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Weyl- and Horn-type inequalities for cyclically compact operators

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Abstract: A variant of Weyl- and Horn-type inequalities for cyclically compact operators on Kaplansky–Hilbert modules is given.

Key words: Kaplansky–Hilbert module, Horn inequality, Weyl inequality, cyclically compact operator

1. Introduction

Kaplansky–Hilbert modules (or AW^*-modules) were initiated and investigated by Kaplansky ([7]), who proved some deep and elegant results for such structures, which share many properties common to Hilbert spaces. In [9, 10], Kusraev introduced the cyclical compactness notion and proved some important structural properties of cyclically compact sets and operators. In addition, cyclically compact operators hold a general form in Kaplansky–Hilbert modules, which resembles the Schmidt representation of compact operators on a Hilbert space (see also [11, 12]). A detailed investigation of the class of cyclically compact operators has henceforth arisen due intrinsically to Kusraev's results. With this motivation in hand, cyclically compact operators have been recently investigated by the author in a series of papers [2–5] (see also [8]).

Two well-known inequalities, the Weyl and the Horn inequalities, which derive an interesting connection between the sequences of eigenvalues and singular numbers of compact operators acting on a Hilbert space, were proved in [6, 13] (cf. [1]). In the present note, we will derive vector-valued versions of these inequalities. The main technical tool used in the work is the functional representations of Kaplansky–Hilbert modules and bounded linear operators on them (see [12, Theorem 7.4.12, 7.5.10, and 7.5.12]). The unexplained terminology and notation can be found in [12].

2. Preliminaries

A Stone algebra is a commutative C^*-algebra (with unity) whose self-adjoint part is a Dedekind-complete Riesz space with respect to its natural partial ordering. In the literature, Stone algebra is known as commutative AW^*-algebra, a term coined by Kaplansky.

Let Λ be a Stone algebra. The set of all projections of Λ, which is a complete Boolean algebra, is denoted by the symbol $\mathfrak{P}(\Lambda)$. A disjoint subset $(e_i)_{i\in I}$ of $\mathfrak{P}(\Lambda)$ is refered to as a partition of unity if $\mathbf{1} = \sup\{e_i : i \in I\}$, where $\mathbf{1}$ is the unit element of Λ. The symbol $[a] := \inf\{\pi \in \mathfrak{P}(\Lambda) : \pi a = a\}$ is called the support of a in Λ.

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Let X be a unitary module over Λ. The function $\langle \cdot \mid \cdot \rangle : X \times X \to \Lambda$ is said to be a Λ-*valued inner product* if the following are satisfied for each x, y, z in X and a in Λ:

(i) $\langle x \mid x \rangle \geq 0$; $\langle x \mid x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle ax + y \mid z \rangle = a \langle x \mid z \rangle + \langle z \mid y \rangle^*$.

The unitary module X over Λ, which is complete with respect to the scalar-valued norm $|||x||| := \|\langle x \mid x \rangle\|^{\frac{1}{2}}$, is called a *Kaplansky–Hilbert module* over Λ provided that the following are fulfilled:

(1) Let $(e_i)_{i \in I}$ be a partition of unity in $\mathfrak{P}(\Lambda)$, and let x be in X with $e_i x = 0$ for all $i \in I$, then $x = 0$;

(2) Let $(e_i)_{i \in I}$ be a partition of unity in $\mathfrak{P}(\Lambda)$, and let $(x_i)_{i \in I}$ be a norm-bounded family in X, then there is an element $x \in X$ such that $e_i x = e_i x_i$ for all $i \in I$.

Using (1), the element x of (2) is unique: it is denoted as $x = \operatorname{mix}_{i \in I}(e_i x_i)$. A norm-bounded subset C of X will be called mix-complete if for any index set J, $\operatorname{mix}_{j \in J}(e_j x_j)$ is an element of C for each partition of unity $(e_j)_{j \in J}$ in $\mathfrak{P}(\Lambda)$ and $(x_j)_{j \in J}$ in C. As a simple corollary of (2), the closed unit ball of X is mix-complete. In addition to the scalar-valued norm, we can define a vector-valued norm $\lvert x \rvert := \sqrt{\langle x \mid x \rangle}$. Throughout this paper, we assume that if for some $a \in \Lambda$ and for every $x \in X$, $ax = 0$, then $a = 0$.

