Research Article

The local and semilocal convergence analysis of new Newton-like iteration methods

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Abstract: The aim of this paper is to find new iterative Newton-like schemes inspired by the modified Newton iterative algorithm and prove that these iterations are faster than the existing ones in the literature. We further investigate their behavior and finally illustrate the results by numerical examples.

Key words: Modified Newton algorithm, Picard S hybrid iteration, semilocal convergence, local convergence

1. Introduction

Throughout this paper, the following notations will be used:

- C is an open convex subset of a Banach space X,

- F is a Fréchet differentiable operator at each point of C with values in a Banach space Y,

- $B_r[x] = \{y \in X : ||y - x|| \leq r\}$, for any $x \in X$ and $r > 0$,

- $B_r(x) = \{y \in X : ||y - x|| < r\}$, for any $x \in X$ and $r > 0$,

- $B(Y, X)$ is the space of all bounded linear operators from Y to X,

- $\mathbb{N}$ denotes all positive integers including zero.

Many problems that arise in engineering and scientific disciplines can be modeled by the following nonlinear operator equation:

$$F(x) = 0. \tag{1}$$

Several problems about studying the solvability of (1) are brought forward (see [3, 13]). To solve this equation, the iterative approximation method is considered as one of the main tools in fixed point theory. Therefore, many iterative methods have been defined and studied by numerous mathematicians (see [7, 10, 11, 14]).

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In 2011 Sahu et al. [17] introduced the normal-S iteration process for finding solutions of constrained minimization problems and split feasibility problems as follows:

Let E be a normed space, B be a nonempty, convex subset of E and $A : B \to B$ be an operator. Then, for an arbitrary $x_0 \in B$,

$$x_{n+1} = A((1 - \alpha_n)x_n + \alpha_n A x_n) \quad n \in \mathbb{N}, \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Gürsoy [9] introduced Picard-S iterative process as follows:

Let B be a closed convex subset of a Banach space X and $T : B \to B$ be an operator. Then, for an arbitrary $x_0 \in B$

$$\begin{cases} x_{n+1} = T y_n \\ y_n = (1 - \alpha_n) T x_n + \alpha_n T z_n \\ z_n = (1 - \beta_n) x_n + \beta_n T x_n \ (n \in \mathbb{N}), \end{cases} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \in [0,1]$.

For solving nonlinear operator equation (1), many authors present several generalizations of the Newton method (see [3] and [21]). The Newton method is given as follows:

$$\begin{cases} x_0 \in C \\ x_{n+1} = x_n - F_{x_n}^{'-1} F(x_n), \quad \forall n \in \mathbb{N} \end{cases} \tag{4}$$

where $F_x^{'}$ denotes the Fréchet derivative of F at the point $x \in C$.

In the Newton method (4), the functional value of the inverse of the derivative is required at each step. A natural question is how to modify the Newton iteration process (4) so that the computation of the inverse of the derivative at each step in the Newton method (4) can be avoided. Argyros [1], Bartle [4], Dennis [6], and Rheinboldt [16] discussed the following modified Newton method,

$$x_{n+1} = x_n - F_{x_0}^{'-1} F(x_n), \quad n \in \mathbb{N}. \tag{5}$$

Let $x^* \in C$ be a solution of (1) such that $F_{x^*}^{'-1} \in B(Y, X)$. For some $x_0 \in C$, assume that $F_{x^*}^{'-1}$ and F satisfy the following:

$$\left\| F_x' - F_{x_0}' \right\| \leq K_0 \left\| x - x_0 \right\|, \ \forall x \in C \text{ and for some } K_0 > 0, \tag{6}$$

$$\left\| F_{x^*}^{'-1} \left(F_x' - F_{x_0}' \right) \right\| \leq K_1 \left\| x - x_0 \right\|, \ \forall x \in C \text{ and for some } K_1 > 0, \tag{7}$$

and

$$\left\| F_{x^*}^{'-1} \left(F_x' - F_{x^*}' \right) \right\| \leq K_2 \left\| x - x^* \right\|, \ \forall x \in C \text{ and for some } K_2 > 0. \tag{8}$$

In [2] and [15], the authors proved theorems for semilocal and local convergence analysis of (5) to solve the operator equation (4).

Recently, Sahu et al. [18] introduced the following Newton-like S-iteration processes (SIP) for solving equation (1):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F_{x_0}^{'-1}F(y_n) \\ y_n = (1 - \alpha)x_n + \alpha z_n \\ z_n = x_n - F_{x_0}^{'-1}F(x_n) \quad n \in \mathbb{N}, \end{cases} \tag{9}$$

and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F_{y_0}^{'-1}F(y_n) \\ y_n = (1 - \alpha)x_n + \alpha z_n \\ z_n = x_n - F_{x_0}^{'-1}F(x_n) \quad n \in \mathbb{N}, \end{cases} \tag{10}$$

where $\alpha \in (0, 1)$.

In the following theorems, they also proved semilocal as well as local convergence analysis of (9) and (10) and obtained that these iterative algorithms are faster than (5).

Theorem 1 *[18] Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. For some $x_0 \in C$, let $F_{x_0}^{'-1} \in B(Y, X)$ and the operator F satisfy (6) and the following conditions:*

i) *$\left\| F_{x_0}^{'-1}F(x_0) \right\| \leq \eta$, for some $\eta > 0$;*

ii) *$\left\| F_{x_0}^{'-1} \right\| \leq \beta$, for some $\beta > 0$.*

Assume that $\alpha \in (0, 1)$, $h = \eta\beta K_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq C$ such that $r = \frac{1 - \sqrt{1 - 2h}}{h}\eta$. Then, under the above restrictions, the following assertions are true:

a) *The operator $A : B_r[x_0] \to X$, defined by*

$$Ax = x - F_{x_0}^{'-1}F(x), x \in B_r[x_0],$$

is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\beta r K_0$ and the operator of Equation (1) has a unique solution in $B_r[x_0]$.

