i}{n-4}\frac{\partial \varepsilon_{ij}}{\partial \mu_i} + \frac{H(a_i,a_j)}{(\mu_i\mu_j)^{(n-4)/2}}\Big) + c_2\varepsilon$$
$$= O\Big(\frac{|\nabla K(a_i)|^2}{\mu_i^2}\Big) + o\Big(\varepsilon + \frac{1}{\mu_i^2} + \sum_{j=1}^{p}\frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r\neq j}\varepsilon_{rj}\Big),$$

and

$$(F_i)\ c_1 \frac{1}{\mu_i^{n-3}}\frac{\partial H(a_i,a_i)}{\partial a_i} - c_4 \frac{\nabla K(a_i)}{\mu_i} - 2c_1 \sum_{j\neq i}\gamma_i\gamma_j \frac{1}{\mu_i}\Big(\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{\partial H}{\partial a_i}(a_i,a_j)\frac{1}{(\mu_i\mu_j)^{(n-4)/2}}\Big)$$
$$= O\Big(\sum_{k=1}^{p}\frac{1}{(\mu_k d_k)^{n-2}} + \sum_{r\neq j}\varepsilon_{rj}^{\frac{n}{n-4}}\log\varepsilon_{rj}^{-1} + \varepsilon\sum_{r\neq j}\varepsilon_{rj}(\log\varepsilon_{rj}^{-1})^{\frac{n-4}{2}} + \frac{1}{\mu_i^3}\Big).$$

Observe that

$$|\frac{\Delta K(a_i)}{\mu_i^2}| + \frac{|\nabla K(a_i)|^2}{\mu_i^2} = o\Big(\frac{1}{\mu_1^2}\Big)\ \forall i>1 \text{ and } \frac{H(a_i,a_j)}{(\mu_i\mu_j)^{\frac{n-2}{2}}} = o\Big(\frac{1}{(\mu_i d_i)^{n-2}}\Big)\ \forall\, i<j. \tag{4.1}$$

First we start by proving the following crucial proposition, which is a key point in the proof of Theorems 1.4 and 1.5.

Lemma 4.1 *Let $i<j<k$, such that $\mu_i|a_r - a_l| \to +\infty$ for $r,l = i,j,k$. Then $\varepsilon_{ik} = o(\varepsilon_{ij})$ or $\varepsilon_{jk} = o(\varepsilon_{ij})$.*

Proof Assume that there exists $c>0$ such that $\varepsilon_{ik} \geq c\varepsilon_{ij}$ and $\varepsilon_{jk} \geq c\varepsilon_{ij}$. Thus, we derive that $\mu_j|a_i-a_j|^2 \geq c\mu_k|a_i-a_k|^2$ and $\mu_i|a_i-a_j|^2 \geq c\mu_k|a_j-a_k|^2$. Hence, we get $|a_i-a_k|^2/|a_i-a_j|^2 \leq c^{-1}(\mu_j/\mu_k) \to 0$ and $|a_j-a_k|^2/|a_i-a_j|^2 \leq c^{-1}(\mu_i/\mu_k) \to 0$, which is a contradiction, and our lemma follows. □

Proposition 4.2 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n\geq 6$, and the assumptions ($H_1 - H_2$) hold. If $\varepsilon = o(\sum_{i\neq j}\varepsilon_{ij} + \sum(\mu_i d_i)^{4-n} + \mu_1^{-2})$, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0,\varepsilon_0)$, problem (P_ε) has no sign-changing solutions u_ε that satisfy (1.6) and (1.7).*

Proof The proof is based on the estimate (E_i). First, using (E_p), we prove that all the terms containing the index p are small with respect to the others. Hence, we can drop the index p from the other (E_i). Step by step, we derive that all the ε_{ij}s and $(\mu_i d_i)^{4-n}$ for $i\geq 2$ are small with respect to $(\mu_1 d_1)^{4-n}$. Finally, from (E_1), we conclude. More precisely, let $m\leq p$, $T_m = \{i<m \,/\, \mu_i|a_i - a_m| \to +\infty\}$ and $j_m := \max\{j<m,\ j\notin T_m\}$.

We remark that, from Lemma 4.1 and the estimate of ε_{im} if $i \notin T_m$ and $i < m$,

$$\exists\, i_m \in T_m \text{ s.t. } \forall i \in T_m \backslash \{i_m\},\ \varepsilon_{im} = o(\varepsilon_{i_m m}) \text{ and } \forall i \notin T_m,\ i < j_m,\ \varepsilon_{im} = o(\varepsilon_{j_m m}). \tag{4.2}$$

Note that the set T_m can be empty (resp. $T_m = \{1, ..., m-1\}$), and then i_m (resp. j_m) does not appear.