A family $(x_i)_{i \in I} \subset X$ is said to be *bo-summable* if we have an element x and a net $e_i \downarrow 0$ in Λ such that for each i there is a finite $F \subset I$ whose every subset L satisfies $\left\lvert \sum_{j \in L} x_j - x \right\rvert \leq e_i$. Besides, it is conventional to write $x = bo\text{-}\sum_{i \in I} x_i$. Note that if $(x_i)_{i \in I}$ and $(e_i)_{i \in I}$ are chosen as in (2), then we have $\operatorname{mix}_{i \in I}(e_i x_i) = bo\text{-}\sum_{i \in I} e_i x_i$. On the other hand, we write $x = o\text{-}\sum_{i \in I} x_i$ in the case of x, x_i in Λ.

A nonempty subset E of X is called *projection orthonormal* provided that $\langle x \mid y \rangle = 0$, and $\langle x \mid x \rangle$ is a nonzero projection for all distinct $x, y \in E$.

Denote by $\mathrm{Prt}_{\mathbb{N}}$ the family of all sequences in $\mathfrak{P}(\Lambda)$ that are partitions of unity. For $\pi_1, \pi_2 \in \mathrm{Prt}_{\mathbb{N}}$, the expression $\pi_1 \ll \pi_2$ means the following:

$$\text{if } k, l \in \mathbb{N} \text{ and } \pi_1(k) \wedge \pi_2(l) \neq 0, \text{ then } k < l.$$

Let $(s_k)_{k \in \mathbb{N}}$ be a bounded sequence in X and $\pi \in \mathrm{Prt}_{\mathbb{N}}$. Then we can define an element $s_\pi := \operatorname{mix}_{n \in \mathbb{N}}(\pi(n) s_n)$ in X. Using this method, for each sequence $(\pi_k)_{k \in \mathbb{N}} \subset \mathrm{Prt}_{\mathbb{N}}$ with $\pi_k \ll \pi_{k+1}$, one may derive a new sequence $(s_{\pi_k})_{k \in \mathbb{N}}$, which is referred to as cyclic subsequence of $(s_k)_{k \in \mathbb{N}}$. A mix-complete subset K of X is called *cyclically compact* whenever every sequence in K has a cyclic subsequence converging to an element of K. A subset of X that is contained in a cyclically compact set is said to be *relatively cyclically compact.* A continuous Λ-linear operator T from X into Y is called *cyclically compact* provided that the image of the closed unit ball of X under T is relatively cyclically compact in Y.

A nonzero element λ of Λ is called a *global eigenvalue* of an operator T if $Tx = \lambda x$ and $[\lambda] \leq \left[\lvert x \rvert\right]$ hold for some x in X. A sequence $(\lambda_k)_{k \in \mathbb{N}}$ that satisfies the following conditions is called a *global eigenvalue sequence* of T:

(i) λ_k is either a global eigenvalue of T or zero in Λ for each $k \in \mathbb{N}$;

(ii) $|\lambda_k| \leq \lvert T \rvert$, $[\lambda_k] \geq [\lambda_{k+1}]$ $(k \in \mathbb{N})$, and $o\text{-}\lim_{k \to \infty} \lambda_k = 0$;

(iii) $\pi\lambda_{k+m} \neq \pi\lambda_k$ for every nonzero projection $\pi \leq [\lambda_k]$ and for all $m, k \in \mathbb{N}$;

(iv) if λ is a global eigenvalue of T, then there is a partition $(p_k)_{k\in\mathbb{N}}$ of $[\lambda]$ with $\lambda = \text{mix}_{k\in\mathbb{N}}(p_k\lambda_k)$.

It is known [3, Theorem 4.10] that a global eigenvalue sequence exists for all cyclically compact operators on X. Now we recall the definition of the multiplicity of a global eigenvalue λ of cyclically compact operator T. From [3, Corollary 4.4] and subsequent consequences, we have a unique sequence $(\tau_{\lambda,l}(n))_{l,n\in\mathbb{N}}$ in $\mathfrak{P}(\Lambda)$ such that $\tau_{\lambda,l}(n) \leq \tau_{\lambda,l+1}(n)$ and $\tau_{\lambda,l}(n) \wedge \tau_{\lambda,k}(m) = 0$ for all $k, l, m, n \in \mathbb{N}$ with $m \neq n$. Moreover, $\tau_{\lambda,l}(n)\left(\bigcup_{k\in\mathbb{N}} \text{Ker}\,(T-\lambda I)^k\right) = \tau_{\lambda,l}(n)\text{Ker}\,(T-\lambda I)^l$ is an n-homogeneous Kaplansky–Hilbert module over $\tau_{\lambda,l}(n)\Lambda$. The multiplicity of the global eigenvalue λ is defined as follows:

$$\overline{\tau}_\lambda := o\text{-}\sum_{n\in\mathbb{N}} n \sup_{l\in\mathbb{N}} \{\tau_{\lambda,l}(n)\} = \sup_{l,n\in\mathbb{N}} \{n\tau_{\lambda,l}(n)\} \in (\text{Re}\Lambda)^\infty,$$

where the universally complete vector lattice $(\text{Re}\Lambda)^\infty$ is the universal completion of $\text{Re}\Lambda$. One may consult [3, 4] for further information about global eigenvalues.

Now we recall the spaces $C_\#(Q,H)$ and $SC_\#(Q,B(H))$ used in the representation of Kaplansky–Hilbert modules and bounded linear operators on them (see [12, 7.4.8 and 7.5.9] for full details). Let Q be a Stonean space and $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Vector-functions $x : \text{dom}(x) \to H$ and $y : \text{dom}(y) \to H$ are equivalent if $x(t) = y(t)$ is satisfied for all $t \in \text{dom}(x) \cap \text{dom}(y)$ where $\text{dom}(x)$ and $\text{dom}(y)$ are comeager subsets of Q. (Recall that a set is called comeager if its complement is of first category.) The set of the equivalence classes of all bounded continuous vector-functions is represented by $C_\#(Q,H)$, which can be equipped with the form of a module over $C(Q)$, and we can introduce a $C(Q)$-valued inner product in $C_\#(Q,H)$ as follows: if $\widetilde{x}, \widetilde{y} \in C_\#(Q,H)$, then the function

$$q \mapsto \langle x(q), y(q)\rangle \ \ (q \in \text{dom}(x) \cap \text{dom}(y))$$

, being continuous and admitting a unique continuous extention $z \in C(Q)$, assigns $\langle\widetilde{x} \mid \widetilde{y}\rangle := z$. By [12, 7.4.8 (1)], $C_\#(Q,H)$ is a Kaplansky–Hilbert module over $C(Q)$. Furthermore, every Kaplansky–Hilbert module is represented nearly as a direct sum of these spaces [12, 7.4.12].

Denote by $SC_\#(Q,B(H))$ the set of all equivalence classes $\widetilde{u}$ such that the operator-function u : $\text{dom}(u) \to B(H)$ is defined on the comeager subset of Q and is continuous in the strong operator topology, and the set $\left\{\left|\widetilde{uh}\right| : \|h\| \leq 1, h \in H\right\}$ is bounded in $C(Q)$, where $\widetilde{uh} \in C_\#(Q,H)$ is the equivalence class of the function $uh : q \mapsto u(q)h \ \ (q \in \text{dom}(u))$. Since $\left|\widetilde{uh}\right|$ coincides with the function $q \mapsto \|u(q)h\| \ \ (q \in \text{dom}(u))$, the membership $\widetilde{u} \in SC_\#(Q,B(H))$ means that the mapping $q \mapsto \|u(q)\| \ \ (q \in \text{dom}(u))$ is bounded and continuous on some comeager set. Hence, there are an element $\left|\widetilde{u}\right| \in C(Q)$ and a comeager set $Q_0 \subset Q$ satisfying $\left|\widetilde{u}\right|(q) = \|u(q)\| \ \ (q \in Q_0)$. Moreover, $SC_\#(Q,B(H))$ can be equipped with the structure of a $*$-algebra and a unitary $C(Q)$-module, and hence becomes an AW^*-algebra. (See [12, 7.5.10].)

Given $\widetilde{u} \in SC_\#(Q,B(H))$, one may define for each $\widetilde{x} \in C_\#(Q,H)$ the element $\widetilde{u}\widetilde{x} := \widetilde{ux} \in C_\#(Q,H)$, with $ux : q \mapsto u(q)x(q) \ \ (q \in \text{dom}(u) \cap \text{dom}(x))$. Denote the operator $\widetilde{x} \mapsto \widetilde{u}\widetilde{x}$ by $S_{\widetilde{u}}$, and note that $\left|S_{\widetilde{u}}\widetilde{x}\right| \leq \left|\widetilde{u}\right|\left|\widetilde{x}\right|$. By [12, 7.5.12], the set of all continuous Λ-linear operators on X, or shortly $B_\Lambda(X)$, which is an AW^*-algebra of type I with center isomorphic to Λ ([7, Theorem 7]), is represented nearly as a direct sum of $SC_\#(Q,B(H))$ spaces.