b) *The S-operator $A_\alpha : B_r[x_0] \to X$ generated by α and A is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\beta r K_0(1 - \alpha + \alpha\beta r K_0)$.*

Theorem 2 *[18] Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. Suppose that $\lambda \in (0, 1]$ and $x^* \in C$ is a solution of (1) such that $F_{x^*}^{'-1} \in B(Y, X)$. For some $x_0 \in C$, let $F_{x^*}^{'-1}$ and F satisfy the conditions (7) and (8). Assume that $B_{r_1}(x^*) \subseteq C$, where $r_1 = \frac{2}{K_2}$. For $x_0 \in B_r(x^*)$ with $r = \frac{2}{2K_2 + 3K_1}$, let A_λ be an operator defined by*

$$A_\lambda(x) = x - \lambda F_{x_0}^{'-1}F(x), \, \forall x \in B_r(x^*).$$

Then we have the following:

i) *For* $x \in B_r(x^*)$, *we have*

$$||A_\lambda(x) - x^*|| \le (\lambda \delta_x + 1 - \lambda)||x - x^*||,$$

where $\delta_x = \frac{K_1}{2(1-rK_2)} \left(\|x - x^*\| + 2 \|x_0 - x^*\| \right).$

ii) A_λ *is a quasi-contraction and self-operator on* $B_r(x^*)$ *with constant* $1 - (1 - \delta)\lambda$, *where* $\delta = \sup\limits_{x \in B_r(x^*)} \{\delta_x\}.$

Motivated by the above studies, we introduce new Newton-like iteration processes as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F_{x_0}^{'-1} F(y_n) \\ y_n = (1 - \alpha) \left[x_n - F_{x_0}^{'-1} F(x_n) \right] + \alpha[z_n - F_{x_0}^{'-1} F(z_n)] \\ z_n = (1 - \theta) x_n + \theta[x_n - F_{x_0}^{'-1} F(x_n)] \quad n \in \mathbb{N}, \end{cases} \tag{11}$$

and

$$\begin{cases} x_0 \in C, \\ x_{n+1} = y_n - F_{y_0}^{'-1} F(y_n) \\ y_n = (1 - \alpha) \left[x_n - F_{x_0}^{'-1} F(x_n) \right] + \alpha[z_n - F_{z_0}^{'-1} F(z_n)] \\ z_n = (1 - \theta) x_n + \theta[x_n - F_{x_0}^{'-1} F(x_n)] \quad n \in \mathbb{N}, \end{cases} \tag{12}$$

where $\alpha, \theta \in (0, 1)$.

In the present study, we obtain semilocal and local convergence results of (11) and (12). Moreover, we compare the rates of convergence of the modified Newton method (5), the SIP of Newton-like iterative processes (9)–(10), and our Newton-like iterative processes (11)-(12).

2. Preliminaries

Definition 1 *Let* C *be a nonempty subset of normed space* X. *A mapping* $T : C \to X$ *is said to be:*

i) *Contraction if there exists a constant* $\delta \in (0, 1)$ *such that*

$$\|Tx - Ty\| \le \delta \|x - y\|, \quad \forall x, y \in C.$$

ii) *Quasi-contraction [19] if there exists a constant* $\delta \in (0, 1)$ *and* $F_T = \{x \in C : Tx = x\} \ne \varnothing$ *such that*

$$\|Tx - p\| \le \delta \|x - p\|, \quad \forall x \in C \text{ and } p \in F_T.$$

Definition 2 *[17] Let* C *be a nonempty convex subset of a normed space* X *and* $T : C \to C$ *an operator. The operator* $G : C \to C$ *is said to be S-operator generated by* $\alpha \in (0, 1)$, T, *and identity mapping* I *if*

$$G = T \left[(1 - \alpha) I + \alpha T \right].$$

Definition 3 *[5] Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be nonnegative real convergent sequences with limits* a *and* b, *respectively. Then* $\{a_n\}_{n=0}^{\infty}$ *converges faster than* $\{b_n\}_{n=0}^{\infty}$ *if*

$$\lim_{n \to \infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.$$

Lemma 1 *[1] Let R be a bounded linear operator on a Banach space X. Then the following assumptions are equivalent:*

    i) There is a bounded linear operator S on X such that S^{-1} exists and

$$\|S - R\| = \frac{1}{\|S^{-1}\|}.$$

   ii) R^{-1} exists.

    Furthermore, if R^{-1} exists, then

$$\left\|R^{-1}\right\| \leq \frac{\left\|S^{-1}\right\|}{1 - \|1 - S^{-1}R\|} \leq \frac{\left\|S^{-1}\right\|}{1 - \|S^{-1}\|\,\|S - R\|}.$$

Lemma 2 *[19] Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Let $x^* \in D$ be a solution of (1) such that $F_{x^*}'^{-1} \in B(Y, X)$ and the operator F satisfies the conditions (8). Assume that $B_r(x^*) \subseteq D$, where $r = \frac{1}{K_2}$. Then, for any $x \in B_r(x^*)$, F_x is invertible and the following estimate holds:*

$$\left\|\left(F_{x^*}'^{-1}F_x'\right)^{-1}\right\| \leq \frac{1}{1 - K_2\|x - x^*\|}.$$

Lemma 3 *[12] Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping. Then T has a unique fixed point in X.*

Lemma 4 *[20] Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Then, for all $x, y \in D$, we have*

$$Fx - Fy = \int_0^1 F_{y+t(x-y)}'\,(x - y)\,dt.$$

3. Main results

3.1. Semilocal convergence analysis

In this subsection, we give semilocal convergence analysis of algorithm (11).

Theorem 3 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$ let $F_{x_0}'^{-1} \in B(Y, X)$. Assume that $F_{x_0}'^{-1}$ and F satisfy (6), (7), and (8) with the following conditions:*

    i) $\left\|F_{x_0}'^{-1}F(x_0)\right\| \leq \eta$, for some $\eta > 0$.

   ii) $\left\|F_{x_0}'^{-1}\right\| \leq \beta$, for some $\beta > 0$.