Now, using (4.1) and (4.2) with $m = p$, the equation (E_p) becomes

$$(E'_p) \qquad c_1 \frac{H(a_p, a_p)}{\mu_p^{n-4}} - c_1 \gamma_{i_p} \gamma_p \varepsilon_{i_p p} - \gamma_{j_p} \gamma_p \varepsilon_{j_p p} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj}\Big).$$

Observe that, if $\mu_{j_p}|a_{i_p} - a_p| \to +\infty$, then $\varepsilon_{i_p p} = o(\varepsilon_{j_p p})$, and if $\mu_{j_p}|a_{i_p} - a_p|$ is bounded, it follows that $j_p < i_p$ and $\mu_{j_p}|a_{j_p} - a_{i_p}|$ is bounded and therefore $\varepsilon_{j_p p} = o(\varepsilon_{j_p i_p})$.

Hence, there exists $i_0 \in \{i_p, j_p\}$ such that

$$(E'_p) \qquad c_1 \frac{H(a_p, a_p)}{\mu_p^{n-4}} - c_1 \gamma_{i_0} \gamma_p \varepsilon_{i_0 p} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj}\Big).$$

Now, if $d_p > c d_{i_0}$ for some positive constant c, we get

$$\frac{H(a_p, a_p)}{\mu_p^{n-4}} = o\Big(\sum_{j=1}^{p-1} \frac{1}{(\mu_j d_j)^{n-4}}\Big) \text{ and } \varepsilon_{i_0 p} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p-1} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj}\Big).$$

In the other case, $d_p/d_{i_0} \to 0$, this implies that $|a_{i_0} - a_p| \sim d_{i_0}$ and $\varepsilon_{i_0 p} = (\mu_{i_0}\mu_p |a_{i_0} - a_p|^2)^{(4-n)/2} = o((\mu_{i_0} d_{i_0})^{4-n})$. Then from (E'_p), we derive, for $i < p$

$$\varepsilon_{ip} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p-1} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj}\Big) \text{ and } \frac{H(a_p, a_p)}{\mu_p^{n-4}} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p-1} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r \neq j} \varepsilon_{rj}\Big).$$

Thus, we remove the index p from the system and we repeat the same argument with $p-1$, ..., 2. Hence, we derive that

$$\frac{H(a_i, a_i)}{\mu_i^{n-4}} = o\Big(\frac{1}{\mu_1^2} + \frac{1}{(\mu_1 d_1)^{n-4}}\Big) \text{ for } i > 1 \text{ and } \varepsilon_{ij} = o\Big(\frac{1}{\mu_1^2} + \frac{1}{(\mu_1 d_1)^{n-4}}\Big) \text{ for } i \neq j.$$

Thus, (E_1) and (F_1) become

$$(E'_1) \qquad c_1 \frac{H(a_1, a_1)}{\mu_1^{n-4}} - \frac{c_3}{n^2} \frac{\Delta K(a_1)}{K(a_1)\mu_1^2} = O\Big(\frac{|\nabla K(a_i)|^2}{\mu_i^2}\Big) + o\Big(\frac{1}{\mu_1^2} + \frac{1}{(\mu_1 d_1)^{n-4}}\Big),$$

$$(F'_1) \qquad c_4 \frac{|\nabla K(a_1)|}{\mu_1} + O\Big(\frac{1}{(\mu_1 d_1)^{n-3}}\Big) = o\Big(\frac{1}{(\mu_1 d_1)^{n-3}} + \frac{1}{\mu_1}\Big).$$

Finally, if $|\nabla K(a_1)| > c$, using $(E'_1 + F'_1)$, we get a contradiction.

If $|\nabla K(a_1)| \to 0$, using (E'_1) and the assumption (H_1), we also derive a contradiction.

The proof of our proposition is thereby completed. □

Proof of Theorem 1.4 Arguing by contradiction, let us assume that problem (P_ε) has solutions (u_ε) as stated in Theorem 1.4.

From the definition of q_1 and q_2, we have $\varepsilon_{q_1q_2} \geq c(\mu_{q_1}/\mu_{q_2})^{(n-4)/2}$, and this implies that, for $i < q_1$,

$$\varepsilon_{ip} \leq c(\mu_i/\mu_p)^{(n-4)/2} = o(\varepsilon_{q_1q_2}). \tag{4.3}$$

Regarding the equation (E_p), using (4.1), (4.3), and the fact that $\gamma_i \neq \gamma_p$ for $i \geq q_1$, we obtain

$$c_1 \frac{H(a_p, a_p)}{\mu_p^{n-4}} + c_2\varepsilon + c_1 \sum_{i=q_1}^{p} \varepsilon_{ip} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj}\Big).$$

This gives an estimate of ε, and using Proposition 4.2, we derive a contradiction. Hence, our theorem is proved. □