3. The main result

Whenever a cyclically compact operator T is given, the family $(s_k(T))_{k\in\mathbb{N}}$ in Λ satisfying the representation theorem [12, 8.5.6. Theorem] is called the singular numbers of T. Moreover, sequences $(\widetilde{e}_k)_{k\in\mathbb{N}}$ and $(\widetilde{f}_k)_{k\in\mathbb{N}}$ in $C_\#(Q,H)$ will verify the statements of [12, 8.5.6. Theorem] for the cyclically compact operator U on $C_\#(Q,H)$.

Proposition 3.1 *Let $U = S_{\widetilde{u}}$ be a cyclically compact operator on $C_\#(Q,H)$. Then for some comeager subset $Q_0 \subset Q$, the numbers $s_k(U)(q)$ are the singular numbers of the compact operator $u(q)$ on H for each $q \in Q_0$. Moreover, the following representation is satisfied for all $q \in Q_0$:*

$$u(q)h = \sum_{k=1}^{\infty} s_k(U)(q)\,\langle h, e_k(q)\rangle\, f_k(q).$$

Proof Let $\widetilde{x} \in C_\#(Q,H)$ and $n \in \mathbb{N}$. Since $(s_k(U))_{k\in\mathbb{N}}$ is a decreasing sequence, we have

$$\begin{aligned}
\left|U\widetilde{x} - \sum_{k=1}^{n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle\, \widetilde{f}_k\right|^2 &= \left| bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle\, \widetilde{f}_k \right|^2 \\
&= \left\langle bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle \widetilde{f}_k \,\middle|\, bo\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)\,\langle \widetilde{x} \mid \widetilde{e}_k\rangle \widetilde{f}_k \right\rangle \\
&= o\text{-}\sum_{k\in\mathbb{N}_n} s_k(U)^2\,|\langle \widetilde{x} \mid \widetilde{e}_k\rangle|^2\, \left|\widetilde{f}_k\right| \\
&\le s_{n+1}(U)^2 \left(o\text{-}\sum_{k\in\mathbb{N}_n} |\langle \widetilde{x} \mid \widetilde{e}_k\rangle|^2 \right) \le s_{n+1}(U)^2\, |x|^2,
\end{aligned}$$

where $\mathbb{N}_n := \{k \in \mathbb{N} : k > n\}$. As $\inf_{k\in\mathbb{N}} s_k(U) = 0$ holds in $C(Q)$, there is a comeager set Q_1 in Q with $\inf_{k\in\mathbb{N}} s_k(U)(q) = 0$ for all $q \in Q_1$. Define

$$Q_0 := Q_1 \cap \operatorname{dom}(u) \cap \left(\bigcap_{k\in\mathbb{N}} \operatorname{dom}(e_k) \right) \cap \left(\bigcap_{k\in\mathbb{N}} \operatorname{dom}(f_k) \right),$$

and note that Q_0 is a comeager set in Q. We then see that $\{e_k(q) : k \in \mathbb{N}, e_k(q) \neq 0\}$ and $\{f_k(q) : k \in \mathbb{N}, f_k(q) \neq 0\}$ are orthonormal sets in H for all $q \in Q_0$. Let now $h \in H$ be given. Define the function $z : q \mapsto h$ $(q \in Q)$, and note that $|\widetilde{z}|(q) = \|h\|$ $(q \in Q)$ and $\operatorname{dom}(z) = Q$. Therefore, for each $q \in Q_0$, the inequality

$$\begin{aligned}
\left\| u(q)h - \sum_{k=1}^{n} s_k(U)(q)\,\langle h, e_k(q)\rangle\, f_k(q) \right\| &= \left| U\widetilde{z} - \sum_{k=1}^{n} s_k(U)\,\langle \widetilde{z} \mid \widetilde{e}_k\rangle\, \widetilde{f}_k \right|(q) \\
&\le s_{n+1}(U)(q)\, |\widetilde{z}|(q) = s_{n+1}(U)(q)\, \|h\|
\end{aligned}$$

holds. Thus, we deduce that for each $q \in Q_0$, one has

$$u(q)h = \sum_{k=1}^{\infty} s_k(U)(q)\,\langle h, e_k(q)\rangle\, f_k(q),$$

from which the numbers $s_k(U)(q)$ are the singular numbers of the compact operator $u(q)$ by [1, Theorem 15.7.1]. □

The following result is a vector version of Weyl- and Horn-type inequalities for cyclically compact operators on Kaplansky–Hilbert modules.