Assume that $\gamma, \alpha, \theta \in (0,1)$, $h = \eta\beta K_0 < \frac{1}{2}$, and $B_r[x_0] \subseteq D$ such that $r = \frac{1-\sqrt{1-2h}}{h}\eta$. Then, under the above restrictions, the following assertions are true.

a) The S^p-operator $K_\alpha : B_r[x_0] \to X$ generated by α, θ, A, and A_θ is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\gamma^2[1 - \alpha(1 + \gamma\theta)(1 - \gamma)]$ such that

$$K_\alpha x = A[(1 - \alpha)Ax + \alpha AA_\theta x].$$

b) Equation (1) has a unique solution $x^* \in B_r[x_0]$.

c) The sequence $\{x_n\}$ generated by Algorithm (11) is in $B_r[x_0]$ and it converges strongly to x^*.

d) The following error estimate holds:

$$||x_{n+1} - x^*|| \leq \kappa^{2(n+1)}||x_0 - x^*||, \forall n \in \mathbb{N},$$

where $\kappa = \gamma[1 - \alpha(1 + \gamma\theta)(1 - \gamma)]$ and $\gamma = \beta r K_0$.

Proof By Theorem (1) and Theorem (2), we know that the following inequalities are provided for (12) and (11), and we have

$$\|Ax - Ay\| \leq \gamma\|x - y\|,$$

$$\|A_\lambda x - x^*\| \leq (1 - \lambda(1 - \delta_x))\|x - x^*\|,$$

and

$$\|A_\theta x - A_\theta y\| \leq \gamma(1 - \theta(1 - \gamma))\|x - y\|.$$

a) We show that K_α is a contraction as follows:

$$
\begin{aligned}
\|K_\alpha x - K_\alpha y\| &= \|A[(1 - \alpha)Ax + \alpha AA_\theta x] - A[(1 - \alpha)Ay + \alpha AA_\theta y]\| \\
&\leq \gamma\|(1 - \alpha)Ax + \alpha AA_\theta x - (1 - \alpha)Ay - \alpha AA_\theta y\| \\
&\leq \gamma(1 - \alpha)\|Ax - Ay\| + \gamma\alpha\|AA_\theta x - AA_\theta y\| \\
&\leq \gamma^2(1 - \alpha)\|x - y\| + \gamma^3\alpha(1 - \theta(1 - \gamma))\|x - y\| \\
&= \gamma^2[(1 - \alpha) + \gamma\alpha(1 - \theta(1 - \gamma))]\|x - y\| \\
&= \gamma^2[1 - \alpha(1 + \gamma\theta)(1 - \gamma)]\|x - y\|.
\end{aligned}
$$

b) It is clear from Theorem (1).

c) From (11), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|K_\alpha x_n - K_\alpha x^*\| \\
&\leq \gamma^2[1 - \alpha(1 + \gamma\theta)(1 - \gamma)]\|x_n - x^*\|.
\end{aligned}
$$

By induction, we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq \gamma^2 \left[1 - \alpha \left(1 + \gamma\theta\right)\left(1 - \gamma\right)\right] \|x_n - x^*\| \\
\|x_n - x^*\| &\leq \gamma^4 \left[1 - \alpha \left(1 + \gamma\theta\right)\left(1 - \gamma\right)\right]^2 \|x_{n-1} - x^*\| \\
\|x_{n-1} - x^*\| &\leq \gamma^6 \left[1 - \alpha \left(1 + \gamma\theta\right)\left(1 - \gamma\right)\right]^3 \|x_{n-2} - x^*\| \\
\vdots \;\; &\leq \;\; \vdots \\
\|x_{n+1} - x^*\| &\leq \gamma^{2(n+1)} \left[1 - \alpha \left(1 + \gamma\theta\right)\left(1 - \gamma\right)\right]^{n+1} \|x_0 - x^*\|.
\end{aligned}
$$

This implies that $x \to x^*$ as $n \to \infty$.

d) Since $\kappa = \gamma \left[1 - \alpha \left(1 + \gamma\theta\right)\left(1 - \gamma\right)\right]$ and $\gamma = \beta r K_0$, we have

$$
\|x_{n+1} - x^*\| \leq \kappa^{2(n+1)} \|x_0 - x^*\|, \forall n \in \mathbb{N}.
$$

$\square$

3.2. Local convergence analysis

The following theorems present results about the local convergence analysis of (11) and (12).

Theorem 4 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Assume that $\alpha, \theta \in (0,1)$ and $x^* \in D$ is a solution of (1). For some $x_0 \in D$ let $F'^{-1}_{x_0} \in B(Y,X)$, and $F'^{-1}_{x_0}$ and F satisfy (7) and (8). Assuming that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$, then we have the following:*

i) *For initial $x_0 \in B_r(x^*)$ with $r = \frac{2}{2K_2 + 3K_1}$, the sequence $\{x_n\}$ generated by algorithm (11) is in $B_r(x^*)$ and converges strongly to the unique solution x^* in $B_{r_1}(x^*)$.*

ii) *The following error estimate holds:*

$$
\|x_{n+1} - x^*\| \leq (\kappa')^{2(n+1)} \|x_0 - x^*\|, \forall n \in \mathbb{N},
$$

where $\kappa' = \delta_0 \left[1 - \alpha \left(1 + \delta_0\theta\right)\left(1 - \delta_0\right)\right]$ and $\delta_0 = \frac{\|x_0 - x^*\|}{r}$.