Proof of Theorem 1.5 Arguing by contradiction, let us assume that (P_ε) has solutions (u_ε) as stated in Theorem 1.5. From the definition of q, we have $\varepsilon_{q(q+1)} \geq c(\mu_q/\mu_{q+1})^{(n-4)/2}$, and this implies that

$$\varepsilon_{i(q+1)} = o(\varepsilon_{q(q+1)}) \text{ for } i < q \quad \text{and} \quad \varepsilon_{(q+1)i} = o(\varepsilon_{q(q+1)}) \text{ for } i > q+1. \tag{4.4}$$

Now, regarding the equation (E_{q+1}), using (4.1), (4.4), and the fact that $\gamma_q \neq \gamma_{q+1}$, we have

$$c_1 \frac{H(a_{q+1}, a_{q+1})}{\mu_{q+1}^{n-4}} + c_2\varepsilon + c_1\varepsilon_{q(q+1)} = o\Big(\frac{1}{\mu_1^2} + \sum_{j=1}^{p} \frac{1}{(\mu_j d_j)^{n-4}} + \sum_{r\neq j} \varepsilon_{rj}\Big).$$

Then we get an estimate of ε, and using Proposition 4.2, we derive a contradiction, and the proof of our theorem is thereby completed. □

References

[1] Bahri A. Critical Point at Infinity in Some Variational Problems. Volume 182 of Pitman Research Notes in Mathematics Series. Harlow, UK: Longman, 1989.

[2] Bahri A, Coron JM. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of topology of the domain. Commun Pur Appl Math 1988; 41: 255-294.

[3] Bahri A, Li YY, Rey O. On a variational problem with lack of compactness: the topological effect of the critical points at infinity. Calc Var Partial Dif 1995; 3: 67-94.

[4] Bartsch T, Weth T. A note on additional properties of sign-changing solutions to superlinear elliptic equations. Topol Method Nonl An 2003; 22: 1-14.

[5] Ben Ayed M, El Mehdi K. On a biharmonic equation involving nearly critical exponent. Nonlinear Differ Equ Appl 2006; 13: 485-509.

[6] Ben Ayed M, El Mehdi K. The Paneitz curvature problem on lower dimensional spheres. Ann Glob Anal Geom 2007; 31: 1-36.

[7] Ben Ayed M, El Mehdi K, Grossi M, Rey O. A nonexistence result of single peaked solutions to a supercritical nonlinear problem. Commun Contemp Math 2003; 2: 179-195.

[8] Ben Ayed M, Ghoudi R. Profile and existence of sign-changing solutions to an elliptic subcritical equation. Commun Contemp Math 2008; 10: 1183-1216.

[9] Ben Ayed M, Ould Bouh K. Nonexistence results of sign-changing solution to a supercritical nonlinear problem. Commun Pur Appl Anal 2008; 7: 1057-1075.

[10] Chang SA. On a Paneitz operator – a fourth order differential operator in conformal geometry. In: Christ M, Kenig C, Sadorsky C, editors. Harmonic Analysis and Partial Differential Equations; Essays in Honor of Alberto P. Calderon. Chicago, IL, USA: University of Chicago Press, 1999, pp. 127-150.

[11] Chou KS, Geng D. Asymptotics of positive solutions for a biharmonic equation involving critical exponent. Differ Integral Equ 2000; 13: 921-940.

[12] Del Pino M, Felmer P, Musso M. Two bubbles solutions in the supercritical Bahri-Coron's problem. Calc Var Partial Dif 2003; 16: 113-145.

[13] Djadli Z, Hebey E, Ledoux M. Paneitz type operators and applications. Calc Duke Math J 2000; 104: 129-169.

[14] Ebobisse F, Ould Ahmedou M. On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent. Nonlinear Anal-Theor 2003; 52: 1535-1552.

[15] El Mehdi K, Hammami M. Blowing up solutions for a biharmonic equation with critical nonlinearity. Asymptotic Anal 2005; 45: 485-509.

[16] Gazzola F, Grunau HC, Squassina M. Existence and nonexistence results for critical growth biharmonic elliptic equations. Cal Var Partial Dif 2003; 18: 117-143.

[17] Ghoudi R. Blowing up of sign-changing solutions to an elliptic subcritical equation. J Partial Dif Equ 2012; 25: 368-388.

[18] Ould Bouh K. Nonexistence result of sign-changing solutions for a supercritical problem of the scalar curvature type. Adv Nonlinear Stud 2012; 12: 149-171.

[19] Ould Bouh K. Sign-changing solutions of a fourth-order elliptic equation with supercritical exponent. Electron J Diff Equ 2014; 77: 1-13.

[20] Pistoia A, Weth T. Sign-changing bubble-tower solutions in a slightly subcritical semilinear Dirichlet problem. Ann I H Poincare-An 2007; 24: 325-340.

[21] Rey O. The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent. J Funct Anal 1990; 89: 1-52.

[22] van der Vorst RCAM. Variational identities and applications to differential systems. Arch Rat Mech Anal 1991; 116: 375-398.

[23] van der Vorst RCAM. Fouth order elliptic equations with critical growth. C R Acad Sci Paris 1995; 320: 295-299.