Theorem 3.2 *Let T be a cyclically compact operator on X and $(\lambda_k(T))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of T with the multiplicity sequence $(\overline{\tau}_k(T))_{k\in\mathbb{N}}$. Then the following hold:*

(1) (Weyl inequality) *If $(\pi s_k(T))_{k\in\mathbb{N}}$ is o-summable in Λ for some projection π, then one has*

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(T)|\lambda_k(T)| \le o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(T).$$

(2) (Horn inequality) *Suppose that T_k is a cyclically compact operator on X for $1 \le k \le K$. Then*

$$\prod_{i=1}^{N} s_i(T_K\cdots T_1) \le \prod_{k=1}^{K}\prod_{i=1}^{N} s_i(T_k) \quad (N\in\mathbb{N}).$$

Proof The proof will be carried over for the case $X = C_{\#}(Q,H)$ and $T = S_{\widetilde{u}}$. The general case is obtained directly by invoking the functional representation of Kaplansky–Hilbert modules and bounded linear operators on them given in [12, Theorems 7.4.12. and 7.5.12.]. Let $(\lambda_k(U))_{k\in\mathbb{N}}$ be a global eigenvalue sequence of U with the multiplicity sequence $(\overline{\tau}_k(U))_{k\in\mathbb{N}}$. From [3, Corollary 4.8, Theorem 4.10] and Proposition 3.1, there exists a comeager set Q_0 such that for each $q\in Q_0$, the following statements hold:

(i) the numbers $s_k(U)(q)$ are the singular numbers of compact operator $u(q)$;

(ii) $\mathrm{Sp}^*(u(q)) = \{\lambda_n(U)(q) : \lambda_n(U)(q) \neq 0\}$;

(iii) $\lambda_n(U)(q) \neq \lambda_m(U)(q)$ if $\lambda_n(U)(q)\neq 0$ or $\lambda_m(U)(q)\neq 0$ for $n\neq m$;

(iv) if $\lambda_k(U)(q)\neq 0$, then $\overline{\tau}_k(U)(q) = m(\lambda_k(U)(q)) \in \mathbb{N}$, where $m(\lambda_k(U)(q))$ is the algebraic multiplicity of $\lambda_k(U)(q)$.

Moreover, $s_k(U)(q)\neq 0$ implies $\|e_k(q)\| = \|f_k(q)\| = 1$.

(1) Let $(\pi s_k(U))_{k\in\mathbb{N}}$ be o-summable for some projection π. Using (i), (ii), (iii), (iv), and Weyl's inequality for the compact operator $u(q)$, one observes that

$$\sum_{k=1}^{\infty} \overline{\tau}_k(U)(q)|\lambda_k(U)(q)| = \sum_{k=1}^{\infty} m(\lambda_k(U)(q))|\lambda_k(U)(q)| \le \sum_{k=1}^{\infty} s_k(U)(q)$$

holds on a comeager set Q_0. This implies

$$o\text{-}\sum_{k\in\mathbb{N}} \pi\overline{\tau}_k(U)|\lambda_k(U)| \le o\text{-}\sum_{k\in\mathbb{N}} \pi s_k(U)$$

since $\sum_{k=1}^{\infty} \pi(q)s_k(U)(q)$ is finite for each $q\in Q$.

(2) Observe that $S_{\widetilde{u}_K} \cdots S_{\widetilde{u}_1} = S_{\widetilde{u}_K \cdots \widetilde{u}_1}$ and $(u_K \cdots u_1)(q) = u_K(q) \cdots u_1(q)$. Define $U_k := S_{\widetilde{u}_k}$, $1 \leq k \leq K$. Using (i), there exists a comeager set Q_0 such that for each $q \in Q_0$ the numbers $s_k(U_K \cdots U_1)(q)$ and $s_k(U_k)(q)$ are the singular numbers of compact operators $u_K(q) \cdots u_1(q)$ and $u_k(q)$ $(1 \leq k \leq K)$, respectively. Therefore, from Horn's inequality for compact operators $u_k(q)$ with $1 \leq k \leq K$, one gets the validity of

$$\prod_{i=1}^{N} s_i(U_K \cdots U_1)(q) \leq \prod_{k=1}^{K} \prod_{i=1}^{N} s_i(U_k)(q)$$

for all $q \in Q_0$. Thus, the desired inequality follows.

□

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