Proof

i) First we show that x^* is a unique solution of (1) in $B_{r_1}(x^*)$. For contradiction, suppose that y^* is another solution of (1) in $B_{r_1}(x^*)$. Then we have

$$
0 = F(x^*) - F(y^*) = \int_0^1 F'_{y^* + t(x^* - y^*)}\left(x^* - y^*\right) dt.
$$

Define an operator L such that

$$
L(h) = \int_0^1 F'_{y^* + t(x^* - y^*)} h \, dt, \forall h \in X.
$$

Hence, we have

$$\|F'_{x^*} - L\| = \left\| \int_0^1 F'_{x^*} dt - \int_0^1 F'_{y^*+t(x^*-y^*)} dt \right\|$$

$$= \left\| \int_0^1 \left(F'_{x^*} - F'_{y^*+t(x^*-y^*)} \right) dt \right\|$$

and

$$\left\| I - F'^{-1}_{x^*} L \right\| = \left\| \int_0^1 F'^{-1}_{x^*} \left(F'_{x^*} - F'_{y^*+t(x^*-y^*)} \right) dt \right\|$$

$$\leq \int_0^1 \left\| F'^{-1}_{x^*} \left(F'_{x^*} - F'_{y^*+t(x^*-y^*)} \right) \right\| dt$$

$$\leq \frac{K_2}{2} \|x^* - y^*\|$$

$$< \frac{r_1 K_2}{2} = 1.$$

By Lemma 1, L is an invertible operator and hence $x^* = y^*$ is a contradiction. It implies that x^* is the unique solution of (1) in $B_{r_1}(x^*)$.

Now we examine that $x_n \to x^*$ as $n \to \infty$. By Theorem 3, we know that K_α is a quasi-contraction self-operator on mapping $B_r(x^*)$. Therefore, $x_n \in B_r(x^*)$ and

$$\|x_{n+1} - x^*\| = \|K_\alpha(x_n) - K_\alpha(x^*)\| \tag{13}$$

$$\leq \delta^2_{x_n} [1 - \alpha(1 + \delta_{x_n}\theta)(1 - \delta_{x_n})] \|x_n - x^*\|,$$

where δ_{x_n} is defined in Theorem 2-b. Since $\delta_{x_n} < 1$, then for all $n \in \mathbb{N}_0$ we obtain

$$\|x_{n+1} - x^*\| \leq \|x_0 - x^*\|.$$

From the definition of δ_x, we have

$$\delta_{x_n} = \frac{K_1}{2(1 - rK_2)} (\|x_n - x^*\| + 2\|x_0 - x^*\|)$$

$$\leq \frac{3K_1 \|x_0 - x^*\|}{2(1 - rK_2)}$$

$$\leq \frac{\|x_0 - x^*\|}{r}$$

$$= \delta_0.$$

Hence, from (13), we obtain

$$\|x_{n+1} - x^*\| \leq \left(\delta_0^2 [1 - \alpha(1 + \delta_0\theta)(1 - \delta_0)] \right)^{n+1} \|x_0 - x^*\|, \tag{14}$$

which implies that $x_n \to x^*$ as $n \to \infty$. This complete the proof.

ii) We conclude from (14) that for all $n \in \mathbb{N}$, we have

$$\|x_{n+1} - x^*\| \leq (\kappa')^{2(n+1)} \|x_0 - x^*\|.$$

$\square$

Theorem 5 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Assume that $\alpha, \theta \in (0,1)$ and $x^* \in D$ is solution of (1). For some $x_0 \in D$, let $F_{x^*}'^{-1} \in B(Y,X)$, and $F_{x^*}'^{-1}$ and F satisfy the conditions (7) and (8), and for all $x \in D$, some $K_3 > 0$,*

$$\left\|F_{x^*}'^{-1}\left(F_x' - F_{y_0}'\right)\right\| \leq K_3 \|x - y_0\|, \tag{15}$$

such that $K_3 \leq K_1$. Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$ and $x_0, y_0, z_0 \in B_r(x^*)$, where $r = \frac{2}{2K_2 + 3K_1}$.

For all $x \in B_r(x^*)$, consider three operators such that

$$V_{y_0}(x) = x - F_{y_0}'^{-1} F(x),$$

$$V_{z_0}(x) = x - F_{z_0}'^{-1} F(x),$$

and

$$\Im_\alpha(x) = V_{y_0}\left[(1-\alpha) A(x) + \alpha V_{z_0} A_\theta(x)\right].$$

Then we have the following:

i) *For all $x \in B_r(x^*)$, we obtain*

$$\|V_{y_0}(x) - x^*\| \leq \delta_x' \|x - x^*\|,$$

where $\delta_x' = \frac{K_3}{2(1-rK_2)}\left(\|x - x^*\| + 2\|y_0 - x^*\|\right)$.

ii) *V_{y_0} is a quasi-contraction and self-operator on $B_r(x^*)$ with constant δ', where $\delta' = \sup\limits_{x \in B_r(x^*)} \{\delta_x'\}$.*

iii) *For all $x \in B_r(x^*)$, we obtain*

$$\|V_{z_0}(x) - x^*\| \leq \delta_x'' \|x - x^*\|,$$

where $\delta_x'' = \frac{K_3}{2(1-rK_2)}\left(\|x - x^*\| + 2\|z_0 - x^*\|\right)$.

iv) *V_{z_0} is a quasi-contraction and self-operator on $B_r(x^*)$ with constant δ'', where $\delta'' = \sup\limits_{x \in B_r(x^*)} \{\delta_x''\}$.*

v) *The sequence $\{x_n\}$ generated by Algorithm 12 is in $B_r(x^*)$ and it converges strongly to the a unique solution x^* in $B_{r_1}(x^*)$.*

vi) *The following error estimate holds:*

$$\|x_{n+1} - x^*\| \leq (\kappa'')^{(n+1)} \|x_0 - x^*\|,$$

where $\kappa'' = \delta_0'\left((1-\alpha)\delta_0 + \alpha\delta_0''(1 - \theta(1 - \delta_0))\right)$ and $\delta_0' = \delta_0'' = \frac{3K_3\|x_0 - x^*\|}{2(1-rK_2)}$.

Proof

$i)$ For all $x \in B_r\left(x^*\right)$, by Lemma 2 and (15), we obtain

$$
\begin{aligned}
\left\|V_{y_0}(x) - x^*\right\| &= \left\|x - F_{y_0}'^{-1} F\left(x\right) - x^*\right\| \\
&= \left\|F_{y_0}'^{-1}\left[F\left(x\right) - F\left(x^*\right) - F_{y_0}'\left(x - x^*\right)\right]\right\| \\
&\leq \left\|\left(F_{x^*}'^{-1} F_{y_0}'\right)^{-1}\right\| \int_0^1 \left\|F_{x^*}'^{-1}\left(F_{x^*+t(x-x^*)}' - F_{y_0}'\left(x - x^*\right)\right)\right\| dt \\
&\leq \frac{K_3 \left\|x - x^*\right\| \left[\left\|x - x^*\right\| + 2\left\|y_0 - x^*\right\|\right]}{2\left(1 - rK_2\right)} \\
&\leq \delta_x' \left\|x - x^*\right\|.
\end{aligned}
$$

$ii)$

$$
\begin{aligned}
\delta' &= \sup_{x \in B_r(x^*)} \left\{\delta_x'\right\} = \frac{K_3 \left[\displaystyle\sup_{x \in B_r(x^*)} \left\|x - x^*\right\| + 2\left\|y_0 - x^*\right\|\right]}{2\left(1 - rK_2\right)} \\
&\leq \frac{K_3 \left[r + 2\left\|y_0 - x^*\right\|\right]}{2\left(1 - rK_2\right)} \\
&\leq \frac{K_3 \left[r + 2r\right]}{2\left(1 - rK_2\right)} \\
&< \frac{3rK_1}{2\left(1 - rK_2\right)} \\
&= 1.
\end{aligned}
$$

Thus, the operator V_{y_0} is a quasi-contraction self-mapping on $B_r(x^*)$.

$iii)$ Similarly, for all $x \in B_r\left(x^*\right)$, by Lemma 2 and (15), we obtain

$$
\begin{aligned}
\left\|V_{z_0}(x) - x^*\right\| &= \left\|x - F_{z_0}'^{-1} F\left(x\right) - x^*\right\| \\
&= \left\|F_{z_0}'^{-1}\left[F\left(x\right) - F\left(x^*\right) - F_{z_0}'\left(x - x^*\right)\right]\right\| \\
&\leq \left\|\left(F_{x^*}'^{-1} F_{z_0}'\right)^{-1}\right\| \int_0^1 \left\|F_{x^*}'^{-1}\left(F_{x^*+t(x-x^*)}' - F_{z_0}'\left(x - x^*\right)\right)\right\| dt \\
&\leq \frac{K_3 \left\|x - x^*\right\| \left[\left\|x - x^*\right\| + 2\left\|z_0 - x^*\right\|\right]}{2\left(1 - rK_2\right)} \\
&\leq \delta_x'' \left\|x - x^*\right\|.
\end{aligned}
$$

iv)

$$\delta'' = \sup_{x \in B_r(x^*)} \{\delta_x''\} = \frac{K_3 \left[\sup_{x \in B_r(x^*)} \|x - x^*\| + 2\|z_0 - x^*\| \right]}{2(1 - rK_2)}$$

$$\leq \frac{K_3 \left[r + 2\|z_0 - x^*\| \right]}{2(1 - rK_2)}$$

$$\leq \frac{K_3 \left[r + 2r \right]}{2(1 - rK_2)}$$

$$< \frac{3rK_1}{2(1 - rK_2)}$$

$$= 1.$$

Thus, the operator V_{z_0} is a quasi-contraction self-mapping on $B_r(x^*)$.

v) If we define an operator A_θ by

$$A_\theta(x) = x - \theta F_{x_0}'^{-1} F(x), \qquad \forall x \in B_r(x^*),$$

then A_θ is a quasi-contraction self-map on $B_r(x^*)$ and the following satisfies:

$$\|A_\theta(x) - x^*\| \leq (1 - \theta(1 - \delta_x))\|x - x^*\|, \qquad \forall x \in B_r(x^*).$$

As in Theorem 4, x^* is the unique solution of (1) in $B_{r_1}(x^*)$. Therefore, by Algorithm 12, we obtain

$$y_0 = (1 - \alpha)Ax + \alpha V_{z_0} A_\theta x_0 \in B_r(x^*)$$

and

$$z_0 = A_\theta x_0 \in B_r(x^*).$$

Also, we can rearrange Algorithm 12 as follows:

$$x_{n+1} = \Im_\alpha(x_n) = V_{y_0}\left[(1 - \alpha)A(x_n) + \alpha V_{z_0} A_\theta(x_n)\right].$$

Now we show that the sequence $\{x_n\}$ generated by Algorithm 12 is in $B_r(x^*)$ and it converges strongly to the unique solution x^* in $B_{r_1}(x^*)$. For all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\Im_\alpha(x_n) - x^*\| = \|V_{y_0}\left[(1 - \alpha)A(x_n) + \alpha V_{z_0} A_\theta(x_n)\right] - x^*\| \\
&\leq \delta_{y_n}' \|(1 - \alpha)A(x_n) + \alpha V_{z_0} A_\theta(x_n) - x^*\| \qquad (16) \\
&\leq \delta_{y_n}'(1 - \alpha)\|A(x_n) - x^*\| + \delta_{y_n}'\alpha\|V_{z_0} A_\theta(x_n) - x^*\| \\
&\leq \delta_{y_n}'(1 - \alpha)\delta_x\|x_n - x^*\| + \delta_{y_n}'\alpha\delta_{z_n}''\|A_\theta(x_n) - x^*\| \\
&\leq \delta_{y_n}'(1 - \alpha)\delta_{x_n}\|x_n - x^*\| + \delta_{y_n}'\alpha\delta_{z_n}''(1 - \theta(1 - \delta_{x_n}))\|x_n - x^*\| \\
&= \delta_{y_n}'\left[(1 - \alpha)\delta_{x_n} + \alpha\delta_{z_n}''(1 - \theta(1 - \delta_{x_n}))\right]\|x_n - x^*\|.
\end{aligned}
$$

From the definitions of δ_x'' and δ_x', we obtain

$$
\begin{aligned}
\delta_{z_n}'' &= \frac{K_3}{2(1 - rK_2)} \left(\|A_\theta(x_n) - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{K_3}{2(1 - rK_2)} \left((1 - \theta(1 - \delta_{x_n})) \|x_n - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{K_3}{2(1 - rK_2)} \left(\|x_0 - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{3K_1 \|x_0 - x^*\|}{2(1 - rK_2)} \\
&= \delta_0'' \\
&\leq \frac{\|x_0 - x^*\|}{r} \\
&= \delta_0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\delta_{y_n}' &= \frac{K_3}{2(1 - rK_2)} \left(\|(1 - \alpha)A(x_n) + \alpha V_{z_0} U_\theta(x_n) - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{K_3}{2(1 - rK_2)} \left((1 - \alpha)\delta_{x_n} + \alpha\delta_{z_n}''(1 - \theta(1 - \delta_{x_n})) \|x_n - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{K_3}{2(1 - rK_2)} \left(\|x_0 - x^*\| + 2\|x_0 - x^*\| \right) \\
&\leq \frac{3K_1 \|x_0 - x^*\|}{2(1 - rK_2)} \\
&= \delta_0' \\
&\leq \frac{\|x_0 - x^*\|}{r} \\
&= \delta_0.
\end{aligned}
$$

Hence, from (16), we obtain

$$
\|x_{n+1} - x^*\| \leq [\delta_0'((1 - \alpha)\delta_0 + \alpha\delta_0''(1 - \theta(1 - \delta_0)))]^{n+1} \|x_0 - x^*\|, \tag{17}
$$

and it implies that $x_n \to x^*$ as $n \to \infty$.

$vi)$ For all $n \in \mathbb{N}_0$, it is concluded from (17) that $\|x_{n+1} - x^*\| \leq (\kappa'')^{(n+1)} \|x_0 - x^*\|$.

$\square$

3.3. The rate of convergence

Theorem 6 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$ let $F_{x_0}'^{-1} \in B(Y, X)$. Assume that $F_{x_0}'^{-1}$ and F satisfy (6) with the following conditions:*

i) $\left\| F_{x_0}'^{-1} F(x_0) \right\| \le \eta$, *for some* $\eta > 0$.

ii) $\left\| F_{x_0}'^{-1} \right\| \le \beta$, *for some* $\beta > 0$.

Assume that $\gamma, \alpha, \theta \in (0,1)$, $h = \eta \beta K_0 < \frac{1}{2}$, and $B_r[x_0] \subseteq D$ such that $r = \frac{1 - \sqrt{1 - 2h}}{h} \eta$.

Under the above restrictions, for given $u_0 = x_0 \in D$, consider the iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ defined by (11) and (9), respectively. Then $\{x_n\}_{n=0}^{\infty}$ converges to x^* faster than $\{u_n\}_{n=0}^{\infty}$ does in $B_r[x_0]$.

Proof Respectively, by Theorems 1 and Theorem 3, we have the following inequalities:

$$\left\| u_{n+1} - x^* \right\| \le \gamma^{n+1} \left(1 - \alpha(1 - \gamma)\right)^{n+1} \left\| u_0 - x^* \right\|,$$

and

$$\left\| x_{n+1} - x^* \right\| \le \gamma^{2(n+1)} \left[1 - \alpha(1 + \gamma\theta)(1 - \gamma)\right]^{n+1} \left\| x_0 - x^* \right\|.$$

Define

$$a_n = \gamma^{2(n+1)} \left[1 - \alpha(1 + \gamma\theta)(1 - \gamma)\right]^{n+1} \left\| x_0 - x^* \right\|,$$

$$b_n = \gamma^{n+1} \left(1 - \alpha(1 - \gamma)\right)^{n+1} \left\| u_0 - x^* \right\|,$$

and

$$
\begin{aligned}
\wp_n &= \frac{a_n}{b_n} = \frac{\gamma^{2(n+1)} \left[1 - \alpha(1 + \gamma\theta)(1 - \gamma)\right]^{n+1} \left\| x_0 - x^* \right\|}{\gamma^{n+1} \left(1 - \alpha(1 - \gamma)\right)^{n+1} \left\| u_0 - x^* \right\|} \\
&= \gamma^{n+1} \left(\frac{1 - \alpha(1 + \gamma\theta)(1 - \gamma)}{1 - \alpha(1 - \gamma)} \right)^{n+1}.
\end{aligned}
$$

Since $\gamma, \alpha, \theta \in (0,1)$, we have

$$\left[\gamma \left(\frac{1 - \alpha(1 + \gamma\theta)(1 - \gamma)}{1 - \alpha(1 - \gamma)} \right) \right] < 1.$$

Therefore, $\lim_{n \to \infty} \wp_n = 0$. From Definition 3, we obtain that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{u_n\}_{n=0}^{\infty}$. $\square$

Theorem 7 *Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. Assume that $\alpha, \theta \in (0,1)$ and $x^* \in D$ is a solution of (1). For some $x_0 \in D$, let $F_{x^*}'^{-1} \in B(Y, X)$, and $F_{x^*}'^{-1}$ and F satisfy the conditions (7) and (8), and for all $x \in D$, some $K_3 > 0$,*

$$\left\| F_{x^*}'^{-1} \left(F_x' - F_{y_0}' \right) \right\| \le K_3 \left\| x - y_0 \right\|, \tag{18}$$

such that $K_3 \le K_1$. Assume that $B_{r_1}(x^*) \subseteq D$, where $r_1 = \frac{2}{K_2}$ and $x_0, y_0, z_0 \in B_r(x^*)$, where $r = \frac{2}{2K_2 + 3K_1}$.

Under the above restrictions, for given $u_0 = x_0 \in D$, consider the iterative sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ defined by (12) and (10), respectively. Then $\{x_n\}_{n=0}^{\infty}$ converges to x^* faster than $\{u_n\}_{n=0}^{\infty}$ does in $B_{r_1}(x^*)$.

Proof Respectively, by Theorems 2 and Theorem 5, we have the following inequalities:

$$\|u_{n+1} - x^*\| \leq \left(\delta_0' \left(1 - \alpha \left(1 - \delta_0\right)\right)\right)^{n+1} \|u_0 - x^*\|,$$

and

$$\|x_{n+1} - x^*\| \leq \left[\delta_0' \left(\left(1 - \alpha\right)\delta_0 + \alpha\delta_0'' \left(1 - \theta \left(1 - \delta_0\right)\right)\right)\right]^{n+1} \|x_0 - x^*\|.$$

Define

$$a_n = \left[\delta_0' \left(\left(1 - \alpha\right)\delta_0 + \alpha\delta_0'' \left(1 - \theta \left(1 - \delta_0\right)\right)\right)\right]^{n+1} \|x_0 - x^*\|,$$

$$b_n = \left(\delta_0' \left(1 - \alpha \left(1 - \delta_0\right)\right)\right)^{n+1} \|u_0 - x^*\|,$$

and

$$
\begin{aligned}
\wp_n &= \frac{a_n}{b_n} = \frac{\left[\delta_0' \left(\left(1 - \alpha\right)\delta_0 + \alpha\delta_0'' \left(1 - \theta \left(1 - \delta_0\right)\right)\right)\right]^{n+1} \|x_0 - x^*\|}{\left(\delta_0' \left(1 - \alpha \left(1 - \delta_0\right)\right)\right)^{n+1} \|u_0 - x^*\|} \\
&= \frac{\left[\left(1 - \alpha\right)\delta_0 + \alpha\delta_0'' \left(1 - \theta \left(1 - \delta_0\right)\right)\right]^{n+1}}{\left(1 - \alpha \left(1 - \delta_0\right)\right)^{n+1}}.
\end{aligned}
$$

Since

$$\delta_0'' \leq \delta_0$$

and $\gamma,\ \alpha, \theta \in (0,1)$, we have

$$\frac{\left[\left(1 - \alpha\right)\delta_0 + \alpha\delta_0'' \left(1 - \theta \left(1 - \delta_0\right)\right)\right]^{n+1}}{\left(1 - \alpha \left(1 - \delta_0\right)\right)^{n+1}} < 1.$$

Therefore, $\lim_{n\to\infty} \wp_n = 0$. From Definition 3, we obtain that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{u_n\}_{n=0}^{\infty}$. $\square$

Example 1 *[18] Let $X = \mathbb{R}$, $C = (-1, 1)$, and $F : C \to \mathbb{R}$ an operator defined by*

$$F(x) = e^x - 1, \quad \forall x \in C.$$

Then F is Fréchet differentiable and its Fréchet derivative F_x' at any point $x \in C$ is given by

$$F_x' = e^x.$$

For $x_0 = 0,26$, we get

$$F_x'^{-1} = \frac{1}{e^{0,26}}.$$

It is easy to see that $\beta = 0.771051585803566, \eta = 0.228948414196434$, and $K_0 = 2.718281828459046$. Then

$$h = \eta\beta K_0 < \frac{1}{2}.$$

Therefore, we have:
i) $\left\|F_{x_0}'^{-1} F\left(x_0\right)\right\| \leq \eta_0,$

ii) $\left\|F_{x_0}'^{-1}\right\| \le \beta_0$,

*iii)* $\left\|F_x' - F_{x_0}'\right\| \le K_0 \|x - x_0\|$.

Consequently, all conditions of Theorem 3 hold. Therefore, the sequence $\{x_n\}$ generated by (11) is in $B_r[x_0]$ and converges to $x^* \in B_r[x_0]$. For given $u_0 = x_0 \in C$, Figure 1, Figure 2, and the Table show that the sequence $\{x_n\}$ defined by (11) is faster than the sequence $\{u_n\}$ generated by (9).

**Table.** Comparison rate of convergence

x_n	Sahu Newton type	Picard-S Newton type
x_0	0.260000000000000	0.260000000000000
x_1	0.017117996475976	0.004551903139995
x_2	0.001613569978360	0.000134045775368
x_3	0.000155439797062	0.000003977964746
x_4	0.000015004477715	0.000000118077485
x_5	0.000001448654022	0.000000003504904
x_6	0.000000139867460	0.000000000104036
x_7	0.000000013504220	0.000000000003088
x_8	0.000000001303834	0.000000000000092
x_9	0.000000000125885	0.000000000000003
x_{10}	0.000000000012154	0.000000000000000
x_{11}	0.000000000001174	0.000000000000000
x_{12}	0.000000000000113	0.000000000000000
x_{13}	0.000000000000011	0.000000000000000
x_{14}	0.000000000000001	0.000000000000000
x_{15}	0.000000000000000	0.000000000000000

The Table shows that our iteration reaches a fixed point at the 10th step while the Sahu Newton-like iteration reaches it at the 15th step.

Figures 1 and 2 are the graphical presentations of the Table.
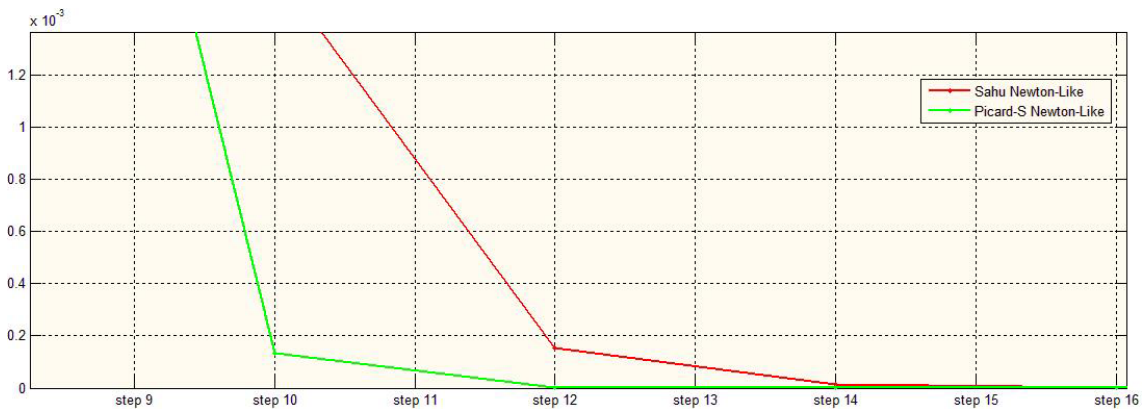
**Figure 1**. Comparison of the iteration processes defined by (1.9) and (1.11).
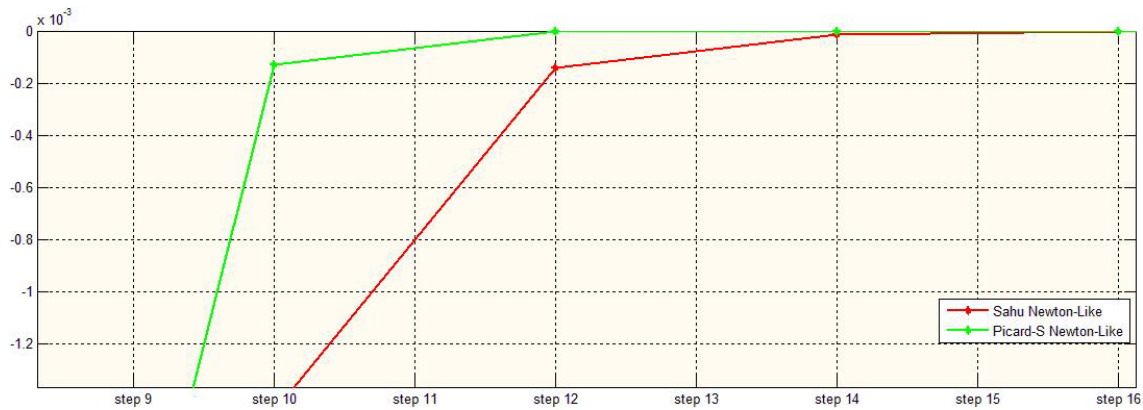
Figure 2. Derivative comparison of the iteration processes defined by (1.9) and (1.11).

Corollary 1 (9) *is shown to be faster than the iteration method of* (5). *In Example* 1, (11) *iterations were shown to be faster than* (9). *For this reason, it is proved that the* (11) *iteration method is faster than the* (5) *iteration method.*

An interesting result is that Newton's method works for complex valued functions. Having seen that Newton's method behaves differently for different starting points, converging to different roots or possibly not converging at all, what happens at problem areas in the complex plane? For example, consider the case of starting points that are equidistant to multiple different roots. Using the Newton method one can attempt to visualize the convergence of each possible starting complex number, resulting in a fractal pattern.

Example 2 *[8] Figures 3 and 4 present examples of fractal polynomiography obtained using the two iteration methods* 9 *and* 11 *by taking the following parameters:* $P(z) = z^7 + z^2 - 1$, $M = 15$, *standard convergence test with* $\varepsilon = 0.001$, $A = [-1.5, 1.5] \times [-1.5, 1.5]$, *and fixing the coefficients as follows: Picard-S:* $\alpha = 0.9 + i$, $\beta = 0.6 - 0.5i$, *Sahu* $\alpha = 0.9 + i$.
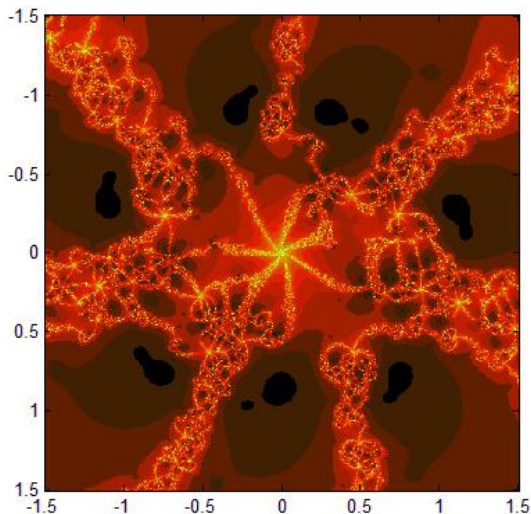
Figure 3. Picard-S-Newton-like.
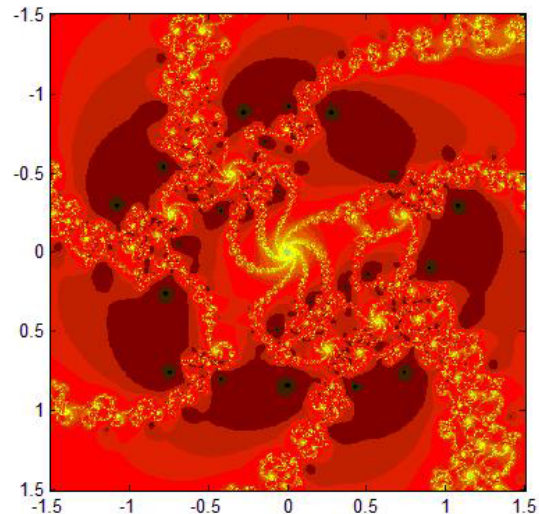
Figure 4. Sahu-Newton-like.

We notice that in each polynomiograph we find seven main areas with different forms and colors. The fractal boundaries represent the seven solutions of the polynomial. Further, the colors in Figures 3 and 4 show how long it took the starting point to reach the convergence point. In the darker colored spots the convergence is faster than in lighter colored ones.

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